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Dedicated to the memory of
Academician I. V. Skrypnik

The transmission problem for quasi-linear elliptic second order equations in a conical domain

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Abstract. We investigate the behavior of weak solutions to the transmission problem for quasi-linear elliptic divergence second order equations in a neighborhood of the boundary conical point.

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1. Introduction

The transmission problems often appear in different fields of physics and technics. For instance, one of the importance problem of the electrodynamics of solid media is the electromagnetic processes research in ferromagnetic media with different dielectric constants. These problems appear as well as in solid mechanics if a body consists of composite materials.

In this work we obtain estimates of weak solutions of the nonlinear elliptic transmission problem near conical boundary point. Namely, for weak solutions of this problem we establish the possible exponent in the local bound of the solution modulus. Earlier the quasi-linear transmission problem was investigated only in smooth domains (see works of M. V. Borsuk [1], V. Ja. Rivkind, N. N. Ural'tseva [15], N. Kutev, P. L. Lions [11]). Later other mathematicians are studied transmission problems

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in non-smooth domains in some particular linear cases (see the references cited in [5, 13, 14]). General *linear* interface problems in polygonal and polyhedral domains was considered in [13, 14]. Regularity results in terms of weighted Sobolev–Kondratiev spaces are obtained in [5] for two and three dimensional transmission problems for the Laplace operator. D. Kapanadze and B.-W. Schulze studied boundary-contact problems with conical [8] singularities and edge [9] singularities at the interfaces for general linear any order elliptic equations (as well as systems). They constructed parametrix and showed regularity with asymptotics of solutions in weighted Sobolev–Kondratiev spaces. We knew only one paper, namely the D. Knees work [10], that concerns with the study of the regularity of weak solutions of special nonlinear transmission problem on polyhedral domains. A principal new feature of our work is the consideration of estimates of solutions for elliptic *general divergence quasi-linear* second order equations in n-dimensional conic domains.

Let $G \subset \mathbb{R}^n$, $n \geq 2$ be a bounded domain with boundary ∂G that is a smooth surface everywhere except at the origin $\mathcal{O} \in \partial G$ and near the point $\mathcal{O}$ it is a conical surface with vertex at $\mathcal{O}$. We assume that $G = G_+ \cup G_- \cup \Sigma_0$ is divided into two subdomains G_+ and G_- by a $\Sigma_0 = G \cap \{x_n = 0\}$, where $\mathcal{O} \in \overline{\Sigma}_0$.

We consider the elliptic transmission problem

$$\begin{cases} -\frac{d}{dx_i} a_i(x, u, \nabla u) + b(x, u, \nabla u) = 0, & x \in G \setminus \Sigma_0; \\ \\ [u]_{\Sigma_0} = 0, & \\ \mathcal{S}[u] \equiv \left[\frac{\partial u}{\partial \nu}\right]_{\Sigma_0} + \frac{1}{|x|^{m-1}} \sigma\left(\frac{x}{|x|}\right) u \cdot |u|^{q+m-2} = h(x, u), & x \in \Sigma_0; \\ \\ \mathcal{B}[u] \equiv \frac{\partial u}{\partial \nu} + \frac{1}{|x|^{m-1}} \gamma\left(\frac{x}{|x|}\right) u \cdot |u|^{q+m-2} = g(x, u), & x \in \partial G \setminus \mathcal{O} \end{cases} \tag{QL}$$

(summation over repeated indices from 1 to n is understood); here:

- $$u(x) = \begin{cases} u_+(x), & x \in G_+, \\ u_-(x), & x \in G_-; \end{cases}$$
  $$a_i(x, u, u_x) = \begin{cases} a_i^+(x, u_+, \nabla u_+), & x \in G_+, \\ a_i^-(x, u_-, \nabla u_-), & x \in G_- \end{cases} \qquad \text{etc.;}$$
- $[u]_{\Sigma_0} = u_+(x)\big|_{\Sigma_0} - u_-(x)\big|_{\Sigma_0}$, where $u_\pm(x)\big|_{\Sigma_0} = \lim\limits_{G_\pm \ni y \to x \in \Sigma_0} u_\pm(y)$;

- $\frac{\partial u}{\partial \nu} = a_i(x, u, u_x)\cos(\overrightarrow{n}, x_i)$, where $\overrightarrow{n}$ denotes the unite outward with respect to G_+ (or G) normal to Σ_0 (respectively $\partial G \setminus \mathcal{O}$);
- $\left[\frac{\partial u}{\partial \nu}\right]_{\Sigma_0}$ denotes the saltus of the co-normal derivative of the function $u(x)$ on crossing Σ_0, i.e.

$$\left[\frac{\partial u}{\partial \nu}\right]_{\Sigma_0} = a_i^+(x, u_+, \nabla u_+)\cos(\overrightarrow{n}, x_i)\big|_{\Sigma_0} - a_i^-(x, u_-, \nabla u_-)\cos(\overrightarrow{n}, x_i)\big|_{\Sigma_0}.$$

We obtain estimates of weak solutions to problem (QL) near a conical boundary point. Such estimates are *sharp* in the case of quasi-linear equations with semi-linear principal part, namely for the transmission problem

$$\begin{cases} -\frac{d}{dx_i}\left(|u|^q a^{ij}(x)u_{x_j}\right) + b(x, u, \nabla u) = 0, \quad q \geq 0, & x \in G \setminus \Sigma_0; \\ [u]_{\Sigma_0} = 0, \quad \mathcal{S}[u] \equiv \left[\frac{\partial u}{\partial \nu}\right]_{\Sigma_0} + \frac{1}{|x|}\sigma\left(\frac{x}{|x|}\right)u \cdot |u|^q = h(x, u), & x \in \Sigma_0; \\ \mathcal{B}[u] \equiv \frac{\partial u}{\partial \nu} + \frac{1}{|x|}\gamma\left(\frac{x}{|x|}\right)u \cdot |u|^q = g(x, u), & x \in \partial G \setminus \mathcal{O} \end{cases} \tag{WL}$$

(summation over repeated indices from 1 to n is understood).

Remark 1.1. The problem (WL) is the problem (QL) with $a_i(x, u, \nabla u) = |u|^q a^{ij}(x)u_{x_j}$, $i = 1, \ldots n$ and $m = 2$.

We introduce the following notations:

- S^{n-1} : a unit sphere in $\mathbb{R}^n$ centered at $\mathcal{O}$;
- $(r, \omega), \omega = (\omega_1, \omega_2, \ldots, \omega_{n-1})$: the spherical coordinates of $x \in \mathbb{R}^n$ with pole $\mathcal{O}$;
- $\mathcal{C}$: the rotational cone $\{x_1 > r\cos\frac{\omega_0}{2}\}$ with the vertex at $\mathcal{O}$;
- $\partial\mathcal{C}$: the lateral surface of $\mathcal{C}$: $\{x_1 = r\cos\frac{\omega_0}{2}\}$;
- Ω : a domain on the unit sphere S^{n-1} with smooth boundary $\partial\Omega$ obtained by the intersection of the cone $\mathcal{C}$ with the sphere S^{n-1};
- $\Omega_+ = \Omega \cap \{x_n > 0\}$, $\Omega_- = \Omega \cap \{x_n < 0\} \Longrightarrow \Omega = \Omega_+ \cup \Omega_- \cup \sigma_0$; $\sigma_0 = \Sigma_0 \cap \Omega$;
- $\partial\Omega = \partial\mathcal{C} \cap S^{n-1}$, $\partial_\pm\Omega = \overline{\Omega_\pm} \cap \partial\mathcal{C}$, $\partial\Omega_\pm = \partial_\pm\Omega \cup \sigma_0$;

- $G_a^b = \{(r,\omega) \mid 0 \le a < r < b;\ \omega \in \Omega\} \cap G$: a layer in $\mathbb{R}^n$;
- $\Gamma_a^b = \{(r,\omega) \mid 0 \le a < r < b; \omega \in \partial\Omega\} \cap \partial G$: the lateral surface of layer G_a^b;
- $\Sigma_a^b = G_a^b \cap \{x_n = 0\} \subset \Sigma_0$;
- $G_d = G \setminus G_0^d,\ \Gamma_d = \partial G \setminus \Gamma_0^d,\ \Sigma_d = \Sigma_0 \setminus \Sigma_0^d,\ d > 0$;
- $\Omega_\rho = G_0^d \cap \{|x| = \rho\};\ 0 < \rho < d.$

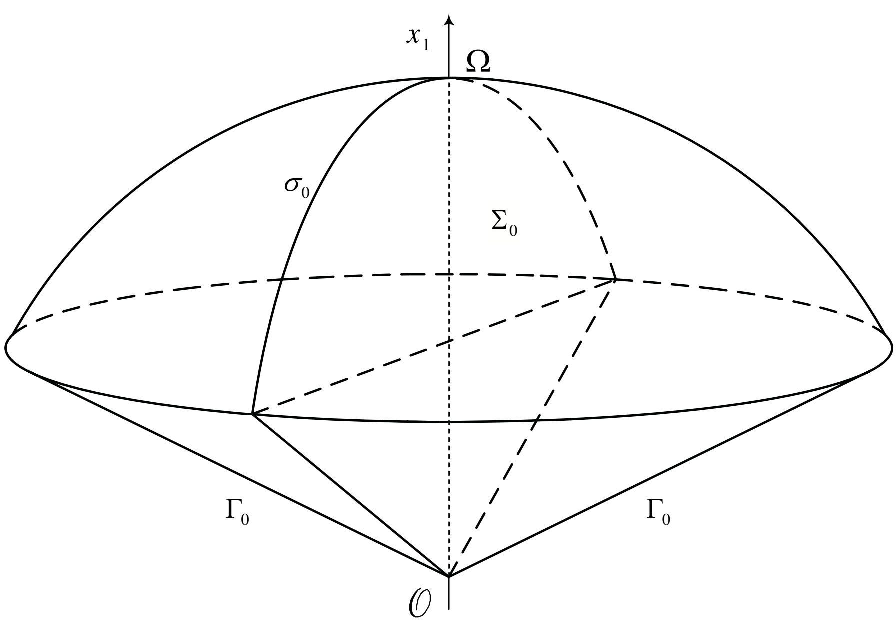

Fig. 1

We use the standard function spaces: $C^k(\overline{G_\pm})$ with the norm $|u_\pm|_{k,G_\pm}$, Lebesgue space $L_m(G_\pm),\ m \ge 1$ with the norm $\|u_\pm\|_{m,G_\pm}$, the Sobolev space $W^{k,m}(G_\pm)$ with the norm $\|u_\pm\|_{k,m;G_\pm}$, and introduce their direct sums $\mathbf{C}^k(\overline{G}) = C^k(\overline{G_+}) \dotplus C^k(\overline{G_-})$, $\mathbf{L}_m(G) = L_m(G_+) \dotplus L_m(G_-)$,

$\mathbf{W}^{k,m}(G) = W^{k,m}(G_+) \dotplus W^{k,m}(G_-)$. We define the weighted Sobolev spaces: $\mathbf{V}^k_{m,\alpha}(G)$ for integer $k \geq 0$ and real α as the space of distributions $u \in \mathcal{D}'(G)$ with the finite norm

$$\|u\|_{\mathbf{V}^k_{m,\alpha}(G)} = \Bigg(\int\limits_{G_+} \sum_{|\beta|=0}^{k} r^{\alpha+m(|\beta|-k)} |D^\beta u_+|^m \, dx \Bigg)^{\frac{1}{m}} + \Bigg(\int\limits_{G_-} \sum_{|\beta|=0}^{k} r^{\alpha+m(|\beta|-k)} |D^\beta u_-|^m \, dx \Bigg)^{\frac{1}{m}}$$

and $\mathbf{V}^{k-\frac{1}{m}}_{m,\alpha}(\partial G)$ as the space of functions φ, given on ∂G, with the norm $\|\varphi\|_{\mathbf{V}^{k-\frac{1}{m}}_{m,\alpha}(\partial G)} = \inf \|\Phi\|_{\mathbf{V}^k_{m,\alpha}(G)}$, where the infimum is taken over all functions Φ such that $\Phi|_{\partial G} = \varphi$ in the sense of traces. We denote

$$\mathbf{W}^k(G) \equiv \mathbf{W}^{k,2}(G), \quad \overset{\circ}{\mathbf{W}}{}^k_\alpha(G) \equiv \mathbf{V}^k_{2,\alpha}(G), \quad \overset{\circ}{\mathbf{W}}{}^{k-\frac{1}{2}}_\alpha(\partial G) \equiv \mathbf{V}^{k-\frac{1}{2}}_{2,\alpha}(\partial G).$$

Definition 1.1. *The function $u(x)$ is called a weak solution of the problem (QL) provided that $u(x) \in \mathbf{C}^0(\overline{G}) \cap \mathbf{W}^{1,m}(G)$ and satisfies the integral identity*

$$\int\limits_G \{a_i(x,u,u_x)\eta_{x_i} + b(x,u,u_x)\eta(x)\}\, dx + \int\limits_{\Sigma_0} \frac{\sigma(\omega)}{r^{m-1}} u|u|^{q+m-2}\eta(x)\, ds + \int\limits_{\partial G} \frac{\gamma(\omega)}{r^{m-1}} u|u|^{q+m-2}\eta(x)\, ds = \int\limits_{\partial G} g(x,u)\eta(x)\, ds + \int\limits_{\Sigma_0} h(x,u)\eta(x)\, ds \quad (II)$$

for all functions $\eta(x) \in \mathbf{C}^0(\overline{G}) \cap \mathbf{W}^{1,m}(G)$.

Lemma 1.1. *Let $u(x)$ be a weak solution of (QL). For any function $\eta(x) \in \mathbf{C}^0(\overline{G}) \cap \mathbf{W}^{1,m}(G)$ the equality*

$$\int\limits_{G_0^\varrho} \big\{a_i(x,u,u_x)\eta_{x_i} + b(x,u,u_x)\eta(x)\big\}\, dx = \int\limits_{\Omega_\varrho} a_i(x,u,u_x)\cos(r,x_i)\eta(x) d\Omega_\varrho$$

$$+\int\limits_{\Gamma_0^\varrho}\Big(g(x,u)-\frac{\gamma(\omega)}{r^{m-1}}u|u|^{q+m-2}\Big)\eta(x)\,ds$$

$$+\int\limits_{\Sigma_0^\varrho}\Big(h(x,u)-\frac{\sigma(\omega)}{r^{m-1}}u|u|^{q+m-2}\Big)\eta(x)\,ds \quad (II)_{loc}$$

holds for a.e. $\varrho\in(0,d)$.

Proof. The proof is analogous to the proof of Lemma 5.2 [3, pp. 167–170]. □

Now we formulate our assumptions and main results.

Problem (WL).

Regarding the equation we assume that the following *conditions* are satisfied:

Let $q\ge 0,\ 0\le\mu<q+1,\ s>1,\ f_1\ge 0,\ g_1\ge 0,\ h_1\ge 0,\ \beta\ge s-2$ be given numbers;

(a) the condition of the uniform ellipticity:

$$a_\pm\xi^2\le a_\pm^{ij}(x)\xi_i\xi_j\le A_\pm\xi^2,\quad \forall x\in\overline{G_\pm},\ \forall\xi\in\mathbb{R}^n;\ a_\pm,A_\pm=\text{const}>0,$$

$a^{ij}(0)=a\delta_i^j$, where δ_i^j is the Kronecker symbol;

$$a=\begin{cases}a_+, & x\in G_+,\\ a_-, & x\in G_-;\end{cases}\quad\text{we denote}\quad\begin{cases}a_*=\min\{a_+,\,a_-\}>0,\\ a^*=\max\{a_+,\,a_-\}>0,\\ A^*=\max(A_-,A_+);\end{cases}$$

(b) $a^{ij}(x)\in\mathbf{C}^0(\overline{G})$ and the inequality $\left(\sum_{i,j=1}^n|a_\pm^{ij}(x)-a_\pm^{ij}(y)|^2\right)^{\frac12}\le\mathcal{A}(|x-y|)$ holds for $x,y\in\overline{G}$, where $\mathcal{A}(r)$ is a monotonically increasing, nonnegative function, *continuous at* 0, $\mathcal{A}(0)=0$;

(c) $|b(x,u,u_x)|\le a\mu|u|^{q-1}|\nabla u|^2+b_0(x);\ b_0(x)\in L_{p/2}(G),\ n<p<2n;$

(d) $\sigma(\omega)\ge\nu_0>0$ on σ_0; $\gamma(\omega)\ge\nu_0>0$ on ∂G;

(e) $\frac{\partial h(x,u)}{\partial u}\le 0,\ \frac{\partial g(x,u)}{\partial u}\le 0;$

(f) $|b_0(x)|\le f_1|x|^\beta,\ |g(x,0)|\le g_1|x|^{s-1},\ |h(x,0)|\le h_1|x|^{s-1}.$

We assume without loss of generality that there exists $d > 0$ such that G_0^d is *a rotational cone* with the vertex at $\mathcal{O}$ and the aperture $\omega_0 \in (0, 2\pi)$ thus

$$\Gamma_0^d = \left\{(r,\omega)\big|x_1^2 = \cot^2\frac{\omega_0}{2}\sum_{i=2}^{n} x_i^2;\ r \in (0,d),\ \omega_1 = \frac{\omega_0}{2},\ \omega_0 \in (0,2\pi)\right\}. \tag{1.1}$$

Our main result about problem (WL) is the following statement. Let

$$\lambda = \frac{2 - n + \sqrt{(n-2)^2 + 4\vartheta}}{2}, \tag{1.2}$$

where ϑ is the smallest positive eigenvalue of the problem $(NEVP)$ with $m = 2$ (see Subsection 2.1).

