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Discontinuous Birkhoff theorem

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Abstract. A topological space X is called totally recurrent if every mapping $f : X \to X$ has a recurrent point. We prove that a Hausdorff space X is totally recurrent if and only if X is either finite or a one-point compactification of an infinite discrete space.

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Let X be a topological space, $f : X \to X$. A point $x \in X$ is said to be *recurrent* if, x is a limit point of the orbit $\{f^m(x) : m \in \omega\}$. By Birkhoff Theorem ([1, 2]), every continuous mapping of a compact space has a recurrent point.

We say that a topological space X is *totally recurrent* if **every** mapping $f : X \to X$ has a recurrent point.

Example 1. Every finite space is totally recurrent.

Example 2. Let X be an infinite discrete space, $\dot{X} = X \cup \{\infty\}$ be a one-point compactification of X, $f : \dot{X} \to \dot{X}$. If ∞ is not a recurrent point of f then there exist a neighbourhood U of x and $n \in \omega$ such that $f^m(x) \notin U$ for every $m > n$. Since $\dot{X} \setminus U$ is finite, at least one point of $\dot{X} \setminus U$ is recurrent, so $\dot{X}$ is totally recurrent.

Example 3. Let X be an infinite set endowed with a topology in which a subset U is open if and only if $X \setminus U$ is finite. Every point of X is a limit point of any infinite subset of X. It follows that X is a totally recurrent T_1-space.

To prove our main result we need some auxiliary lemmas.

Lemma 1. *Every closed subspace F of a totally recurrent space X is totally recurrent.*

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Proof. We take an arbitrary mapping $f : F \to F$, fix some point $y \in F$ and define a mapping $h : X \to X$ by the rule

$$h|_F = f|_F, \quad h|_{X \setminus F} \equiv y.$$

Since X is totally recurrent, there exists a recurrent point x of h. By the definition of h, we have $x \in F$, so x is a recurrent point of f and F is totally recurrent. $\square$

Lemma 2. *Let X be a topological space, γ be a limit ordinal. Assume that there exists a family $\{F_\alpha : \alpha < \gamma\}$ of non-empty closed subspaces of X such that $F_0 = X$ and*

(i) $F_\alpha \supset F_\beta$ *for all* $\alpha < \beta < \gamma$*;*

(ii) $F_\beta = \bigcap_{\alpha<\beta} F_\alpha$ *for every limit ordinal* $\beta < \gamma$*;*

(iii) $\bigcap_{\alpha<\gamma} F_\alpha = \varnothing$*.*

Then X is not totally recurrent.

Proof. For every $\alpha < \gamma$, we fix some point $x_\alpha \in F_\alpha \setminus F_{\alpha+1}$ and define a mapping $f : X \to X$ by the rule $f|_{F_\alpha \setminus F_{\alpha+1}} \equiv x_{\alpha+1}$. Given an arbitrary point $x \in X$, we choose the minimal ordinal β such that $x \notin F_\beta$. By (ii), β is not a limit ordinal, so $\beta = \alpha + 1$ for some $\alpha < \gamma$ and $x \in F_\alpha \setminus F_{\alpha+1}$. By definition of f, we have $f^n(x) \in F_{\alpha+1}$ for every natural number n, so $f^n(x) \notin X \setminus F_{\alpha+1}$. Since $X \setminus F_{\alpha+1}$ is a neighbourhood of X, we conclude that x is not a recurrent point of f. $\square$

Lemma 3. *Let X be a topological space. Assume that there exist two families $\{F_n : n \in \omega\}, \{H_n : n \in \omega\}$ of closed subspaces of X such that $F_0 \cap H_0 = \varnothing$ and $F_n \supset F_{n+1}$, $H_n \supset H_{n+1}$ for every $n \in \omega$. Then X is not totally recurrent.*

Proof. For every $n \in \omega$, we fix some points $x_n \in F_n \setminus F_{n+1}$, $y_n \in H_n \setminus H_{n+1}$ and define a mapping $f : X \to X$ by the rule

$$f|_{F_n \setminus F_{n+1}} \equiv x_{n+1}, \quad f|_{H_n \setminus H_{n+1}} \equiv y_{n+1},$$

$$f|_{\cap_{n\in\omega} F_n} \equiv y_0, \quad f|_{\cap_{n\in\omega} H_n} \equiv x_0, \quad f|_{X \setminus (F_0 \cup H_0)} \equiv x_0.$$

It is a routine verification that f has no recurrent points. $\square$

Lemma 4. *Let X be an infinite totally recurrent space such that every infinite closed subspace of X has an infinite proper closed subspace. Then the following statements hold* (*i*) $F \cap H \neq \varnothing$ *for any two closed infinite subspaces of* X*;*

(*ii*) *there exists a non-empty finite subset* A *of* X *such that* $X \setminus U$ *is finite for every open subset* U *of* X *containing* A*.*

Proof. (*i*) follows directly from Lemma 3.

(*ii*) Using the assumption of lemma, we can construct inductively, a family $\{F_\alpha : \alpha < \gamma\}$ of infinite closed subspaces of X satisfying (*i*), (*ii*) of Lemma 2 and such that $\bigcap_{\alpha<\gamma} F_\alpha$ is finite. Put $A = \bigcap_{\alpha<\gamma} F_\alpha$. By Lemma 2, A is non-empty. Let U be an open subset of X such that $A \subseteq U$. Assume that $X \setminus U$ is infinite and, for every $\alpha < \gamma$, put $H_\alpha = (X \setminus U) \cap F_\alpha$. By (*i*) of Lemma 4, $H_\alpha \neq \varnothing$ for every $\alpha < \gamma$. Then the family $\{H_\alpha : \alpha < \gamma\}$ of closed subsets of $X \setminus U$ satisfies Lemma 2, so $X \setminus U$ is not totally recurrent contradicting Lemma 1. □

Theorem 1. *Let* X *be an infinite Hausdorff totally recurrent space. Then* X *is a one-point compactification of a discrete space.*

Proof. Let $A = \{a_1, \dots, a_n\}$ be a subset of X given by Lemma 4. Since X is Hausdorff, every point from $X \setminus A$ is isolated, so it suffices to show that A has only one non-isolated point in X. We assume the contrary that a_1, a_2 are non-isolated, and choose pairwise disjoint open sets $U_1, \dots, U_n$ containing $a_1, \dots, a_n$. Since every point from $X \setminus (U_1, \dots, U_n)$ is isolated then $U_1, \dots, U_n$ are closed, so U_1, U_2 are infinite disjoint closed subsets of X and we get a contradiction to Lemma 4. □

Example 3 shows that this theorem does not hold for T_1-spaces.

References

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