Theorem 1.1. *Let u be a weak solution of the problem (WL), the assumptions (a)–(f) are satisfied with $\mathcal{A}(r)$ Dini-continuous at zero. Let us assume that $M_0 = \max_{x\in\overline{G}} |u(x)|$ is known. Then there are $d \in (0,1)$ and a constant $C_0 > 0$ depending only on $n, a_*, A^*, p, q, \lambda, \mu, f_1, h_1, g_1, \nu_0, s$, $M_0, \operatorname{meas} G, \operatorname{diam} G$ and on the quantity $\int_0^1 \frac{\mathcal{A}(r)}{r}dr$ such that $\forall\, x \in G_0^d$*

$$|u(x)| \le C_0\big(\|u\|_{2(q+1),G} + f_1 + g_1 + h_1\big) \times \begin{cases} |x|^{\frac{\lambda(1+q-\mu)}{(q+1)^2}}, & \text{if } s > \lambda\frac{1+q-\mu}{1+q}, \\ |x|^{\frac{\lambda(1+q-\mu)}{(q+1)^2}} \ln^{3/2(q+1)}\big(\frac{1}{|x|}\big), & \text{if } s = \lambda\frac{1+q-\mu}{1+q}, \\ |x|^{\frac{s}{q+1}}, & \text{if } s < \lambda\frac{1+q-\mu}{1+q}. \end{cases} \tag{1.3}$$

Suppose, in addition, that coefficients of the problem (WL) are satisfied such conditions, which guarantee the local a-priori estimate $|\nabla u|_{0,G'} \le M_1$ for any smooth $G' \subset\subset \overline{G} \setminus \{\mathcal{O}\}$ (see for example [1, §4] *or* [15]*). Then for $\forall\, x \in G_0^d$*

$$|\nabla u(x)| \le C_1 \cdot \begin{cases} |x|^{\frac{\lambda(1+q-\mu)}{(q+1)^2}-1}, & \text{if } s > \lambda\frac{1+q-\mu}{1+q}, \\ |x|^{\frac{\lambda(1+q-\mu)}{(q+1)^2}-1} \ln^{3/2(q+1)}\big(\frac{1}{|x|}\big), & \text{if } s = \lambda\frac{1+q-\mu}{1+q}, \\ |x|^{\frac{s}{q+1}-1}, & \text{if } s < \lambda\frac{1+q-\mu}{1+q} \end{cases} \tag{1.4}$$

with $C_1 = c_1\Big(\|u\|_{2(q+1),G} + f_1 + g_1 + h_1\Big)$, where c_1 depends on M_0, M_1 and C_0 from above.

Problem (QL).

Regarding the equation we assume that the following *conditions* are satisfied:

Let

$$a = \begin{cases} a_+, & x \in G_+, \\ a_-, & x \in G_-, \end{cases} \qquad a_\pm > 0;$$

$$a_* = \min\{a_+,\, a_-\} > 0, \qquad a^* = \max\{a_+,\, a_-\} > 0;$$

$$1 < m < n,\; mn > p > n > m,\; q \geq 0,\; k_1 \geq 0,\; s > 1,\; 0 \leq \mu < \frac{q+m-1}{m-1}$$

be given numbers; $a_0(x), \alpha(x), b_0(x)$ are nonnegative measurable functions; $a_i(x,u,\xi),\ i = 1,\dots,n$; $b(x,u,\xi)$ are Caratheodory functions $G \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ and $h(x,u)$ is a continuously differentiable with respect to the u variable function $\Sigma_0 \times \mathbb{R} \to \mathbb{R}$ but $g(x,u)$ is a continuously differentiable with respect to the u variable function $\partial G \times \mathbb{R} \to \mathbb{R}$ possessing the properties:

1) $a_i(x,u,\xi)\xi_i \geq a|u|^q|\xi|^m - a_0(x);\ a_0(x) \in \mathbf{L}_{p/m}(G);$

2) $\sqrt{\sum_{i=1}^n a_i^2(x,u,\xi)} + \sqrt{\sum_{i=1}^n \left|\frac{\partial a_i(x,u,\xi)}{\partial x_i}\right|^2} \leq a|u|^q|\xi|^{m-1} + \alpha(x);$

   $\alpha(x) \in \mathbf{L}_{\frac{p}{m-1}}(G);$

3a) $|b(x,u,\xi)| \leq a\mu|u|^{q-1}|\xi|^m + b_0(x);\ b_0(x) \in \mathbf{L}_{\frac{p}{m}}(G);$

3b) $b(x,u,\xi) = \beta(x,u) + \widetilde{b}(x,u,\xi), \quad u \cdot \beta(x,u) \geq a|u|^{q+m};$

   $|\widetilde{b}(x,u,\xi)| \leq a\mu|u|^{q-1}|\xi|^m + b_0(x),\ b_0(x) \in \mathbf{L}_{\frac{p}{m}}(G);$

4) $\frac{\partial h(x,u)}{\partial u} \leq 0, \quad \frac{\partial g(x,u)}{\partial u} \leq 0;$

5) $\sigma(\omega) \geq \nu_0 \geq 0$ on σ_0; $\quad \gamma(\omega) \geq \nu_0 \geq 0$ on ∂G.

In addition, suppose that the functions $a_i(x,u,\xi)$ are continuously differentiable with respect to the u, ξ variables in $\mathfrak{M}_{d,M_0} = \overline{G_0^d} \times [-M_0, M_0] \times \mathbb{R}^n$ and satisfy in $\mathfrak{M}_{d,M_0}$

6) $(m-1)u\frac{\partial a_i(x,u,\xi)}{\partial u} = q\frac{\partial a_i(x,u,\xi)}{\partial \xi_j}\xi_j;\ i = 1,\dots,n;$

7) $\left|a_i(x,u,u_x)\cos(\overrightarrow{n},x_i) - a|u|^q|\nabla u|^{m-2}\frac{\partial u}{\partial \overrightarrow{n}}\right|\Big|_{\Omega_d} \leq k_1|x|^{s-1},\ |x| \leq d.$

Our main result about problem (QL) is the following statement.

Theorem 1.2. *Let u be a weak solution of the problem (QL) and assumptions 1)–7) are satisfied. Let us assume that $M_0 = \max_{x\in\overline{G}} |u(x)|$ is known. Let ϑ be the smallest positive eigenvalue of the problem $(NEVP)$ (see Subsection 2.1). Suppose, in addition, that $h(x,0) \in L_\infty(\Sigma_0)$, $g(x,0) \in L_\infty(\partial G)$ and there exist real numbers $k_s \geq 0$, $K \geq 0$ such that*

$$k_s =: \sup_{\varrho>0} \varrho^{1-n-s}\Bigg\{ \int\limits_{G_0^\varrho} r^{\frac{q}{q+m-1}} |a_0(x)|^{\frac{m(q+m-1)}{(m-1)(q+m)}}\, dx + \int\limits_{G_0^\varrho} r^{\frac{1}{m-1}} |b_0(x)|^{\frac{m}{m-1}} + \int\limits_{\Sigma_0^\varrho} \frac{1}{r}|h(x,0)|^{\frac{m}{m-1}}\, ds + \int\limits_{\Gamma_0^\varrho} \frac{1}{r}|g(x,0)|^{\frac{m}{m-1}}\, ds \Bigg\}; \quad (1.5)$$

$$K =: \sup_{\varrho>0} \frac{\varrho^{\frac{n}{m}-1}}{\psi(\varrho)} \Big\{ \varrho^{m(1-\frac{n}{p})\frac{q+m-1}{(m-1)(q+m)}} \|a_0\|_{\frac{p}{m},G_0^\varrho}^{\frac{q+m-1}{(m-1)(q+m)}} + \varrho^{1-\frac{n}{p}} \|\alpha(x)\|_{\frac{p}{m-1},G_0^\varrho}^{\frac{1}{m-1}} + \varrho^{(1-\frac{n}{p})\frac{m}{m-1}} \|b_0(x)\|_{\frac{p}{m},G_0^\varrho}^{\frac{1}{m-1}} + \varrho\Big(\|g(x,0)\|_{\infty,\Gamma_0^\varrho}^{\frac{1}{m-1}} + \|h(x,0)\|_{\infty,\Sigma_0^\varrho}^{\frac{1}{m-1}} \Big) \Big\}, \quad (1.6)$$

where

$$\psi(\varrho) = \begin{cases} \varrho^{\frac{\vartheta^{\frac{1}{m}}(m)}{C(m)}\cdot\frac{q+(m-1)(1-\mu)}{q+m-1}}, & s > \frac{\vartheta^{\frac{1}{m}}(m)}{C(m)}\cdot\frac{q+(m-1)(1-\mu)}{q+m-1}; \\ \varrho^{\frac{\vartheta^{\frac{1}{m}}(m)}{C(m)}\cdot\frac{q+(m-1)(1-\mu)}{q+m-1}} \ln\frac{d}{\varrho}, & s = \frac{\vartheta^{\frac{1}{m}}(m)}{C(m)}\cdot\frac{q+(m-1)(1-\mu)}{q+m-1}; \\ \varrho^s, & s < \frac{\vartheta^{\frac{1}{m}}(m)}{C(m)}\cdot\frac{q+(m-1)(1-\mu)}{q+m-1}, \end{cases} \quad (1.7)$$

$$C(m) = (m-1)^{\frac{m-1}{m}} \max\{1; 2^{\frac{m-2}{2(m-1)}}\}. \quad (1.8)$$

Then there are $d \in (0,1)$ and a constant $C_0 > 0$ independent of u such that

$$|u(x)| \leq C_0\big(|x|^{1-\frac{n}{m}}\psi(|x|)\big)^{\frac{m-1}{q+m-1}}, \qquad \forall\, x \in G_0^d. \quad (1.9)$$

Moreover, if coefficients of the problem (QL) satisfy such conditions which guarantee the local a-priori estimate $|\nabla u|_{0,G'} \leq M_1$ for any smooth $G' \subset\subset \overline{G}\setminus\{\mathcal{O}\}$ (see for example [1, §4] or [15]), then there is a constant $C_1 > 0$ independent of u such that

$$|\nabla u(x)| \leq C_1 |x|^{-\frac{n(m-1)+qm}{m(q+m-1)}} \psi^{\frac{m-1}{q+m-1}}(|x|), \qquad \forall\, x \in G_0^d. \quad (1.10)$$

2. Preliminaries

2.1. The eigenvalue problem

Let $\Omega \subset S^{n-1}$ with smooth boundary $\partial\Omega$ be the intersection of the cone $\mathcal{C}$ with the unit sphere S^{n-1}. Let $\overrightarrow{\nu}$ be the exterior normal to $\partial\mathcal{C}$ at points of $\partial\Omega$ and $\overrightarrow{\tau}$ be the exterior with respect Ω_+ normal to Σ_0 (lying in the tangent to Ω plane). Let $\gamma(\omega)$ be *a positive* bounded piecewise smooth function on $\partial\Omega$, $\sigma(\omega)$ be *a positive* continuous function on Σ. We consider the eigenvalue problem for the m-Laplace–Beltrami operator on the unit sphere:

$$\begin{cases} a\left(\mathrm{div}_\omega(|\nabla_\omega\psi|^{m-2}\nabla_\omega\psi) + \vartheta|\psi|^{m-2}\psi\right) = 0, \qquad \omega \in \Omega, \\ [\psi]_{\sigma_0} = 0, \quad \left[a|\nabla_\omega\psi|^{m-2}\frac{\partial\psi}{\partial\overrightarrow{\tau}}\right]_{\sigma_0} + \sigma(\omega)|\psi|^{m-2}\psi\Big|_{\sigma_0} = 0; \\ a|\nabla_\omega\psi|^{m-2}\frac{\partial\psi}{\partial\overrightarrow{\nu}} + \gamma(\omega)|\psi|^{m-2}\psi\Big|_{\partial\Omega} = 0, \end{cases} \qquad (NEVP)$$

which consists of the determination of all values ϑ (eigenvalues) for which $(NEVP)$ has a non-zero weak solutions (eigenfunctions); here

$$a = \begin{cases} a_+, & x \in \Omega_+, \\ a_-, & x \in \Omega_-, \end{cases}$$

$a_\pm$ are positive constants.

Definition 2.1. *Non-zero function ψ is called a* **weak** *eigenfunction of the problem $(NEVP)$ provided that $\psi \in \mathbf{C}^0(\overline{\Omega}) \cap \mathbf{W}^{1,m}(\Omega)$ and satisfies the integral identity*

$$\int_\Omega a\Big\{|\nabla_\omega\psi|^{m-2}\frac{1}{q_i}\frac{\partial\psi}{\partial\omega_i}\frac{\partial\eta}{\partial\omega_i} - \vartheta|\psi|^{m-2}\psi\eta\Big\}\,d\Omega + \int_{\sigma_0}\sigma(\omega)|\psi|^{m-2}\psi\eta\,d\sigma + \int_{\partial\Omega}\gamma(\omega)|\psi|^{m-2}\psi\eta\,d\sigma = 0$$

for all $\eta(x) \in \mathbf{C}^0(\overline{\Omega}) \cap \mathbf{W}^{1,m}(\Omega)$; here: $q_1 = 1,\ q_i = (\sin\omega_1 \dots \sin\omega_{i-1})^2$; $i \ge 2$.

Remark 2.1. We observe that $\vartheta = 0$ is *not* an eigenvalue of $(NEVP)$. In fact, setting $\eta = \psi$ and $\vartheta = 0$ we have

$$\int_\Omega a|\nabla_\omega\psi|^m\,d\Omega + \int_{\sigma_0}\sigma(\omega)|\psi|^m\,d\sigma + \int_{\partial\Omega}\gamma(\omega)|\psi|^m\,d\sigma = 0 \implies \psi \equiv 0,$$

since $a > 0,\ \sigma(\omega) > 0,\ \gamma(\omega) > 0.$

We characterize the first eigenvalue $\vartheta(m)$ of the eigenvalue problem for m-Laplacian by

$$\vartheta(m) = \inf_{\substack{\psi \in W^{1,m}(\Omega) \\ \psi \neq 0}} \frac{\int_\Omega a|\nabla_\omega \psi|^m \, d\Omega + \int_{\sigma_0} \sigma(\omega)|\psi|^m \, d\sigma + \int_{\partial\Omega} \gamma(\omega)|\psi|^m \, d\sigma}{\int_\Omega a|\psi|^m \, d\Omega}. \tag{2.1}$$

Theorem 2.1. *Let $\Omega \subset S^{n-1}$ be a bounded domain with smooth boundary $\partial\Omega$. Let $\gamma(\omega)$, $\omega \in \partial\Omega$ be a positive bounded piecewise smooth function, $\sigma(\omega)$ be a positive continuous function on σ_0. There exist the eigenvalue $\vartheta > 0$ and the corresponding weak eigenfunction ψ.*

Proof. The proof is similar to Theorem 8.20 [3]. □

Now from the variational principle we obtain

The Friedrichs–Wirtinger type inequality. *Let ϑ be the smallest positive eigenvalue of the problem $(NEVP)$ (it exists according to Theorem 2.1). Let $\Omega \subset S^{n-1}$. Let $\psi \in \mathbf{W}^{1,m}(\Omega)$ and satisfies the boundary and conjunction conditions from $(NEVP)$ in the weak sense. Let $\gamma(\omega)$ be a positive bounded piecewise smooth function on $\partial\Omega$, $\sigma(\omega)$ be a positive continuous function on σ_0. Then*

$$\int_\Omega a|\psi|^m \, d\Omega \le \frac{1}{\vartheta}\left\{ \int_\Omega a|\nabla_\omega \psi|^m \, d\Omega + \int_{\sigma_0} \sigma(\omega)|\psi|^m \, d\sigma + \int_{\partial\Omega} \gamma(\omega)|\psi|^m \, d\sigma \right\} \tag{$(W)_m$}$$

with the sharp constant $\frac{1}{\vartheta}$.

Remark 2.2. In the case $m = 2$ we consider the value λ by (1.2); therefore the Friedrichs–Wirtinger inequality will be written in the following form

$$\lambda(\lambda + n - 2) \int_\Omega a\psi^2(\omega) \, d\Omega \le \int_\Omega a|\nabla_\omega \psi|^2 \, d\Omega + \int_{\sigma_0} \sigma(\omega)\psi^2(\omega) \, d\sigma + \int_{\partial\Omega} \gamma(\omega)\psi^2(\omega) \, d\sigma, \quad \forall\, \psi(\omega) \in \mathbf{W}^1(\Omega) \tag{$(W)_2$}$$

satisfying the boundary and conjunction conditions from $(NEVP)$ in the weak sense; $\sigma(\omega) \ge \nu_0 > 0,\ \gamma(\omega) \ge \nu_0 > 0.$

Corollary 2.1. *Let ϑ be the smallest positive eigenvalue of the problem $(NEVP)$. Let $v(x) \in \mathbf{W}^{1,m}(G_0^d)$ and $v(\cdot,\omega)$ satisfies the boundary and conjunction conditions from $(NEVP)$ in the weak sense. Let $\gamma(\omega)$ be a positive bounded piecewise smooth function on $\partial\Omega$, $\sigma(\omega)$ be a positive continuous function on σ_0. Then for any $\varrho \in (0,d)$ and $\forall\, \alpha$*

$$\int\limits_{G_0^\varrho} ar^\alpha |v|^m\, dx \le \frac{1}{\vartheta}\Bigg\{ \int\limits_{G_0^\varrho} ar^{\alpha+m}|\nabla v|^m\, dx + \int\limits_{\Sigma_0^\varrho} r^{\alpha+m}\frac{\sigma(\omega)}{r^{m-1}}|v|^m\, ds + \int\limits_{\Gamma_0^\varrho} r^{\alpha+m}\frac{\gamma(\omega)}{r^{m-1}}|v|^m\, ds \Bigg\}, \quad (2.2)$$

provided that integrals on the right are finite. In particular,

$$\int\limits_{G_0^\varrho} a|v|^m\, dx \le \frac{\varrho^m}{\vartheta(m)}\Bigg\{ \int\limits_{G_0^\varrho} a|\nabla v|^m\, dx + \int\limits_{\Sigma_0^\varrho} \frac{\sigma(\omega)}{r^{m-1}}|v|^m\, ds + \int\limits_{\Gamma_0^\varrho} \frac{\gamma(\omega)}{r^{m-1}}|v|^m\, ds \Bigg\} \quad (H-W)_m$$

as well

$$\int\limits_{G} a|v|^m\, dx \le C(m,n,\vartheta,G)\Bigg\{ \int\limits_{G} a|\nabla v|^m\, dx + \int\limits_{\Sigma_0} \frac{\sigma(\omega)}{r^{m-1}}|v|^m\, ds + \int\limits_{\partial G} \frac{\gamma(\omega)}{r^{m-1}}|v|^m\, ds \Bigg\}. \quad (F-W)_m$$

Proof. We consider the inequality $(W)_m$ for the function $v(r,\omega)$, multiply it by $r^{\alpha+n-1}$ and integrate over $r \in (0,\varrho)$. Hence it follows the desired (2.2). Setting $\alpha = 0$ we get $(H-W)_m$. The inequality $(F-W)_m$ is the consequence of $(H-W)_m$ and the Poincaré inequality. □

Corollary 2.2. *Let $v \in \mathbf{C}^0(\overline{G}) \cap \overset{\circ}{\mathbf{W}}{}^1_{\alpha-2}(G)$ and $v(\cdot,\omega)$ satisfies the boundary and conjunction conditions from $(NEVP)$ for $m=2$ in the weak sense. Let $\sigma(\omega)$, $\omega \in \sigma_0$; $\gamma(\omega)$, $\omega \in \partial\Omega$ be positive bounded piecewise smooth functions with $\sigma(\omega) \ge \nu_0 > 0$, $\gamma(\omega) \ge \nu_0 > 0$. Then*

$$\int\limits_{G_0^d} ar^{\alpha-4}v^2\, dx \le \frac{1}{\lambda(\lambda+n-2)}\Bigg\{ \int\limits_{G_0^d} ar^{\alpha-2}|\nabla v|^2\, dx$$

$$+\int\limits_{\Sigma_0^d} r^{\alpha-3}\sigma(\omega)v^2(x)\,ds+\int\limits_{\Gamma_0^d} r^{\alpha-3}\gamma(\omega)v^2(x)\,ds\Bigg\},\qquad \forall\,\alpha,\quad (2.3)$$

where λ *is defined by* (1.2).

Proof. Multiplying $(W)_2$ by $r^{n-5+\alpha}$ and integrating over $r\in(0,d)$ we obtain the required (2.3). □

2.2. One auxiliary integral inequality

Lemma 2.1. *Let* G_0^d *be the conical domain,* $\nabla v(\varrho,\cdot)\in\mathbf{W}^{1,m}(\Omega)$ *for almost all* $\varrho\in(0,d)$ *and*

$$V(\varrho)=\int\limits_{G_0^\varrho} a|\nabla v|^m\,dx+\int\limits_{\Sigma_0^\varrho}\frac{\sigma(\omega)}{r^{m-1}}|v|^m\,ds+\int\limits_{\Gamma_0^\varrho}\frac{\gamma(\omega)}{r^{m-1}}|v|^m\,ds<\infty.\quad (2.4)$$

Let $\vartheta(m)$ *be the smallest positive eigenvalue of the problem* $(NEVP)$ *and* $\gamma(\omega)$ *be a positive bounded piecewise smooth function on* $\partial\Omega$*,* $\sigma(\omega)$ *be a positive continuous function on* σ_0*. Then for almost all* $\varrho\in(0,d)$

$$\int\limits_{\Omega_\varrho} av\frac{\partial v}{\partial r}|\nabla v|^{m-2}d\Omega_\varrho\le(m-1)^{\frac{m-1}{m}}\max\{1;2^{\frac{m-2}{2(m-1)}}\}\cdot\frac{\varrho}{m\vartheta^{\frac{1}{m}}}V'(\varrho).\quad (2.5)$$

Proof. Writing $V(\varrho)$ in spherical coordinates

$$\begin{aligned}V(\varrho)=\int\limits_0^\varrho r^{n-1}\Bigg(\int\limits_\Omega a|\nabla v(r,\omega)|^m\,d\Omega\Bigg)dr\\ +\int\limits_0^\varrho r^{n-m-1}\Bigg(\int\limits_{\partial\Omega}\gamma(\omega)|v(r,\omega)|^m\,d\sigma\Bigg)dr\\ +\int\limits_0^\varrho r^{n-m-1}\Bigg(\int\limits_{\sigma_0}\sigma(\omega)|v(r,\omega)|^m\,d\sigma\Bigg)dr\end{aligned}$$

and differentiating with respect to ϱ we obtain

$$\begin{aligned}V'(\varrho)=\varrho^{n-1}\int\limits_\Omega a|\nabla v(\varrho,\omega)|^m\,d\Omega\\ +\varrho^{n-m-1}\Bigg(\int\limits_{\partial\Omega}\gamma(\omega)|v(\varrho,\omega)|^m\,d\sigma+\int\limits_{\sigma_0}\sigma(\omega)|v(\varrho,\omega)|^m\,d\sigma\Bigg).\end{aligned}$$

Now using the Young inequality with $p = m,\ p' = \frac{m}{m-1}$ we have

$$\int\limits_{\Omega_\varrho} av\frac{\partial v}{\partial r}|\nabla v|^{m-2}\,d\Omega_\varrho = \varrho^{n-1}\int\limits_{\Omega} av\frac{\partial v}{\partial r}|\nabla v|^{m-2}\Big|_{r=\varrho}\,d\Omega$$
$$= \varrho^n\int\limits_{\Omega} a\Big(\frac{v}{\varrho}\Big)\Big(\frac{\partial v}{\partial r}|\nabla v|^{m-2}\Big)\Big|_{r=\varrho}\,d\Omega \leq$$
$$\leq \varrho^n\int\limits_{\Omega} a\Big\{\frac{\varepsilon}{m}\Big(\frac{v}{\varrho}\Big)^m + \frac{m-1}{m}\varepsilon^{-\frac{1}{m-1}}\Big(\frac{\partial v}{\partial r}\Big)^{\frac{m}{m-1}}|\nabla v|^{(m-2)\frac{m}{m-1}}\Big\}\Big|_{r=\varrho}\,d\Omega,$$
$$\forall\,\varepsilon > 0.$$

Further, applying the Friedrichs–Wirtinger type inequality $(W)_m$ we obtain

$$\int\limits_{\Omega_\varrho} av\frac{\partial v}{\partial r}|\nabla v|^{m-2}d\Omega_\varrho$$
$$\leq \frac{\varepsilon\varrho^n}{m\vartheta(m)}\Big\{\int\limits_{\sigma_0}\sigma(\omega)\Big|\frac{v(\varrho,\omega)}{\varrho}\Big|^m\,d\sigma + \int\limits_{\partial\Omega}\gamma(\omega)\Big|\frac{v(\varrho,\omega)}{\varrho}\Big|^m\,d\sigma\Big\}$$
$$+\frac{1}{m}\varrho^n\int\limits_{\Omega} a\Big\{\frac{\varepsilon}{\vartheta(m)}\Big|\frac{\nabla_\omega v}{\varrho}\Big|^m + (m-1)\varepsilon^{-\frac{1}{m-1}}\Big|\frac{\partial v}{\partial r}\Big|^{\frac{m}{m-1}}|\nabla v|^{(m-2)\frac{m}{m-1}}\Big\}\Big|_{r=\varrho}\,d\Omega.$$

But, by $|\nabla v|^2 = v_r^2 + \frac{1}{r^2}|\nabla_\omega v|^2$ and $\big|\frac{\nabla_\omega v}{\varrho}\big| \leq |\nabla v|$, we get from above

$$\int\limits_{\Omega_\varrho} av\frac{\partial v}{\partial r}|\nabla v|^{m-2}\,d\Omega_\varrho$$
$$\leq \frac{\varepsilon\varrho^{n-m}}{m\vartheta(m)}\Big\{\int\limits_{\sigma_0}\sigma(\omega)|v(\varrho,\omega)|^m\,d\sigma + \int\limits_{\partial\Omega}\gamma(\omega)|v(\varrho,\omega)|^m\,d\sigma\Big\}$$
$$+\frac{1}{m}\varrho^n\int\limits_{\Omega} a|\nabla v|^{(m-2)\frac{m}{m-1}}\Big\{\frac{\varepsilon}{\vartheta(m)}\Big(\Big|\frac{\nabla_\omega v}{\varrho}\Big|^2\Big)^{\frac{m}{2(m-1)}}$$
$$+ (m-1)\varepsilon^{-\frac{1}{m-1}}(v_r^2)^{\frac{m}{2(m-1)}}\Big\}\Big|_{r=\varrho}\,d\Omega.$$

Now we choose

$$\varepsilon = \langle(m-1)\vartheta(m)\rangle^{\frac{m-1}{m}} \implies (m-1)\varepsilon^{-\frac{1}{m-1}} = \frac{\varepsilon}{\vartheta(m)} \tag{2.6}$$

and therefore

$$\int\limits_{\Omega_\varrho} av\frac{\partial v}{\partial r}|\nabla v|^{m-2}d\Omega_\varrho$$

$$\leq \frac{\varepsilon\varrho^{n-m}}{m\vartheta(m)}\Bigg\{\int\limits_{\sigma_0}\sigma(\omega)|v(\varrho,\omega)|^m\,d\sigma+\int\limits_{\partial\Omega}\gamma(\omega)|v(\varrho,\omega)|^m\,d\sigma\Bigg\}$$

$$+\frac{\varepsilon}{m\vartheta(m)}\varrho^n\int\limits_{\Omega}a|\nabla v|^{(m-2)\frac{m}{m-1}}\Big\{\Big(\Big|\frac{\nabla_\omega v}{\varrho}\Big|^2\Big)^{\frac{m}{2(m-1)}}+(v_r^2)^{\frac{m}{2(m-1)}}\Big\}\Big|_{r=\varrho}d\Omega.$$

Applying yet the Jensen inequality (see e.g. [7, Theorem 65]), by $|\nabla v|^2 = v_r^2+\frac{1}{r^2}|\nabla_\omega v|^2$, hence it follows

$$\int\limits_{\Omega_\varrho} av\frac{\partial v}{\partial r}|\nabla v|^{m-2}d\Omega_\varrho$$

$$\leq \frac{\varepsilon\varrho^{n-m}}{m\vartheta(m)}\Bigg\{\int\limits_{\sigma_0}\sigma(\omega)|v(\varrho,\omega)|^m\,d\sigma+\int\limits_{\partial\Omega}\gamma(\omega)|v(\varrho,\omega)|^m\,d\sigma\Bigg\}$$

$$+\frac{\varepsilon}{m\vartheta(m)}\cdot\frac{1}{\min\{1;2^{\frac{m}{2(m-1)}-1}\}}\varrho^n\int\limits_{\Omega}a|\nabla v|^m\Big|_{r=\varrho}d\Omega.$$

Substituting here ε from (2.6) we get the desired inequality (2.5). □

In the case $m=2$ we establish the more exact statement:

Lemma 2.2. *Let G_0^d be the conical domain and $\nabla v(\varrho,\cdot)\in \mathbf{L}_2(\Omega)$ a.e. $\varrho\in(0,d)$. Assume that for a.e. $\varrho\in(0,d)$*

$$V(\rho)=\int\limits_{G_0^\varrho}ar^{2-n}|\nabla v|^2\,dx+\int\limits_{\Sigma_0^\varrho}r^{1-n}\sigma(\omega)v^2(x)\,ds+\int\limits_{\Gamma_0^\varrho}r^{1-n}\gamma(\omega)v^2(x)\,ds<\infty. \quad (2.7)$$

Then

$$\int\limits_{\Omega}a\Big(\varrho v\frac{\partial v}{\partial r}+\frac{n-2}{2}v^2\Big)\Big|_{r=\varrho}d\Omega\leq\frac{\varrho}{2\lambda}V'(\varrho), \quad (2.8)$$

where λ is defined by (1.2).

Proof. The proof is similar to the [3, Lemma 2.35] proof, if we use inequality $(W)_2$. □

We need also the well known inequalities:

$$\int_{\Gamma} v\,ds \le C\int_{G}(|v|+|\nabla v|)\,dx, \quad \forall\, v(x)\in \mathbf{W}^{1,1}(G),\ \forall\,\Gamma\subseteq\partial G, \tag{2.9}$$

$$\int_{\partial G} v^2\,ds \le \int_{G}\Big(\delta|\nabla v|^2+\frac{1}{\delta}c_0v^2\Big)\,dx, \quad \forall\, v(x)\in \mathbf{W}^{1,2}(G),\ \forall\,\delta>0, \tag{2.10}$$

where C, c_0 depend only on n, G, Γ.

2.3. Quasi-distance $\mathbf{r}_\varepsilon(\mathbf{x})$

Further, we define *the function* $\mathbf{r}_\varepsilon(\mathbf{x})$ as follows. We fix the point $Q=(-1,0,\dots,0)\in S^{n-1}\setminus\bar\Omega$ and consider the unit radius-vector $\vec{l}=\mathcal{O}Q=\{-1,0,\dots,0\}$. We denote by $\vec{r}$ the radius-vector of the point $x\in\bar G$ and introduce the vector $\vec{r}_\varepsilon=\vec{r}-\varepsilon\vec{l}$, $\forall\,\varepsilon>0$. Since $\varepsilon\vec{l}\notin G_0^d$ for all $\varepsilon\in]0,d[$, it follows that $r_\varepsilon(x)=|\vec{r}-\varepsilon\vec{l}|\neq 0$ for all $x\in\bar G$. It is easy to see that $r_\varepsilon(x)$ has the following properties (see in detail [3, §1.4]):

1. $\exists\, h>0$ *such that:* $r_\varepsilon(x)\ge hr$ *and* $r_\varepsilon(x)\ge h\varepsilon$, $\forall\, x\in\overline{G}$, *where* $h=\begin{cases}1, & \text{if } x_1\ge 0\ ,\\ \sin\frac{\omega_0}{2}, & \text{if } x_1<0.\end{cases}$
2. *If* $x\in G_d$*, then* $r_\varepsilon(x)\ge\frac{d}{2}$ *for all* $\varepsilon\in]0,\frac{d}{2}[$.
3. $\lim_{\varepsilon\to0^+} r_\varepsilon(x)=r$*, for all* $x\in\bar G$.
4. $|\nabla r_\varepsilon|^2=1$*, and* $\triangle r_\varepsilon=\frac{n-1}{r_\varepsilon}$.

Lemma 2.3. *Let* $v\in\mathbf{C}^0(\overline{G})\cap\mathbf{W}^1(G)$ *and* $v(\cdot,\omega)$ *satisfies the boundary and conjunction conditions from* (EVP) *in the weak sense. Let* $\sigma(\omega)\ge\sigma_0>0$, $\gamma(\omega)\ge\gamma_0>0$ *and* λ *be as above in* (1.2)*. Then for any* $\varepsilon>0$

$$\int_{G_0^d} ar_\varepsilon^{\alpha-2}r^{-2}v^2\,dx \le \frac{1}{\lambda(\lambda+n-2)}\Bigg\{\int_{G_0^d} ar_\varepsilon^{\alpha-2}|\nabla v|^2\,dx + \int_{\Sigma_0^d} r^{-1}r_\varepsilon^{\alpha-2}\sigma(\omega)v^2(x)\,ds + \int_{\Gamma_0^d} r^{-1}r_\varepsilon^{\alpha-2}\gamma(\omega)v^2\,ds\Bigg\}. \tag{2.11}$$

Proof. Multiplying both sides of the Friedrichs–Wirtinger inequality $(W)_2$ by $(\varrho+\varepsilon)^{\alpha-2}r^{n-3}$ and integrating over $r\in(\frac{\varrho}{2},\varrho)$ we obtain

$$\int\limits_{G^{\varrho}_{\varrho/2}} a(\varrho+\varepsilon)^{\alpha-2} r^{-2} v^2\,dx$$

$$\leq \frac{1}{\lambda(\lambda+n-2)}\Bigg\{\int\limits_{G^{\varrho}_{\varrho/2}} a(\varrho+\varepsilon)^{\alpha-2}|\nabla v|^2\,dx + \int\limits_{\Gamma^{\varrho}_{\varrho/2}} r^{-1}(\varrho+\varepsilon)^{\alpha-2}\gamma(\omega) v^2\,ds$$

$$+ \int\limits_{\Sigma^{\varrho}_{\varrho/2}} r^{-1}(\varrho+\varepsilon)^{\alpha-2}\sigma(\omega) v^2\,ds\Bigg\}, \qquad \forall\,\varepsilon>0$$

or since $\varrho+\varepsilon\sim r_\varepsilon$

$$\int\limits_{G^{\varrho}_{\varrho/2}} a r_\varepsilon^{\alpha-2} r^{-2} v^2\,dx \leq \frac{1}{\lambda(\lambda+N-2)}\Bigg\{\int\limits_{G^{\varrho}_{\varrho/2}} a r_\varepsilon^{\alpha-2}|\nabla v|^2\,dx$$

$$+ \int\limits_{\Sigma^{\varrho}_{\varrho/2}} r^{-1} r_\varepsilon^{\alpha-2}\sigma(\omega) v^2\,ds + \int\limits_{\Gamma^{\varrho}_{\varrho/2}} r^{-1} r_\varepsilon^{\alpha-2}\gamma(\omega) v^2\,ds\Bigg\}, \quad \forall\,\varepsilon>0.$$

Letting $\rho=2^{-k}d,\ (k=0,1,2,\dots)$ and summing the obtained inequalities over all k we get the desired inequality (2.11). □

3. Local estimate at the boundary. The maximum principle

Applying the Moser iteration method (see e.g. [6, §§ 8.6, 8.10] or [4]) we derive a result asserting the local boundedness (near the conical point) of the weak solution of problem (QL).

Theorem 3.1. *Let $u(x)$ be a weak solution of the problem (QL). Let assumptions* 1), 2), 3*a*), 4), 5), 6), (1.6) *be satisfied and, in addition, $h(x,0)\in L_\infty(\Sigma_0)$, $g(x,0)\in \mathbf{L}_\infty(\partial G)$. Then the inequality*

$$\sup_{x\in G_0^{\varkappa\varrho}}|u(x)| \leq \frac{C}{(1-\varkappa)^{n\varsigma/t}}\Big\{\varrho^{-n\varsigma/t}\|u\|_{\frac{t}{\varsigma},G_0^{\varrho}} + \varrho^{\frac{m\varsigma(p-n)}{p(m-1+\varsigma)}}\cdot\|a_0(x)\|^{\frac{\varsigma}{m-1+\varsigma}}_{\frac{p}{m},G_0^{\varrho}}$$

$$+\varrho^{\varsigma(1-\frac{n}{p})}\|\alpha(x)\|^{\frac{\varsigma}{m-1}}_{\frac{p}{m-1},G_0^{\varrho}} + \varrho^{\varsigma(1-\frac{n}{p})\frac{m}{m-1}}\|b_0(x)\|^{\frac{\varsigma}{m-1}}_{\frac{p}{m},G_0^{\varrho}}$$

$$+\varrho^{\varsigma}\big(\|g(x,0)\|^{\frac{\varsigma}{m-1}}_{\infty,\Gamma_0^{\varrho}} + \|h(x,0)\|^{\frac{\varsigma}{m-1}}_{\infty,\Sigma_0^{\varrho}}\big)\Big\}, \quad p>n>m \quad (3.1)$$

holds for any $t>0$, $\varkappa\in(0,1)$, $\varrho\in(0,d)$ and $\varsigma=\frac{m-1}{q+m-1}$, where $C=C\big(n,m,p,t,q,a_*,a^*,m_*,m^*,d,\|a_0(x)\|_{\frac{p}{m},G},\|\alpha(x)\|_{\frac{p}{m-1},G},\|b_0(x)\|_{\frac{p}{m},G}\big)$.

We consider also one in a possible case of the deriving $L_\infty(G)-a$ *priori* estimate of the weak solution to problem (QL).

Theorem 3.2. *Let $u(x)$ be a weak solution of (QL) and assumptions* 1), 3*b*), 4), 5) *hold. Suppose, in addition, that* $h(x,0) \in L_{\frac{j}{j-1}}(\Sigma_0)$, $g(x,0) \in \mathbf{L}_{\frac{j}{j-1}}(\partial G)$, $1 < j < \frac{n-1}{m-1}$. *Then there exists the constant* $M_0 > 0$, *depending only on* $\operatorname{meas} G$, $\operatorname{meas} \partial G$, $\operatorname{meas} \Sigma_0$, n, m, $\|b_0(x)\|_{\mathbf{L}_{\frac{p}{m}}(G)}$, $\|h(x,0)\|_{L_{\frac{j}{j-1}}(\Sigma_0)}$, $\|g(x,0)\|_{\mathbf{L}_{\frac{j}{j-1}}(\partial G)}$, μ, q, $\|a_0(x)\|_{\mathbf{L}_{\frac{p}{m}}(G)}$, *such that* $\|u\|_{L_\infty(G)} \le M_0$.

Proof. The proof is similar to [3, Theorem 9.11]. □

4. Global integral estimates

Theorem 4.1. *Let $u(x)$ be a weak solution of the problem (QL). Let us assume that $M_0 = \max_{x\in\overline{G}} |u(x)|$ is known. Let assumptions* 1), 3*a*), 4), 5) *with $\nu_0 > 0$ be satisfied. Suppose, in addition, that*

$$a_0(x) \in \mathbf{L}_1(G), \quad b_0(x) \in \mathbf{L}_1(G), \quad h(x,0) \in L_1(\Sigma_0), \quad g(x,0) \in L_1(\partial G).$$

Then the inequality

$$\int_G a|u|^{\frac{qm}{m-1}} |\nabla u|^m \, dx + \int_{\Sigma_0} \frac{\sigma(\omega)}{r^{m-1}} |u|^{\frac{m}{m-1}(q+m-1)} \, ds + \int_{\partial G} \frac{\gamma(\omega)}{r^{m-1}} |u|^{\frac{m}{m-1}(q+m-1)} \, ds \le c(M_0, a_*, \nu_0, q, m, \mu, n, \operatorname{meas} G) \times \left(\int_G (a_0(x) + b_0(x)) \, dx + \int_{\Sigma_0} |h(x,0)| \, ds + \int_{\partial G} |g(x,0)| \, ds \right) \tag{4.1}$$

holds.

Proof. At first we make the function change

$$u = v|v|^{\varsigma - 1} \text{ with } \quad \varsigma = \frac{m-1}{q+m-1}. \tag{4.2}$$

By virtue of the assumption 6), the identity (II) takes the form:

$$\int_G \left\langle \mathcal{A}_i(x, v_x)\eta_{x_i} + \mathcal{B}(x, v, v_x)\eta \right\rangle dx$$

$$+\int\limits_{\partial G}\frac{\gamma(\omega)}{r^{m-1}}v|v|^{m-2}\eta(x)\,ds+\int\limits_{\Sigma_0}\frac{\sigma(\omega)}{r^{m-1}}v|v|^{m-2}\eta(x)\,ds$$
$$=\int\limits_{\partial G}\mathcal{G}(x,v)\eta(x)\,ds+\int\limits_{\Sigma_0}\mathcal{H}(x,v)\eta(x)\,ds \qquad (4.3)$$

and the identity $(II)_{loc}$ takes the form

$$\int\limits_{G_0^\varrho}\left\langle\mathcal{A}_i(x,v_x)\eta_{x_i}+\mathcal{B}(x,v,v_x)\eta\right\rangle dx+\int\limits_{\Gamma_0^\varrho}\frac{\gamma(\omega)}{r^{m-1}}v|v|^{m-2}\eta(x)\,ds$$
$$+\int\limits_{\Sigma_0^\varrho}\frac{\sigma(\omega)}{r^{m-1}}v|v|^{m-2}\eta(x)\,ds=\int\limits_{\Omega_\varrho}\mathcal{A}_i(x,v_x)\cos(r,x_i)\eta(x)\,d\Omega_\varrho$$
$$+\int\limits_{\Gamma_0^\varrho}\mathcal{G}(x,v)\eta(x)\,ds+\int\limits_{\Sigma_0^\varrho}\mathcal{H}(x,v)\eta(x)\,ds \qquad (4.4)$$

for a.e. $\varrho\in(0,d)$, $v(x)\in\mathbf{C}^0(\overline{G})\cap\mathbf{W}^{1,m}(G)$ and any $\eta(x)\in\mathbf{C}^0(\overline{G})\cap\mathbf{W}^{1,m}(G)$, where

$$\begin{aligned}\mathcal{A}_i(x,v_x)&\equiv a_i(x,v|v|^{\varsigma-1},\varsigma|v|^{\varsigma-1}v_x),\\ \mathcal{B}(x,v,v_x)&\equiv b(x,v|v|^{\varsigma-1},\varsigma|v|^{\varsigma-1}v_x),\\ \mathcal{G}(x,v)\equiv g(x,v|v|^{\varsigma-1}),&\qquad \mathcal{H}(x,v)\equiv h(x,v|v|^{\varsigma-1}).\end{aligned} \qquad (4.5)$$

We verify that coefficients $\mathcal{A}_i,\ i=1,\dots n$ do not depend on v explicit. In fact, by the change (4.2) and the assumption *6)*, we calculate

$$\frac{\partial\mathcal{A}_i}{\partial v}=\frac{\partial a_i(x,u,\xi)}{\partial u}\cdot\frac{\partial}{\partial v}\left(|v^2|^{\frac{\varsigma-1}{2}}\cdot v\right)+\frac{\partial a_i(x,u,\xi)}{\partial\xi_j}\cdot\varsigma v_{x_j}\frac{\partial}{\partial v}\left(|v^2|^{\frac{\varsigma-1}{2}}\right)$$
$$=\varsigma|v|^{\varsigma-1}\frac{\partial a_i}{\partial u}+\varsigma(\varsigma-1)v_{x_j}v|v|^{\varsigma-3}\cdot\frac{\partial a_i}{\partial\xi_j}=\varsigma\cdot\frac{u}{v}\cdot\frac{\partial a_i}{\partial u}+(\varsigma-1)\cdot\frac{\xi_j}{v}\cdot\frac{\partial a_i}{\partial\xi_j}$$
$$=\frac{1}{v}\Big(\varsigma u\cdot\frac{\partial a_i}{\partial u}+(\varsigma-1)\cdot\frac{m-1}{q}u\frac{\partial a_i}{\partial u}\Big)=\frac{u}{v}\cdot\frac{\partial a_i}{\partial u}\cdot\Big(\varsigma+(\varsigma-1)\cdot\frac{m-1}{q}\Big)=0,$$

because of (4.2).

Our assumptions regarding problem (QL) take the form:

1)′ $\mathcal{A}_i(x,v_x)v_{x_i}\ge a\varsigma^{m-1}|\nabla v|^m-\frac{1}{\varsigma}|v|^{1-\varsigma}a_0(x);\quad a_0(x)\in\mathbf{L}_{p/m}(G)$;

2)′ $\sqrt{\sum_{i=1}^{n}\mathcal{A}_i^2(x,v_x)}+\sqrt{\sum_{i=1}^{n}\left|\frac{\partial\mathcal{A}_i(x,v_x)}{\partial x_i}\right|^2}\le a\varsigma^{m-1}|\nabla v|^{m-1}+\alpha(x)$;

$\alpha(x)\in\mathbf{L}_{\frac{p}{m-1}}(G)$;

$3a)'$ $|\mathcal{B}(x,v,v_x)| \le a\mu\varsigma^m |v|^{-1}|\nabla v|^m + b_0(x); \quad b_0(x) \in \mathbf{L}_{\frac{p}{m}}(G),$

$4)'$ $\dfrac{\partial \mathcal{H}(x,v)}{\partial v} \le 0, \quad \dfrac{\partial \mathcal{G}(x,v)}{\partial v} \le 0;$

$7)'$ $\left|\mathcal{A}_i(x,v_x)\cos(\overrightarrow{n},x_i) - a\varsigma^{m-1}|\nabla v|^{m-2}\frac{\partial v}{\partial \overrightarrow{n}}\right|\Big|_{\Omega_d} \le k_1|x|^{s-1}, \ |x|<d;$

We consider integral identity (4.3) and we put $\eta(x) = v(x)$. Then we have

$$\int_G \big\langle \mathcal{A}_i(x,v_x)v_{x_i} + \mathcal{B}(x,v,v_x)v \big\rangle\, dx + \int_{\partial G} \frac{\gamma(\omega)}{r^{m-1}}|v|^m\, ds + \int_{\Sigma_0} \frac{\sigma(\omega)}{r^{m-1}}|v|^m\, ds = \int_{\partial G} \mathcal{G}(x,v)v(x)\, ds + \int_{\Sigma_0} \mathcal{H}(x,v)v(x)\, ds.$$

With regard to assumptions $1)'$, $3a)'$, $5)'$, since $\varsigma^{m-1}(1-\varsigma\mu) < 1$ by (4.2), we obtain

$$\varsigma^{m-1}(1-\varsigma\mu)\left\{ \int_G a|\nabla v|^m\, dx + \int_{\Sigma_0} \frac{\sigma(\omega)}{r^{m-1}}|v|^m\, ds + \int_{\partial G} \frac{\gamma(\omega)}{r^{m-1}}|v|^m\, ds \right\} \le \int_G |v| b_0(x)\, dx + \frac{1}{\varsigma}\int_G |v|^{1-\varsigma}a_0(x)\, dx + \int_{\Sigma_0} |h(x,0)|\cdot|v|\, ds + \int_{\partial G} |g(x,0)|\cdot|v|\, ds.$$

From $M_0 = \sup_G |u(x)|$, by the change (4.2), it follows that $|v(x)| \le M_0^{\frac{1}{\sigma}}$. Therefore we get

$$\int_G a|\nabla v|^m\, dx + \int_{\Sigma_0} \frac{\sigma(\omega)}{r^{m-1}}|v|^m\, ds + \int_{\partial G} \frac{\gamma(\omega)}{r^{m-1}}|v|^m\, ds \le c(M_0,m,q,\mu,\operatorname{meas} G)\cdot\left(\int_G \big(a_0(x)+b_0(x)\big)\, dx + \int_{\Sigma_0} |h(x,0)|\, ds + \int_{\partial G} |g(x,0)|\, ds \right). \quad (4.6)$$

Returning to the function $u(x)$ by (4.2) we obtain the desired estimate (4.1). $\square$

For the problem (WL) we obtain a global estimate for the weighted Dirichlet integral.

Theorem 4.2. *Let $u(x)$ be a weak solution of the problem (WL). Let assumptions (a)–(e) be satisfied with a function $\mathcal{A}(r)$ that is continuous at zero. Suppose, in addition, that*

$$b_0(x) \in \overset{\circ}{\mathbf{W}}{}^0_\alpha(G), \qquad \int\limits_{\Sigma_0} r^{\alpha-1} h^2(x,0)\, ds < \infty,$$
$$\int\limits_{\partial G} r^{\alpha-1} g^2(x,0)\, ds < \infty, \qquad 4-n \le \alpha \le 2.$$

Then $|u(x)|^{q+1} \in \overset{\circ}{\mathbf{W}}{}^1_{\alpha-2}(G)$ and

$$\int\limits_G a\big(r^{\alpha-2}|u|^{2q}|\nabla u|^2 + r^{\alpha-4}|u|^{2(q+1)}\big)\, dx + \int\limits_{\Sigma_0} r^{\alpha-3}\sigma(\omega)|u|^{2(q+1)}\, ds + \int\limits_{\partial G} r^{\alpha-3}\gamma(\omega)|u|^{2(q+1)}\, ds$$
$$\le C\Bigg\{ \int\limits_G \big(|u|^{2(q+1)} + (1+r^\alpha) b_0^2(x)\big)\, dx + \int\limits_{\Sigma_0} r^{\alpha-1} h^2(x,0)\, ds + \int\limits_{\partial G} r^{\alpha-1} g^2(x,0)\, ds \Bigg\}, \tag{4.7}$$

where the constant $C>0$ depends only on $a_*, \alpha, \lambda, \mu, q, n$ and the domain G.

Proof. Making the function change (4.2) we consider the integral identity for the function $v(x)$:

$$\int\limits_G \big\langle \varsigma a^{ij}(x) v_{x_j}\eta_{x_i} + \mathcal{B}(x,v,v_x)\eta \big\rangle\, dx + \int\limits_{\partial G} \frac{\gamma(\omega)}{r} v\eta(x)\, ds + \int\limits_{\Sigma_0} \frac{\sigma(\omega)}{r} v\eta(x)\, ds$$
$$= \int\limits_{\partial G} \mathcal{G}(x,v)\eta(x)\, ds + \int\limits_{\Sigma_0} \mathcal{H}(x,v)\eta(x)\, ds. \tag{$\widetilde{II}$}$$

Setting in this identity $\eta(x) = r_\varepsilon^{\alpha-2} v(x)$, with regard to

$$\eta_{x_i} = r_\varepsilon^{\alpha-2} v_{x_i} + (\alpha-2) r_\varepsilon^{\alpha-3} \frac{x_i - \varepsilon l_i}{r_\varepsilon} v(x)$$

we obtain

$$\begin{aligned}
&\varsigma \int_G a r_\varepsilon^{\alpha-2} |\nabla v|^2 \, dx + \int_{\Sigma_0} r^{-1} r_\varepsilon^{\alpha-2} \sigma(\omega) v^2(x) \, ds \\
&\quad + \int_{\partial G} r^{-1} r_\varepsilon^{\alpha-2} \gamma(\omega) v^2(x) \, ds = \varsigma \frac{2-\alpha}{2} \int_G a r_\varepsilon^{\alpha-4} (x_i - \varepsilon l_i)(v^2)_{x_i} \, dx \\
&\quad + \varsigma(2-\alpha) \int_G \big(a^{ij}(x) - a^{ij}(0)\big) r_\varepsilon^{\alpha-4} (x_i - \varepsilon l_i) v_{x_j} v(x) \, dx \\
&\quad - \varsigma \int_G \big(a^{ij}(x) - a^{ij}(0)\big) r_\varepsilon^{\alpha-2} v_{x_i} v_{x_j} \, dx - \int_G \mathcal{B}(x, v, v_x) r_\varepsilon^{\alpha-2} v(x) \, dx \\
&\quad + \int_{\Sigma_0} r_\varepsilon^{\alpha-2} v(x) \mathcal{H}(x, v) \, ds + \int_{\partial G} r_\varepsilon^{\alpha-2} v(x) \mathcal{G}(x, v) \, ds. \qquad (4.8)
\end{aligned}$$

We transform the first integral on the right by integrating by parts:

$$\begin{aligned}
&\int_G a r_\varepsilon^{\alpha-4} (x_i - \varepsilon l_i) \frac{\partial v^2}{\partial x_i} \, dx \\
&\quad = \int_{G_+} a_+ r_\varepsilon^{\alpha-4} (x_i - \varepsilon l_i) \frac{\partial v_+^2}{\partial x_i} \, dx + \int_{G_-} a_- r_\varepsilon^{\alpha-4} (x_i - \varepsilon l_i) \frac{\partial v_-^2}{\partial x_i} \, dx \\
&= - \int_G a v^2 \frac{\partial}{\partial x_i} \big(r_\varepsilon^{\alpha-4} (x_i - \varepsilon l_i) \big) \, dx + \int_{\partial G_+} a_+ v_+^2 r_\varepsilon^{\alpha-4} (x_i - \varepsilon l_i) \cos(\overrightarrow{n}, x_i) \, ds \\
&+ \int_{\partial G_-} a_- v_-^2 r_\varepsilon^{\alpha-4} (x_i - \varepsilon l_i) \cos(\overrightarrow{n}, x_i) \, ds = - \int_G a v^2 \frac{\partial}{\partial x_i} \big(r_\varepsilon^{\alpha-4} (x_i - \varepsilon l_i) \big) \, dx \\
&\quad + \int_{\partial G} a v^2 r_\varepsilon^{\alpha-4} (x_i - \varepsilon l_i) \cos(\overrightarrow{n}, x_i) \, ds \\
&\quad + [a]_{\Sigma_0} \int_{\Sigma_0} v^2 r_\varepsilon^{\alpha-4} (x_i - \varepsilon l_i) \cos(\overrightarrow{n}, x_i) \, ds, \qquad (4.9)
\end{aligned}$$

because of $[v]_{\Sigma_0} = 0$. By elementary calculation we have:

1) $\frac{\partial}{\partial x_i} \big(r_\varepsilon^{\alpha-4} (x_i - \varepsilon l_i) \big) = n r_\varepsilon^{\alpha-4} + (\alpha-4)(x_i - \varepsilon l_i) r_\varepsilon^{\alpha-5} \frac{x_i - \varepsilon l_i}{r_\varepsilon} = (n + \alpha - 4) r_\varepsilon^{\alpha-4}$;

2) because of $\cos(\overrightarrow{n}, x_i)\big|_{\Sigma_0} = \cos(x_n, x_i) = \delta_i^n$,

$(x_i - \varepsilon l_i)\cos(\overrightarrow{n}, x_i)\big|_{\Sigma_0} = \delta_i^n (x_i - \varepsilon l_i)\big|_{\Sigma_0} = (x_n - \varepsilon l_n)\big|_{\Sigma_0} = x_n\big|_{\Sigma_0} = 0,$

since $\Sigma_0 = \{x_n = 0\} \cap G$ and $l_n = 0$;

3) from the representation $\partial G = \Gamma_0^d \cup \Gamma_d$ and $(x_i - \varepsilon l_i)\cos(\overrightarrow{n}, x_i)\big|_{\Gamma_0^d} = -\varepsilon \sin\frac{\omega_0}{2} \Longrightarrow$

$$\int\limits_{\partial G} av^2 r_\varepsilon^{\alpha-4}(x_i - \varepsilon l_i)\cos(\overrightarrow{n}, x_i)\,ds = -\varepsilon \sin\frac{\omega_0}{2}\int\limits_{\Gamma_0^d} av^2 r_\varepsilon^{\alpha-4}\,ds + \int\limits_{\Gamma_d} av^2 r_\varepsilon^{\alpha-4}(x_i - \varepsilon l_i)\cos(\overrightarrow{n}, x_i)\,ds.$$

Hence and from (4.9) it follows

$$\begin{aligned}\frac{2-\alpha}{2}&\int\limits_G a r_\varepsilon^{\alpha-4}(x_i - \varepsilon l_i)\frac{\partial v^2}{\partial x_i}\,dx \\ &= \frac{(2-\alpha)(4-n-\alpha)}{2}\int\limits_G a r_\varepsilon^{\alpha-4} v^2\,dx - \varepsilon\frac{2-\alpha}{2}\sin\frac{\omega_0}{2}\int\limits_{\Gamma_0^d} av^2 r_\varepsilon^{\alpha-4}\,ds \\ &\quad + \frac{2-\alpha}{2}\int\limits_{\Gamma_d} av^2 r_\varepsilon^{\alpha-4}(x_i - \varepsilon l_i)\cos(\overrightarrow{n}, x_i)\,ds \qquad (4.10)\end{aligned}$$

From (4.8), (4.10) we obtain following equality:

$$\begin{aligned}\varsigma&\int\limits_G a r_\varepsilon^{\alpha-2}|\nabla v|^2\,dx + \varepsilon\varsigma\frac{2-\alpha}{2}\sin\frac{\omega_0}{2}\int\limits_{\Gamma_0^d} av^2 r_\varepsilon^{\alpha-4}\,ds \\ &+ \int\limits_{\Sigma_0} r^{-1} r_\varepsilon^{\alpha-2}\sigma(\omega) v^2(x)\,ds + \int\limits_{\partial G} r^{-1} r_\varepsilon^{\alpha-2}\gamma(\omega) v^2(x)\,ds \\ &= \frac{2-\alpha}{2}\varsigma\int\limits_{\Gamma_d} av^2 r_\varepsilon^{\alpha-4}(x_i - \varepsilon l_i)\cos(\overrightarrow{n}, x_i)\,ds \\ &+ \varsigma\frac{(2-\alpha)(4-n-\alpha)}{2}\int\limits_G a r_\varepsilon^{\alpha-4} v^2\,dx \\ &+ \varsigma(2-\alpha)\int\limits_G \big(a^{ij}(x) - a^{ij}(0)\big) r_\varepsilon^{\alpha-4}(x_i - \varepsilon l_i) v_{x_j} v(x)\,dx\end{aligned}$$

$$-\varsigma\int\limits_G \big(a^{ij}(x)-a^{ij}(0)\big)r_\varepsilon^{\alpha-2}v_{x_i}v_{x_j}\,dx-\int\limits_G \mathcal{B}(x,v,v_x)r_\varepsilon^{\alpha-2}v(x)\,dx$$
$$+\int\limits_{\Sigma_0} r_\varepsilon^{\alpha-2}v(x)\mathcal{H}(x,v)\,ds+\int\limits_{\partial G} r_\varepsilon^{\alpha-2}v(x)\mathcal{G}(x,v)\,ds. \quad (4.11)$$

Now we estimate the integral over Γ_d. Because on $\Gamma_d:\ r_\varepsilon \ge hr \ge hd \Rightarrow (\alpha-3)\ln r_\varepsilon \le (\alpha-3)\ln(hd)$, since $\alpha \le 2$, we have $r_\varepsilon^{\alpha-3}|_{\Gamma_d} \le (hd)^{\alpha-3}$ and therefore:

$$\frac{2-\alpha}{2}\varsigma\int\limits_{\Gamma_d} av^2 r_\varepsilon^{\alpha-4}(x_i-\varepsilon l_i)\cos(\overrightarrow{n},x_i)\,ds \le \frac{2-\alpha}{2}\varsigma\int\limits_{\Gamma_d} ar_\varepsilon^{\alpha-3}v^2\,ds$$
$$\le \frac{2-\alpha}{2}\varsigma\,(hd)^{\alpha-3}\int\limits_{\Gamma_d} av^2\,ds \le c\int\limits_{G_d}(v^2+|\nabla v|^2)\,dx, \quad (4.12)$$

by (2.10). Now, in virtue of assumption (c) and the function change (4.2), we have

$$|v\cdot\mathcal{B}(x,v,v_x)| \le a\mu\varsigma^2|\nabla v|^2+b_0(x)|v|. \quad (4.13)$$

Therefore using the Cauchy inequality we deduce the following

$$\int\limits_G \mathcal{B}(x,v,v_x)r_\varepsilon^{\alpha-2}v(x)\,dx$$
$$\le \mu\varsigma^2\int\limits_G ar_\varepsilon^{\alpha-2}|\nabla v|^2\,dx+\int\limits_G r_\varepsilon^{\alpha-2}|v|b_0(x)\,dx$$
$$\le \mu\varsigma^2\int\limits_G ar_\varepsilon^{\alpha-2}|\nabla v|^2\,dx+\frac{\delta}{2}\int\limits_G ar^{-2}r_\varepsilon^{\alpha-2}v^2\,dx$$
$$+\frac{1}{2\delta a_*}\int\limits_G r^2 r_\varepsilon^{\alpha-2}b_0^2(x)\,dx, \qquad \forall\,\delta>0. \quad (4.14)$$

Now we use the representation $G=G_0^d\cup G_d$. At first we estimate integrals over G_0^d. By assumption (b) and the Cauchy inequality, we obtain:

$$\int\limits_{G_0^d}\big(a^{ij}(x)-a^{ij}(0)\big)\big(r_\varepsilon^{\alpha-2}v_{x_i}v_{x_j}+r_\varepsilon^{\alpha-4}(x_i-\varepsilon l_i)v(x)v_{x_j}\big)\,dx$$
$$\le \mathcal{A}(d)\int\limits_{G_0^d} a\big(r_\varepsilon^{\alpha-2}|\nabla v|^2+r_\varepsilon^{\alpha-3}|\nabla v|\cdot|v(x)|\big)\,dx$$

$$\leq \frac{3}{2}\mathcal{A}(d)\int\limits_{G_0^d} a\big(r_\varepsilon^{\alpha-2}|\nabla v|^2 + r_\varepsilon^{\alpha-4}v^2\big)\,dx. \quad (4.15)$$

Now we estimate integrals over G_d. By assumptions (a), the Cauchy inequality and taking into account that $r_\varepsilon \geq hd$ for $r \geq d$, we obtain:

$$\int\limits_{G_d} \big(a^{ij}(x) - a^{ij}(0)\big)\big(r_\varepsilon^{\alpha-2}v_{x_i}v_{x_j} + r_\varepsilon^{\alpha-4}(x_i - \varepsilon l_i)v(x)v_{x_j}\big)\,dx$$

$$\leq A^*\int\limits_{G_d}\Big(\frac{3}{2}r_\varepsilon^{\alpha-2}|\nabla v|^2 + r_\varepsilon^{\alpha-4}|v|^2\Big)\,dx$$

$$\leq C(A^*, h, \alpha, d)\int\limits_{G_d}\big(|\nabla v|^2 + v^2\big)\,dx. \quad (4.16)$$

Next, in virtue of assumption (e),

$$v\mathcal{G}(x,v) = v\mathcal{G}(x,0) + v^2\cdot\int\limits_0^1 \frac{\partial\mathcal{G}(x,\tau v)}{\partial(\tau v)}\,d\tau \leq |g(x,0)|\cdot|v|. \quad (4.17)$$

Further, by the Cauchy inequality and because of $\gamma(\omega) \geq \nu_0 > 0$,

$$|g(x,0)|\cdot|v| = \Big(r^{\frac{1}{2}}\frac{1}{\sqrt{\gamma(\omega)}}|g(x,0)|\Big)\Big(r^{-\frac{1}{2}}\sqrt{\gamma(\omega)}|v|\Big)$$

$$\leq \frac{\delta}{2}r^{-1}\gamma(\omega)v^2 + \frac{1}{2\delta\nu_0}rg^2(x,0), \quad \forall\,\delta > 0;$$

taking into account that $r_\varepsilon \geq hr$ (see Subsection 2.3) we obtain

$$\int\limits_{\partial G} r_\varepsilon^{\alpha-2}v\mathcal{G}(x,v)\,ds \leq \frac{\delta}{2}\int\limits_{\partial G} r_\varepsilon^{\alpha-2}\frac{1}{r}\gamma(\omega)v^2\,ds$$

$$+\frac{1}{2\delta\nu_0}\int\limits_{\partial G} r^{\alpha-1}g^2(x,0)\,ds, \quad \forall\,\delta > 0. \quad (4.18)$$

Similarly, because of $\sigma(\omega) \geq \nu_0 > 0$,

$$\int\limits_{\Sigma_0} r_\varepsilon^{\alpha-2}v\mathcal{H}(x,v)\,ds \leq \frac{\delta}{2}\int\limits_{\Sigma_0} r_\varepsilon^{\alpha-2}\frac{1}{r}\sigma(\omega)v^2\,ds$$

$$+\frac{1}{2\delta\nu_0}\int\limits_{\Sigma_0} r^{\alpha-1}h^2(x,0)\,ds, \quad \forall\,\delta > 0. \quad (4.19)$$

Taking into account that $\varsigma \le 1$ and $4-n \le \alpha \le 2$ as a result from (4.11)–(4.19) we obtain:

$$\varsigma(1-\mu\varsigma)\int\limits_G a r_\varepsilon^{\alpha-2}|\nabla v|^2\,dx + \int\limits_{\Sigma_0}\frac{1}{r}r_\varepsilon^{\alpha-2}\sigma(\omega)v^2(x)\,ds$$
$$+\int\limits_{\partial G}\frac{1}{r}r_\varepsilon^{\alpha-2}\gamma(\omega)v^2(x)\,ds \le \frac{3}{2}\mathcal{A}(d)\int\limits_{G_0^d} a\left(r_\varepsilon^{\alpha-2}|\nabla v|^2 + r_\varepsilon^{\alpha-4}v^2\right)\,dx$$
$$+\frac{\delta}{2}\int\limits_G a r^{-2}r_\varepsilon^{\alpha-2}v^2\,dx + C\int\limits_{G_d}\left(|\nabla v|^2+v^2\right)\,dx + \frac{1}{2a_*\delta}\int\limits_G r^\alpha b_0^2(x)\,dx$$
$$+\frac{1}{2\delta\nu_0}\left\{\int\limits_{\partial G} r^{\alpha-1}g^2(x,0)\,ds + \int\limits_{\Sigma_0} r^{\alpha-1}h^2(x,0)\,ds\right\}$$
$$+\frac{\delta}{2}\left\{\int\limits_{\Sigma_0} r_\varepsilon^{\alpha-2}\frac{1}{r}\sigma(\omega)v^2 ds + \int\limits_{\partial G} r_\varepsilon^{\alpha-2}\frac{1}{r}\gamma(\omega)v^2\,ds\right\},\quad \forall\,\delta>0. \tag{4.20}$$

In virtue of the inequality $r_\varepsilon \ge hr$, we have $r_\varepsilon^{\alpha-4} \le h^{-2}r^{-2}r_\varepsilon^{\alpha-2}$. Hence, by Lemma 2.3, from (4.20) it follows

$$\varsigma(1-\mu\varsigma)\left\{\int\limits_G a r_\varepsilon^{\alpha-2}|\nabla v|^2\,dx + \int\limits_{\Sigma_0}\frac{1}{r}r_\varepsilon^{\alpha-2}\sigma(\omega)v^2(x)\,ds\right.$$
$$\left.+\int\limits_{\partial G}\frac{1}{r}r_\varepsilon^{\alpha-2}\gamma(\omega)v^2(x)\,ds\right\} \le c(\lambda,\omega_0)\left(\delta+\mathcal{A}(d)\right)\left\{\int\limits_G a r_\varepsilon^{\alpha-2}|\nabla v|^2\,dx\right.$$
$$\left.+\int\limits_{\Sigma_0} r^{-1}r_\varepsilon^{\alpha-2}\sigma(\omega)v^2(x)\,ds + \int\limits_{\partial G} r^{-1}r_\varepsilon^{\alpha-2}\gamma(\omega)v^2\,ds\right\}$$
$$+C\int\limits_G\left(|\nabla v|^2+v^2\right)\,dx + \frac{1}{2a_*\delta}\int\limits_G r^\alpha b_0^2(x)\,dx$$
$$+\frac{1}{2\delta\nu_0}\int\limits_{\partial G} r^{\alpha-1}g^2(x,0)\,ds + \frac{1}{2\delta\nu_0}\int\limits_{\Sigma_0} r^{\alpha-1}h^2(x,0)\,ds,$$
$$\forall\,\delta>0,\ \forall\,\varepsilon>0. \tag{4.21}$$

Because of $0\le\mu<1+q$, we can choose $\delta = \frac{1}{4c(\lambda,\omega_0)}(1-\mu\varsigma)$ and next $d>0$ such that, by the continuity of $\mathcal{A}(r)$ at zero, $c(\lambda,\omega_0)\mathcal{A}(d) \le \frac{1}{4}(1-\mu\varsigma)$ Thus, from (4.21) we get

$$\int\limits_G a r_\varepsilon^{\alpha-2}|\nabla v|^2\,dx+\int\limits_{\Sigma_0}\frac{1}{r}r_\varepsilon^{\alpha-2}\sigma(\omega)v^2(x)\,ds+\int\limits_{\partial G}\frac{1}{r}r_\varepsilon^{\alpha-2}\gamma(\omega)v^2(x)\,ds$$
$$\le C(a_*,\alpha,\lambda,\mu,q,n,d)\left\{\int\limits_G(|\nabla v|^2+v^2)\,dx+\int\limits_G r^\alpha b_0^2(x)\,dx\right.$$
$$\left.+\frac{1}{\nu_0}\int\limits_{\partial G}r^{\alpha-1}g^2(x,0)\,ds+\frac{1}{\nu_0}\int\limits_{\Sigma_0}r^{\alpha-1}h^2(x,0)\,ds\right\},\quad\forall\varepsilon>0. \quad (4.22)$$

We observe that the right hand side of (4.22) does not depend on ε. Therefore we can perform the passage to the limit as $\varepsilon\to+0$ by the Fatou Theorem. Hence it follows that $v(x)\in\overset{\circ}{\mathbf{W}}{}^1_{\alpha-2}(G)$ and

$$\int\limits_G a r^{\alpha-2}|\nabla v|^2\,dx+\int\limits_{\Sigma_0}r^{\alpha-3}\sigma(\omega)v^2(x)\,ds+\int\limits_{\partial G}r^{\alpha-3}\gamma(\omega)v^2(x)\,ds$$
$$\le C(a_*,\alpha,\lambda,\mu,q,n,d)\left\{\int\limits_G(|\nabla v|^2+v^2)\,dx+\int\limits_G r^\alpha b_0^2(x)\,dx\right.$$
$$\left.+\frac{1}{\nu_0}\int\limits_{\partial G}r^{\alpha-1}g^2(x,0)\,ds+\frac{1}{\nu_0}\int\limits_{\Sigma_0}r^{\alpha-1}h^2(x,0)\,ds\right\}. \quad (4.23)$$

Further, returning to the integral identity $(\widetilde{II})$ and setting in it $\eta(x)=v(x)$, we get

$$\int\limits_G\left\langle\varsigma a^{ij}(x)v_{x_j}v_{x_i}+\mathcal{B}(x,v,v_x)v\right\rangle dx+\int\limits_{\partial G}\frac{\gamma(\omega)}{r}v^2\,ds+\int\limits_{\Sigma_0}\frac{\sigma(\omega)}{r}v^2\,ds$$
$$=\int\limits_{\partial G}\mathcal{G}(x,v)v\,ds+\int\limits_{\Sigma_0}\mathcal{H}(x,v)v\,ds.$$

By the ellipticity condition (a), inequalities (4.13), (4.18)–(4.19) for $\alpha=2,\ \delta=2$, hence it follows

$$\varsigma(1-\mu\varsigma)\int\limits_G a|\nabla v|^2\,dx$$
$$\le c(a_*,\nu_0,\operatorname{diam}G)\left\{\int\limits_G\left(|v|^2+b_0^2(x)\right)\,dx+\int\limits_{\partial G}r^{\alpha-1}g^2(x,0)\,ds\right.$$
$$\left.+\int\limits_{\Sigma_0}r^{\alpha-1}h^2(x,0)\,ds\right\}. \quad (4.24)$$

Now, using the inequality (2.3) and returning to the function $u(x)$, by means of the function change (4.2), from (4.23)–(4.24) we get the desired estimate (4.7). □

5. Local integral weighted estimates

Now we will obtain a local estimate for the weighted Dirichlet integral.

Theorem 5.1. *Let $u(x)$ be a weak solution of the problem (QL) and $M_0 = \max_{x\in\overline{G}} |u(x)|$ be known. Let $\vartheta(m)$ be the smallest positive eigenvalue of $(NEVP)$. Let assumptions of Theorem 4.1 and 7) be satisfied. Suppose, in addition, that there exists real number $k_s \geq 0$ defined by (1.5). Then there are $d \in (0,1)$ and a constant $c > 0$ independent of u and depending only on $m, n, s, q, d, \vartheta(m), k_1, k_s, \operatorname{meas}\Omega$ and M_0 such that for any $\varrho \in (0,d)$*

$$\int\limits_{G_0^\varrho} a|u|^{\frac{qm}{m-1}}|\nabla u|^m\,dx + \int\limits_{\Sigma_0^\varrho} \frac{\sigma(\omega)}{r^{m-1}}|u|^{\frac{m}{m-1}(q+m-1)}\,ds + \int\limits_{\Gamma_0^\varrho} \frac{\gamma(\omega)}{r^{m-1}}|u|^{\frac{m}{m-1}(q+m-1)}\,ds \leq c\psi^m(\varrho) \quad (5.1)$$

where $\psi(\varrho)$ is defined by (1.7)–(1.8).

Proof. By virtue of Theorem 4.1 (see (4.6)), we have that

$$V(\varrho) = \int\limits_{G_0^\varrho} a|\nabla v|^m\,dx + \int\limits_{\Sigma_0^\varrho} \frac{\sigma(\omega)}{r^{m-1}}|v|^m\,ds + \int\limits_{\Gamma_0^\varrho} \frac{\gamma(\omega)}{r^{m-1}}|v|^m\,ds < \infty, \quad \varrho \in (0,d). \quad (5.2)$$

Therefore we can set $\eta(x) = v(x)$ in the identity (4.4) after the making the change (4.2)):

$$\int\limits_{G_0^\varrho} \big\langle \mathcal{A}_i(x,v_x)v_{x_i} + \mathcal{B}(x,v,v_x)v(x)\big\rangle\,dx + \int\limits_{\Sigma_0^\varrho} \frac{\sigma(\omega)}{r^{m-1}}|v|^m\,ds + \int\limits_{\Gamma_0^\varrho} \frac{\gamma(\omega)}{r^{m-1}}|v|^m\,ds = \int\limits_{\Omega_\varrho} \mathcal{A}_i(x,v_x)\cos(r,x_i)\cdot v(x)\,d\Omega_\varrho$$

$$+\int_{\Gamma_0^\varrho} \mathcal{G}(x,v)\cdot v(x)\,ds + \int_{\Sigma_0^\varrho} \mathcal{H}(x,v)\cdot v(x)\,ds. \quad (5.3)$$

By assumptions $1)'$, $3a)'$, $4)'$, $5)$, $7)'$ and since $\varsigma\mu < 1$, we obtain

$$(1-\varsigma\mu)\varsigma^{m-1}V(\varrho) \le \int_{G_0^\varrho} |v| b_0(x)\,dx + \frac{1}{\varsigma}\int_{G_0^\varrho} |v|^{1-\varsigma} a_0(x)\,dx$$
$$+ k_1 \int_{\Omega_\varrho} r^{s-1}|v|\,d\Omega_\varrho + \varsigma^{m-1}\int_{\Omega_\varrho} a|\nabla v|^{m-2}\cdot v\frac{\partial v}{\partial r}\,d\Omega_\varrho$$
$$+\int_{\Sigma_0^\varrho} |h(x,0)|\cdot|v|\,ds + \int_{\Gamma_0^\varrho} |g(x,0)|\cdot|v|\,ds. \quad (5.4)$$

Applying Lemma 2.1 from (5.4) it follows that

$$(1-\varsigma\mu)\varsigma^{m-1}V(\varrho)$$
$$\le \varsigma^{m-1}(m-1)^{\frac{m-1}{m}} \max\{1; 2^{\frac{m-2}{2(m-1)}}\}\cdot\frac{\varrho}{m\vartheta^{\frac{1}{m}}(m)}V'(\varrho)$$
$$+\frac{1}{\varsigma}\int_{G_0^\varrho} |v|^{1-\varsigma}a_0(x)\,dx + \int_{G_0^\varrho}|v|b_0(x)\,dx + k_1\varrho^{s-1}\int_{\Omega_\varrho}|v|\,d\Omega_\varrho$$
$$+\int_{\Sigma_0^\varrho} |h(x,0)|\cdot|v|\,ds + \int_{\Gamma_0^\varrho} |g(x,0)|\cdot|v|\,ds. \quad (5.5)$$

Further, by the Young inequality and inequality (2.2),

$$\frac{1}{\varsigma}\int_{G_0^\varrho}|v|^{1-\varsigma}a_0(x)\,dx = \frac{1}{\varsigma}\int_{G_0^\varrho}\left(r^{\frac{(1-\varsigma)(1-\varsigma-m)}{m}}|v|^{1-\varsigma}\right)\cdot\left(r^{\frac{(1-\varsigma)(m+\varsigma-1)}{m}}a_0(x)\right)dx$$
$$\le \int_{G_0^\varrho}\left(\frac{1-\varsigma}{m\varsigma}r^{1-\varsigma-m}|v|^m + \frac{m+\varsigma-1}{m\varsigma}r^{1-\varsigma}|a_0(x)|^{\frac{m}{m+\varsigma-1}}\right)dx$$
$$\le \frac{1-\varsigma}{a_* m\varsigma\vartheta(m)}\varrho^{1-\varsigma}V(\varrho) + \frac{m+\varsigma-1}{m\varsigma}\int_{G_0^\varrho} r^{1-\varsigma}|a_0(x)|^{\frac{m}{m+\varsigma-1}}\,dx; \quad (5.6)$$

$$\int_{G_0^\varrho}|v|b_0(x)\,dx = \int_{G_0^\varrho}\left(r^{-\frac{1}{m}}|v|\right)\left(r^{\frac{1}{m}}b_0(x)\right)dx$$

$$\le \int\limits_{G_0^\varrho} \Big(\frac{1}{m} r^{-1}|v|^m + \frac{m-1}{m} r^{\frac{1}{m-1}} |b_0(x)|^{\frac{m}{m-1}}\Big)\, dx$$

$$\le \frac{1}{a_* m\vartheta(m)} \varrho^{m-1} V(\varrho) + \frac{m}{m-1} \int\limits_{G_0^\varrho} r^{\frac{1}{m-1}} |b_0(x)|^{\frac{m}{m-1}}\, dx$$

$$= \frac{1}{a_* m\vartheta(m)} \varrho^{m-1} V(\varrho) + \frac{m}{m-1} \int\limits_{G_0^\varrho} r^{\frac{1}{m-1}} |b_0(x)|^{\frac{m}{m-1}}\, dx. \quad (5.7)$$

Now we estimate the integral over Ω_ϱ on the right in (5.5). By the Young inequality with $\forall\,\delta > 0$ and inequality $(H-W)_m$,

$$k_1 \varrho^{s-1} \int\limits_{\Omega_\varrho} |v|\, d\Omega_\varrho \le \widetilde{c} k_1 \varrho^{s-1} \int\limits_{G_0^\varrho} (|v| + |\nabla v|)\, dx$$

$$\le \delta \varrho^{s-1} \int\limits_{G_0^\varrho} (|v|^m + |\nabla v|^m)\, dx + c(k_1, \widetilde{c}, m, \delta) \cdot \operatorname{meas}\Omega \cdot \varrho^{s+n-1}$$

$$\le \frac{\delta}{a_*} \varrho^{s-1} \Big(1 + \frac{\varrho^m}{\vartheta(m)}\Big) V(\varrho) + c(k_1, \widetilde{c}, m, \delta) \cdot \operatorname{meas}\Omega \cdot \varrho^{s-1+n}$$

$$\le \varrho^{s-1} V(\varrho) + c(k_1, \widetilde{c}, m, a_*, \vartheta(m) \operatorname{meas}\Omega) \cdot \varrho^{s-1+n}, \quad (5.8)$$

if we choose $\delta > 0$ in a reasonable way.

Further, by the Young inequality with $\forall\,\delta > 0$, we have

$$\int\limits_{\Sigma_0^\varrho} |v| \cdot |h(x,0)|\, ds \le \frac{\delta}{m} \int\limits_{\Sigma_0^\varrho} |v|^m\, ds + \frac{m-1}{m} \delta^{\frac{1}{1-m}} \int\limits_{\Sigma_0^\varrho} |h(x,0)|^{\frac{m}{m-1}}\, ds.$$

Applying again inequality $(H-W)_m$ we obtain

$$\int\limits_{\Sigma_0^\varrho} |v|^m\, ds \le \widetilde{c} \int\limits_{G_0^\varrho} \big(|v|^m + m|v|^{m-1}|\nabla v|\big)\, dx$$

$$\le \widetilde{c} \int\limits_{G_0^\varrho} (m|v|^m + |\nabla v|^m)\, dx$$

$$\Longrightarrow$$

$$\frac{\delta}{m} \int\limits_{\Sigma_0^\varrho} |v|^m\, ds \le \delta \widetilde{c} \int\limits_{G_0^\varrho} |v|^m\, dx + \frac{\delta}{m} \widetilde{c} \int\limits_{G_0^\varrho} |\nabla v|^m\, ds$$

$$\le \frac{\delta}{a_*} \widetilde{c} \cdot \Big(\frac{\varrho^m}{\vartheta(m)} + \frac{1}{m}\Big) V(\varrho).$$

Therefore, if we choose $\delta = \varrho^{m-1}$, hence it follows that $\forall\, \varepsilon > 0$

$$\int\limits_{\Sigma_0^\varrho} |h(x,0)| \cdot |v|\, ds \le c_1 \varrho^{m-1} V(\varrho) + \frac{m-1}{m} \int\limits_{\Sigma_0^\varrho} \frac{1}{r} |h(x,0)|^{\frac{m}{m-1}}\, ds. \qquad (5.9)$$

where $c_1 = \frac{\widetilde{c}}{a_*}(\frac{1}{\vartheta(m)} + \frac{1}{m})$. In the same way $\forall\, \varepsilon > 0$

$$\int\limits_{\Gamma_0^\varrho} |g(x,0)| \cdot |v|\, ds \le c_1 \varrho^{m-1} V(\varrho) + \frac{m-1}{m} \int\limits_{\Gamma_0^\varrho} \frac{1}{r} |g(x,0)|^{\frac{m}{m-1}}\, ds. \qquad (5.10)$$

Thus, from (5.5)–(5.10) with regard to (1.8) it follows that

$$\begin{aligned} &(1 - \varsigma\mu)\,(1 - \delta(\varrho))\, V(\varrho) \le \frac{\varrho C(m)}{m \vartheta^{\frac{1}{m}}(m)} V'(\varrho) \\ &+ \frac{m+\varsigma-1}{m\varsigma^m} \int\limits_{G_0^\varrho} r^{1-\varsigma} |a_0(x)|^{\frac{m}{m+\varsigma-1}}\, dx + \varsigma^{1-m} \frac{m}{m-1} \int\limits_{G_0^\varrho} r^{\frac{1}{m-1}} |b_0(x)|^{\frac{m}{m-1}}\, dx \\ &+ c^* \varrho^{s-1+n} + \varsigma^{1-m} \frac{m-1}{m} \Bigg(\int\limits_{\Sigma_0^\varrho} \frac{1}{r} |h(x,0)|^{\frac{m}{m-1}}\, ds + \int\limits_{\Gamma_0^\varrho} \frac{1}{r} |g(x,0)|^{\frac{m}{m-1}}\, ds \Bigg), \end{aligned} \qquad (5.11)$$

where $\delta(\varrho) = \mathrm{const}(m, a_*, q, \vartheta(m)) \cdot \big(\varrho + \varrho^{s-1} + \varrho^{m-1} + \frac{q}{q+m-1} \varrho^{\frac{q}{q+m-1}}\big)$, $s > 1,\ c^* = \mathrm{const}(k_1, \widetilde{c}, m, q, a_*, \vartheta(m), \mathrm{meas}\, \Omega)$. We observe that $\int_0 \frac{\delta(\varrho)}{\varrho}\, d\varrho < \infty$. Thus, from (5.11) in virtue of assumption (1.5) we have the Cauchy problem for the differential inequality:

$$\begin{cases} V'(\varrho) - \mathcal{P}(\varrho) V(\varrho) + \mathcal{Q}(\varrho) \ge 0, & 0 < \varrho < d, \\ V(d) \le V_0, \end{cases} \qquad \text{(CP)}$$

where $\mathcal{P}(\varrho) = \frac{1-\delta(\varrho)}{\varrho} \cdot \frac{m\vartheta^{\frac{1}{m}}(m)}{C(m)}(1 - \varsigma\mu),\ \mathcal{Q}(\varrho) = k\varrho^{s+n-2};\ k = \mathrm{const}(k_s, m, q, \vartheta(m))$. We estimate now:

$$\begin{aligned} V(d) &= \int\limits_{G_0^d} a|\nabla v|^m\, dx + \int\limits_{\Sigma_0^d} \frac{\sigma(\omega)}{r^{m-1}} |v|^m\, ds + \int\limits_{\Gamma_0^d} \frac{\gamma(\omega)}{r^{m-1}} |v|^m\, ds \\ &\le \int\limits_{G} a|\nabla v|^m\, dx + \int\limits_{\Sigma_0} \frac{\sigma(\omega)}{r^{m-1}} |v|^m\, ds + \int\limits_{\partial G} \frac{\gamma(\omega)}{r^{m-1}} |v|^m\, ds \\ &\le c(M_0, m, q, \mu, \mathrm{meas}\, G) \cdot \Bigg(\int\limits_{G} (a_0(x) + b_0(x))\, dx + \int\limits_{\Sigma_0} |h(x,0)|\, ds \end{aligned}$$

$$+\int\limits_{\partial G}|g(x,0)|\,ds+1\Bigg)\equiv V_0 \quad (5.12)$$

in virtue of (4.6) The solution of problem (CP) is the following inequality

$$V(\varrho)\le V_0\exp\Bigg(-\int\limits_{\varrho}^{d}\mathcal{P}(\tau)\,d\tau\Bigg)+\int\limits_{\varrho}^{d}\mathcal{Q}(\tau)\exp\Bigg(-\int\limits_{\varrho}^{\tau}\mathcal{P}(\xi)\,d\xi\Bigg)\,d\tau. \quad (5.13)$$

(see [3, Theorem 1.57, §1.10]).

Direct calculations give:

$$-\int\limits_{\varrho}^{\tau}\mathcal{P}(\xi)\,d\xi=-\frac{m\vartheta^{\frac{1}{m}}(m)(1-\varsigma\mu)}{C(m)}\int\limits_{\varrho}^{\tau}\frac{1-\delta(\xi)}{\xi}\,d\xi$$
$$\le\frac{m\vartheta^{\frac{1}{m}}(m)(1-\varsigma\mu)}{C(m)}\Bigg(\ln\frac{\varrho}{\tau}+\int\limits_{0}^{d}\frac{\delta(\xi)}{\xi}\,d\xi\Bigg)$$

$$\Longrightarrow\qquad \exp\Bigg(-\int\limits_{\varrho}^{\tau}\mathcal{P}(\xi)\,d\xi\Bigg)\le\Big(\frac{\varrho}{\tau}\Big)^{\frac{m\vartheta^{\frac{1}{m}}(m)(1-\varsigma\mu)}{C(m)}}\cdot\exp\Bigg(\int\limits_{0}^{d}\frac{\delta(\xi)}{\xi}\,d\xi\Bigg);$$

$$\int\limits_{\varrho}^{d}\mathcal{Q}(\tau)\exp\Bigg(-\int\limits_{\varrho}^{\tau}\mathcal{P}(\xi)\,d\xi\Bigg)\,d\tau$$
$$\le k\exp\Bigg(\int\limits_{0}^{d}\frac{\delta(\xi)}{\xi}\Bigg)\varrho^{\frac{m\vartheta^{\frac{1}{m}}(m)(1-\varsigma\mu)}{C(m)}}\int\limits_{\varrho}^{d}\tau^{ms-1}\tau^{-\frac{m\vartheta^{\frac{1}{m}}(m)(1-\varsigma\mu)}{C(m)}}\,d\tau$$
$$=k\exp\Bigg(\int\limits_{0}^{d}\frac{\delta(\xi)}{\xi}\Bigg)\varrho^{\frac{m\vartheta^{\frac{1}{m}}(m)(1-\varsigma\mu)}{C(m)}}$$
$$\times\begin{cases}\dfrac{d^{m\left(s-\frac{\vartheta^{\frac{1}{m}}(m)(1-\varsigma\mu)}{C(m)}\right)}-\varrho^{m\left(s-\frac{\vartheta^{\frac{1}{m}}(m)(1-\varsigma\mu)}{C(m)}\right)}}{m\left(s-\frac{\vartheta^{\frac{1}{m}}(m)(1-\varsigma\mu)}{C(m)}\right)}, & s\ne\dfrac{\vartheta^{\frac{1}{m}}(m)(1-\varsigma\mu)}{C(m)};\\ \ln\frac{d}{\varrho}, & s=\dfrac{\vartheta^{\frac{1}{m}}(m)(1-\varsigma\mu)}{C(m)}.\end{cases}$$

$$V_0\cdot\exp\Bigg(-\int\limits_{\varrho}^{d}\mathcal{P}(\xi)\,d\xi\Bigg)\le V_0\Big(\frac{\varrho}{d}\Big)^{\frac{m\vartheta^{\frac{1}{m}}(m)(1-\varsigma\mu)}{C(m)}}\cdot\exp\Bigg(\int\limits_{0}^{d}\frac{\delta(\xi)}{\xi}\,d\xi\Bigg)$$

$\Longrightarrow$

$$V(\varrho) \le c \cdot \exp\left(\int\limits_0^d \frac{\delta(\xi)}{\xi}\, d\xi \right) \cdot (V_0 + k) \times \begin{cases} \varrho^{\frac{m\vartheta^{\frac{1}{m}}(m)(1-\varsigma\mu)}{C(m)}}, & s > \frac{\vartheta^{\frac{1}{m}}(m)(1-\varsigma\mu)}{C(m)}; \\ \varrho^{\frac{m\vartheta^{\frac{1}{m}}(m)(1-\varsigma\mu)}{C(m)}} \ln\frac{d}{\varrho}, & s = \frac{\vartheta^{\frac{1}{m}}(m)(1-\varsigma\mu)}{C(m)}; \\ \varrho^{ms}, & s < \frac{\vartheta^{\frac{1}{m}}(m)(1-\varsigma\mu)}{C(m)}, \end{cases} \tag{5.14}$$

where $c = \mathrm{const}(m, s, \vartheta(m))$. Thus we proved the statement of our theorem. □

For problem (WL) we establish the estimate with *best possible exponents.*

Theorem 5.2. *Let $u(x)$ be a weak solution of the problem (WL) and λ be as above in (1.2). Let assumptions (a)–(f), be satisfied with $\mathcal{A}(r)$ that is Dini-continuous at zero.*

Then $|u(x)|^{q+1} \in \overset{\circ}{\mathbf{W}}{}^1_{2-n}(G)$ and there are $d \in (0,1)$ and a constant $C > 0$ depending only on $n, s, \lambda, q, \mu, \nu_0, a_*, G, \Sigma_0$ and on $\int_0^1 \frac{\mathcal{A}(r)}{r} dr$ such that $\forall\, \varrho \in (0, d)$

$$\int\limits_{G_0^\varrho} a\big(r^{2-n}|u|^{2q}|\nabla u|^2 + r^{-n}|u|^{2(q+1)}\big)\, dx + \int\limits_{\Sigma_0^\varrho} r^{1-n}\sigma(\omega)|u|^{2(q+1)}\, ds + \int\limits_{\Gamma_0^\varrho} r^{1-n}\gamma(\omega)|u|^{2(q+1)}\, ds \le C\left(\int\limits_G |u|^{2(q+1)}\, dx + f_1^2 + \frac{1}{\nu_0} g_1^2 + \frac{1}{\nu_0} h_1^2 \right) \times \begin{cases} \varrho^{2\lambda(1-\mu\varsigma)}, & \text{if } s > \lambda(1-\mu\varsigma), \\ \varrho^{2\lambda(1-\mu\varsigma)} \ln^3\left(\frac{1}{\varrho}\right), & \text{if } s = \lambda(1-\mu\varsigma), \\ \varrho^{2s}, & \text{if } s < \lambda(1-\mu\varsigma), \end{cases} \tag{5.15}$$

where $\varsigma = \frac{1}{1+q}$.

Proof. The proof is similar to above. We use sharp inequality (2.8) and therefore we obtain the estimate with best possible exponents. See also proofs of Theorems 4.18, 7.21, 10.39 [3]. □

6. The power modulus of continuity at the conical point for weak solutions

Proof of Theorems 1.1, 1.2. We consider the function $\psi(\varrho),\ 0<\varrho<d$ that is determined by (1.7). By Theorem 3.1 (about the local bound of the weak solution modulus) with $t=m$ for $|v|=|u|^{\frac{1}{\varsigma}}$, we have

$$\sup_{x\in G_0^{\varrho/2}}|v(x)|^m\le C\big\{\varrho^{-n}\|v\|^m_{m,G_0^\varrho}+K^m(\varrho)\big\},\tag{6.1}$$

$$K(\varrho)=\varrho^{\frac{m(p-n)}{p(m-1+\varsigma)}}\cdot\|a_0(x)\|^{\frac{1}{m-1+\varsigma}}_{\frac{p}{m},G_0^\varrho}+\varrho^{(1-\frac{n}{p})\frac{m}{m-1}}\|b_0(x)\|^{\frac{1}{m-1}}_{\frac{p}{m},G_0^\varrho}$$
$$+\varrho^{1-\frac{n}{p}}\|\alpha(x)\|^{\frac{1}{m-1}}_{\frac{p}{m-1},G_0^\varrho}+\varrho\Big(\|g(x,0)\|^{\frac{1}{m-1}}_{\infty,\Gamma_0^\varrho}+\|h(x,0)\|^{\frac{1}{m-1}}_{\infty,\Sigma_0^\varrho}\Big),$$
$$p>n>m.\tag{6.2}$$

Hence, in virtue of inequality $(H-W)_m$ with regard to the notation (2.4), we get

$$\varrho^{-n}\int\limits_{G_0^\varrho}a|v(x)|^m\,dx\le\frac{\varrho^{m-n}}{\vartheta(m)}V(\varrho)\le C\varrho^{m-n}\psi^m(\varrho),\tag{6.3}$$

by inequality (5.14). From (6.1)–(6.3) it follows

$$\sup_{x\in G_0^{\varrho/2}}|v(x)|\le C\big\{\varrho^{1-\frac{n}{m}}\psi(\varrho)+K(\varrho)\big\},\tag{6.4}$$

Now, in virtue of assumption (1.6), it follows $K(\varrho)\le K\varrho^{1-\frac{n}{m}}\psi(\varrho)$. Therefore hence and from (6.4) we get $|v(x)|\le C_0\varrho^{1-\frac{n}{m}}\psi(\varrho),\ x\in G_0^{\varrho/2}$. Putting $|x|=\frac{1}{3}$ we obtain

$$|v(x)|\le C_0|x|^{1-\frac{n}{m}}\psi(|x|),\ x\in G_0^d.\tag{6.5}$$

Finally, because of (4.2), from (6.5) we establish the first desired estimate (1.9).

Repeating verbatim the Theorem 10.35 [3] proof we obtain the inequality

$$|\nabla v(x)|\le C_1|x|^{-\frac{n}{m}}\psi(|x|),\qquad x\in G_0^d.\tag{6.6}$$

But since, by (4.2), $|\nabla u(x)|\le|v|^{\varsigma-1}|\nabla v(x)|$ from (6.5)–(6.6) we establish the second desired estimate (1.10).

The Theorem 1.1 proof is similar. □

7. Example

Here we consider two dimensional transmission problem for the Laplace operator with absorbtion term in an angular domain and investigate the corresponding eigenvalue problem. Suppose $n = 2$, the domain G lies inside the corner

$$G_0 = \Big\{ (r,\omega) \,|r > 0;\ -\frac{\omega_0}{2} < \omega < \frac{\omega_0}{2} \Big\}, \qquad \omega_0 \in]0, 2\pi[;$$

$\mathcal{O} \in \partial G$ and in some neighborhood of $\mathcal{O}$ the boundary ∂G coincides with the sides of the corner $\omega = -\frac{\omega_0}{2}$ and $\omega = \frac{\omega_0}{2}$. We denote

$$\Gamma_\pm = \Big\{(r,\omega) \mid r > 0;\ \omega = \pm\frac{\omega_0}{2}\Big\}, \qquad \Sigma_0 = \{(r,\omega) \mid r > 0;\ \omega = 0\}$$

and we put $\sigma(\omega)|_{\Sigma_0} = \sigma(0) = \sigma = \text{const} > 0$, $\gamma(\omega)|_{\omega=\pm\frac{\omega_0}{2}} = \gamma_\pm = \text{const} > 0$. We consider the following problem:

$$\begin{cases} \frac{d}{dx_i}\left(|u|^q u_{x_i}\right) = a_0 r^{-2} u|u|^q - \mu u|u|^{q-2}|\nabla u|^2, & x \in G_0 \setminus \Sigma_0; \\ [u]_{\Sigma_0} = 0, \quad \left[a|u|^q \frac{\partial u}{\partial n}\right]_{\Sigma_0} + \frac{1}{|x|}\sigma(0)u|u|^q = 0, & x \in \Sigma_0; \\ \alpha_\pm a_\pm |u_\pm|^q \frac{\partial u_\pm}{\partial n} + \frac{1}{|x|}\gamma_\pm u_\pm |u_\pm|^q = 0, & x \in \Gamma_\pm \setminus \mathcal{O} \end{cases} \qquad (AL)$$

where $a_0 \geq 0,\ 0 \leq \mu < 1 + q,\ q \geq 0;\ \alpha_\pm \in \{0;1\};\ a_\pm > 0$. We make the function change (4.2) and consider our problem for the function $v(x)$:

$$\begin{cases} \triangle v + \mu\varsigma v^{-1}|\nabla v|^2 = a_0(1+q)r^{-2}v; \quad \varsigma = \frac{1}{1+q}, & x \in G_0 \setminus \Sigma_0; \\ [v]_{\Sigma_0} = 0, \quad \left[a\frac{\partial v}{\partial n}\right]_{\Sigma_0} + (1+q)\sigma(0)\frac{v(x)}{|x|} = 0, & x \in \Sigma_0; \\ \alpha_\pm a_\pm \frac{\partial v_\pm}{\partial n} + (1+q)\gamma_\pm \frac{v_\pm(x)}{|x|} = 0, & x \in \Gamma_\pm \setminus \mathcal{O}. \end{cases}$$

We want find the exact solution of this problem in the form $v(r,\omega) = r^{\varkappa}\psi(\omega)$. For $\psi(\omega)$ we obtain the problem

$$\begin{cases} \psi''(\omega) + \frac{\mu\varsigma}{\psi(\omega)}\psi'^2(\omega) + \left\{(1+\mu\varsigma)\varkappa^2 - a_0(1+q)\right\} \cdot \psi(\omega) = 0, \\ \qquad\qquad \omega \in \left(-\frac{\omega_0}{2}, 0\right) \cup \left(0, \frac{\omega_0}{2}\right); \\ [\psi]_{\omega=0} = 0, \quad [a\psi'(0)] = (1+q)\sigma(0)\psi(0); \\ \pm\alpha_\pm a_\pm \psi'_\pm\left(\pm\frac{\omega_0}{2}\right) + (1+q)\gamma_\pm\psi_\pm\left(\pm\frac{\omega_0}{2}\right) = 0. \end{cases}$$

We assume that $\varkappa^2 > a_0 \frac{(1+q)^2}{1+q+\mu}$ and define the value

$$\Upsilon = \sqrt{\varkappa^2 - a_0 \frac{(1+q)^2}{1+q+\mu}}. \tag{7.1}$$

We consider separately two cases: $\mu = 0$ and $\mu \neq 0$.

$\boxed{\mu = 0}$: In this case we get

$$\psi_\pm(\omega) = A\cos(\Upsilon\omega) + B_\pm \sin(\Upsilon\omega), \tag{7.2}$$

where constants $A, B_\pm$ it should be determined from conjunction and boundary conditions; namely, they satisfy the system

$$\begin{cases} (1+q)\sigma(0)\cdot A - a_+\Upsilon\cdot B_+ + a_-\Upsilon\cdot B_- = 0; \\ \left\{(1+q)\gamma_+\cos\left(\Upsilon\frac{\omega_0}{2}\right) - \alpha_+ a_+ \Upsilon \sin\left(\Upsilon\frac{\omega_0}{2}\right)\right\}\cdot A + \left\{(1+q)\gamma_+\sin\left(\Upsilon\frac{\omega_0}{2}\right)\right. \\ \qquad\qquad \left. + \alpha_+ a_+\Upsilon\cos\left(\Upsilon\frac{\omega_0}{2}\right)\right\}\cdot B_+ = 0; \\ \left\{(1+q)\gamma_-\cos\left(\Upsilon\frac{\omega_0}{2}\right) - \alpha_- a_- \Upsilon \sin\left(\Upsilon\frac{\omega_0}{2}\right)\right\}\cdot A - \left\{(1+q)\gamma_-\sin\left(\Upsilon\frac{\omega_0}{2}\right)\right. \\ \qquad\qquad \left. + \alpha_- a_-\Upsilon\cos\left(\Upsilon\frac{\omega_0}{2}\right)\right\}\cdot B_- = 0. \end{cases}$$

The Dirichlet problem: $\alpha_\pm = 0,\ \gamma_\pm \neq 0$.
Direct calculations will give

$$\psi_\pm(\omega) = \cos(\Upsilon\omega) \mp \cot\left(\Upsilon\frac{\omega_0}{2}\right)\cdot\sin(\Upsilon\omega), \quad \Upsilon = \begin{cases} \frac{\pi}{\omega_0}, & \text{if } \sigma(0) = 0; \\ \Upsilon^*, & \text{if } \sigma(0) \neq 0, \end{cases}$$

where Υ^* is the least positive root of the transcendence equation

$$\Upsilon\cdot\cot\left(\Upsilon\frac{\omega_0}{2}\right) = -\frac{1+q}{a_+ + a_-}\sigma(0)$$

and from the graphic solution we obtain $\frac{\pi}{\omega_0} < \Upsilon^* < \frac{2\pi}{\omega_0}$.
The corresponding eigenfunctions

$$\psi_\pm(\omega) = \begin{cases} \cos\left(\frac{\pi\omega}{\omega_0}\right), & \text{if } \sigma(0) = 0; \\ \cos(\Upsilon^*\omega) \mp \cot\left(\Upsilon^*\frac{\omega_0}{2}\right)\cdot\sin(\Upsilon^*\omega), & \text{if } \sigma(0) \neq 0. \end{cases}$$

The Neumann problem: $\alpha_\pm = 1,\ \gamma_\pm = 0$.
Direct calculations will give:

$$\Upsilon = \begin{cases} \frac{\pi}{\omega_0}, & \text{if } \sigma(0) = 0; \\ \Upsilon^*, & \text{if } \sigma(0) \neq 0, \end{cases}$$

where Υ^* is the least positive root of the transcendence equation $\Upsilon \cdot \tan\left(\Upsilon\frac{\omega_0}{2}\right) = \frac{1+q}{a_+ + a_-}\sigma(0)$ and from the graphic solution we obtain $0 < \Upsilon^* < \frac{\pi}{\omega_0}$. The corresponding eigenfunctions

$$\psi_\pm(\omega) = \begin{cases} a_\mp \sin\left(\frac{\pi\omega}{\omega_0}\right), & \text{if } \sigma(0) = 0; \\ \cos(\Upsilon^*\omega) \pm \tan\left(\Upsilon^*\frac{\omega_0}{2}\right) \cdot \sin(\Upsilon^*\omega), & \text{if } \sigma(0) \neq 0. \end{cases}$$

Mixed problem: $\alpha_+ = 1,\ \alpha_- = 0;\ \gamma_+ = 0,\ \gamma_- = 1.$

Direct calculations will give: $\Upsilon = \Upsilon^*$, where Υ^* is the least positive root of the transcendence equation

$$a_+ \tan\left(\Upsilon\frac{\omega_0}{2}\right) - a_- \cot\left(\Upsilon\frac{\omega_0}{2}\right) = \frac{1+q}{\Upsilon}\sigma(0).$$

The corresponding eigenfunctions

$$\psi_+(\omega) = \cos(\Upsilon^*\omega) + \tan\left(\Upsilon^*\frac{\omega_0}{2}\right) \cdot \sin(\Upsilon^*\omega), \quad \omega \in \left[0, \frac{\omega_0}{2}\right];$$
$$\psi_-(\omega) = \cos(\Upsilon^*\omega) + \cot\left(\Upsilon^*\frac{\omega_0}{2}\right) \cdot \sin(\Upsilon^*\omega), \quad \omega \in \left[-\frac{\omega_0}{2}, 0\right].$$

The Robin problem: $\alpha_\pm = 1,\ \gamma_\pm \neq 0.$

Direct calculations of the above system will give:

1) $\frac{\gamma_+}{\gamma_-} = \frac{a_+}{a_-} \Longrightarrow \psi_\pm(\omega) = a_\mp \sin(\Upsilon^*\omega)$, where Υ^* is the least positive root of the transcendence equation

$$\Upsilon \cdot \cot\left(\Upsilon\frac{\omega_0}{2}\right) = -(1+q)\frac{\gamma_+}{a_+}$$

and from the graphic solution we obtain $\frac{\pi}{\omega_0} < \Upsilon^* < \frac{2\pi}{\omega_0}$.

2) $\frac{\gamma_+}{\gamma_-} \neq \frac{a_+}{a_-} \Longrightarrow A \neq 0$ and from (7.2) it follows that $\psi_\pm(0) \neq 0$; further see below the general case $\mu \neq 0$.

$\boxed{\mu \neq 0}$: By setting $y(\omega) = \frac{\psi'(\omega)}{\psi(\omega)}$, we arrive at the problem for $y(\omega)$

$$\begin{cases} y' + (1+\mu\varsigma)y^2(\omega) + (1+\mu\varsigma)\varkappa^2 - a_0(1+q) = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad \omega \in \left(-\frac{\omega_0}{2}, 0\right) \cup \left(0, \frac{\omega_0}{2}\right); \\ a_+ y_+(0) - a_- y_-(0) = (1+q)\sigma(0); \\ \pm\alpha_\pm a_\pm y_\pm\left(\pm\frac{\omega_0}{2}\right) + (1+q)\gamma_\pm = 0. \end{cases}$$

Integrating the equation of our problem we find

$$y_\pm(\omega) = \Upsilon \tan\{\Upsilon(C_\pm - (1+\mu\varsigma)\omega)\}, \qquad \forall C_\pm. \tag{7.3}$$

From boundary conditions we have

$$C_\pm = \pm(1+\mu\varsigma)\frac{\omega_0}{2} \mp \frac{1}{\Upsilon}\arctan\frac{(1+q)\gamma_\pm}{\alpha_\pm a_\pm \Upsilon}. \tag{7.4}$$

Finally, in virtue of the conjunction condition, we get the equation for required $\varkappa$:

$$a_+ \cdot \frac{\alpha_+ a_+ \Upsilon \tan\left\{(1+\mu\varsigma)\Upsilon\frac{\omega_0}{2}\right\} - (1+q)\gamma_+}{\alpha_+ a_+ \Upsilon + (1+q)\gamma_+ \tan\left\{(1+\mu\varsigma)\Upsilon\frac{\omega_0}{2}\right\}} + a_- \cdot \frac{\alpha_- a_- \Upsilon \tan\left\{(1+\mu\varsigma)\Upsilon\frac{\omega_0}{2}\right\} - (1+q)\gamma_-}{\alpha_- a_- \Upsilon + (1+q)\gamma_- \tan\left\{(1+\mu\varsigma)\Upsilon\frac{\omega_0}{2}\right\}} = \frac{1+q}{\Upsilon}\sigma(0), \tag{7.5}$$

where $1+\mu\varsigma = \frac{1+q+\mu}{1+q}$. Further, from (7.3) and (7.4) we obtain

$$y_\pm(\omega) = \Upsilon \tan\left\{\Upsilon\frac{1+q+\mu}{1+q}\left(\pm\frac{\omega_0}{2} - \omega\right) \mp \arctan\frac{(1+q)\gamma_\pm}{\alpha_\pm a_\pm \Upsilon}\right\} \tag{7.6}$$

and, because of $(\ln\psi(\omega))' = y(\omega)$, hence it follows

$$\psi_\pm(\omega) = \cos^{\frac{1+q}{1+q+\mu}}\left\{\Upsilon\frac{1+q+\mu}{1+q}\left(\pm\frac{\omega_0}{2} - \omega\right) \mp \arctan\frac{(1+q)\gamma_\pm}{\alpha_\pm a_\pm \Upsilon}\right\}. \tag{7.7}$$

At last, returning to the function u, by (4.2), we establish a solution of (AL)

$$u_\pm(r,\omega) = r^{\frac{\varkappa}{1+q}} \cos^{\frac{1}{1+q+\mu}}\left\{\Upsilon\frac{1+q+\mu}{1+q}\left(\pm\frac{\omega_0}{2} - \omega\right) \mp \arctan\frac{(1+q)\gamma_\pm}{\alpha_\pm a_\pm \Upsilon}\right\}, \tag{7.8}$$

where Υ is defined by (7.1) and $\varkappa$ is the smallest positive root of the transcendence equation (7.5).

If we consider *the Dirichlet problem without the interface:* $\alpha_\pm = 0$, $a_\pm = 1$, $\sigma(0) = 0$, then we can calculate from (7.5) and (7.8)

$$u(r,\omega) = r^{\widetilde{\lambda}} \cos^{\frac{1}{1+q+\mu}}\left(\frac{\pi\omega}{\omega_0}\right); \qquad \widetilde{\lambda} = \frac{\sqrt{(\pi/\omega_0)^2 + a_0(1+q+\mu)}}{1+q+\mu}.$$

It is well known result (see [2, Example 4.6, p. 374]).

Now we can verify that the derived exact solution satisfies the estimate (1.3) of Theorem 1.1. In fact, in our case we have: the value λ for (1.2) is equal $\vartheta = \frac{\pi}{\omega_0}$ and therefore

$$|u(r,\omega)| \le r^{\widetilde{\lambda}} \le r^{\frac{\pi}{\omega_0}\cdot\frac{1}{1+q+\mu}} \le r^{\frac{\pi}{\omega_0}\cdot\frac{1+q-\mu}{(1+q)^2}},$$

since $a_0 \geq 0$ and $\frac{1}{1+q+\mu} \geq \frac{1+q-\mu}{(1+q)^2}$.

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