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Upper bounds on second order operators, acting on metric function

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Abstract. We prove upper bounds on the general second order operator acting on metric function. The suggested approach does not use traditional formulas for deviations of geodesics and Jacobi fields construction and leads to the manifolds generalization of the classical coercitivity and dissipativity conditions for diffusion equations.

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In this paper we turn to the geometric problem of upper estimates on the general second order operators, acting on metric function. We consider operators of form

$$\mathcal{L}f = \frac{1}{2}\sum_{\sigma=1}^{d} A_\sigma(A_\sigma f) + A_0 f, \tag{1}$$

where A_0, A_α represent C^∞ smooth globally defined vector fields on the oriented smooth complete connected Riemannian manifold M without boundary.

The known approaches of differential geometry were mostly invented for the case of Laplace-Beltrami Δ or similar operators [5, 6], because it was hard to find the implicit representations for arbitrary differential operators on metric function, defined as a minimum of length functional

$$\rho^2(x,y) = \inf\left\{ \int_0^1 |\dot\gamma(\ell)|^2 d\ell,\ \gamma(0)=x,\ \gamma(1)=y \right\}. \tag{2}$$

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Corresponding techniques were related with the use of geodesic deviations formulas and related Jacobi fields, with the study of the global geometry of manifold, e.g. [5, 6], survey [7] and references therein.

However, for upper bounds one does not need the precise representations for differential operators on metric (!). Below we develop such estimates and demonstrate, that the traditional approach of geodesic deviations is a little advanced for such simple problem.

The found conditions on coefficients of operator $\mathcal{L}$ generalize the classical dissipativity and coercitivity conditions from the linear base space to manifold. They relate the coefficients of operator with the geometric properties of manifold, without traditional separation of geometry:

- **coercitivity**: $\exists\, o \in M$ such that $\forall\, C \in \mathbb{R}_+$ $\exists\, K_C \in \mathbb{R}^1$ such that $\forall\, x \in M$
$$\langle \widetilde{A_0}(x), \nabla^x \rho^2(x,o)\rangle + C \sum_{\sigma=1}^{d} \|A_\sigma(x)\|^2 \leq K_C(1+\rho^2(x,o)); \quad (3)$$
- **dissipativity**: $\forall\, C, C' \in \mathbb{R}_+$ $\exists\, K_C \in \mathbb{R}^1$ such that $\forall\, x \in M$, $\forall\, h \in T_x M$
$$\langle \nabla\widetilde{A_0}(x)[h], h\rangle + C \sum_{\sigma=1}^{d} \|\nabla A_\sigma(x)[h]\|^2 \\ - C' \sum_{\sigma=1}^{d} \langle R_x(A_\sigma(x), h) A_\sigma(x), h\rangle \leq K_C \|h\|^2, \quad (4)$$

  where $\widetilde{A_0} = A_0 + \frac{1}{2}\sum_{\sigma=1}^{d} \nabla_{A_\sigma} A_\sigma$ and $[R(A,h)A]^m = R_{p\ \ell q}^{\ m} A^p A^\ell h^q$ denotes the curvature operator, related with (1,3)-curvature tensor with components
$$R_{1\ 34}^{\ 2} = \frac{\partial \Gamma_{1\ 3}^{\ 2}}{\partial x^4} - \frac{\partial \Gamma_{1\ 4}^{\ 2}}{\partial x^3} + \Gamma_{1\ 3}^{\ j} \Gamma_{j\ 4}^{\ 2} - \Gamma_{1\ 4}^{\ j} \Gamma_{j\ 3}^{\ 2}. \quad (5)$$

  For simplicity of further calculations we only point the positions of corresponding indexes.

  Notation $\nabla H[h]$ means the directional covariant derivative, defined by
$$(\nabla H(x)[h])^i = \nabla_j H^i(x) \cdot h^j. \quad (6)$$

Main result of article provides

Theorem 1. *Suppose that conditions* (3)–(4) *hold.*

Then there is constant K such that at the points of C^2-regularity of metric distance

$$\left\{ A_0^I + A_0^{II} + \frac{1}{2} \sum_{\sigma=1}^{d} (A_\sigma^I + A_\sigma^{II})^2 \right\} \rho^2(x,y) \le K \rho^2(x,y). \tag{7}$$

Notations A^I, A^{II} mean vector fields, acting on the first and second variables x and y of function $\rho^2(x.y)$ correspondingly, for example $A^{II}\rho^2(x,y) = \langle A(y), \nabla_y \rangle \rho^2(x,y)$.

Similarly $\forall\, C\ \exists\, K_C$ such that

$$\mathcal{L}^I \rho^2(x,o) + C \sum_{\sigma=1}^{d} \frac{(A_\sigma^I \rho^2(x,o))^2}{\rho^2(x,o)} \le K(1 + \rho^2(x,o)). \tag{8}$$

Proof. **Step 1.** First note, that for smooth vector field X in a vicinity of some point z of manifold N and smooth function f on N there are following representations

$$Xf(z) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^\varepsilon Xf(z^s)\, ds = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^\varepsilon \frac{d}{ds} f(z^s)\, ds = \lim_{\varepsilon \to 0} \frac{f(z^\varepsilon) - f(z)}{\varepsilon},$$

$$\begin{aligned} X(Xf)(z) &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \int_0^\varepsilon ds \int_{-s}^{s} X(Xf)(z^\ell)\, d\ell \\ &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \int_0^\varepsilon ds \int_{-s}^{s} \frac{d}{d\ell}(Xf)(z^\ell)\, d\ell \\ &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \int_0^\varepsilon \{(Xf)(z^s) - (Xf)(z^{-s})\}\, ds = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \int_0^\varepsilon \frac{d}{ds}\{f(z^s) + f(z^{-s})\}\, ds \\ &= \lim_{\varepsilon \to 0} \frac{f(z^\varepsilon) + f(z^{-\varepsilon}) - 2f(z)}{\varepsilon^2}. \end{aligned} \tag{9}$$

Here we used notation z^ε for the differential flow along field X: $z^\varepsilon = z + \int_0^\varepsilon X(z^s)\, ds$.

Step 2. In the vicinity of geodesic $\gamma(\ell)$, $\ell \in [0,1]$ from $\gamma(0) = x$ to $\gamma(1) = y$ that minimizes (2) consider smooth vector field H. Introduce a family of paths

$$[0,1] \times (-\delta, \delta) \ni (\ell, s) \to \gamma(\ell, s) \in M$$

such that at $s = 0$ path $\gamma(\ell, s)\big|_{s=0} = \gamma(\ell)$ gives geodesic γ above and parameter s appears as a result of evolution along H:

$$\frac{\partial}{\partial s}\gamma(\ell, s) = H(\gamma(\ell, s)). \tag{10}$$

Note that for $s \neq 0$ each path $\gamma(\ell, s)_{s\text{-fixed}}$ must not be geodesic, unlike in formulas for geodesic deviations.

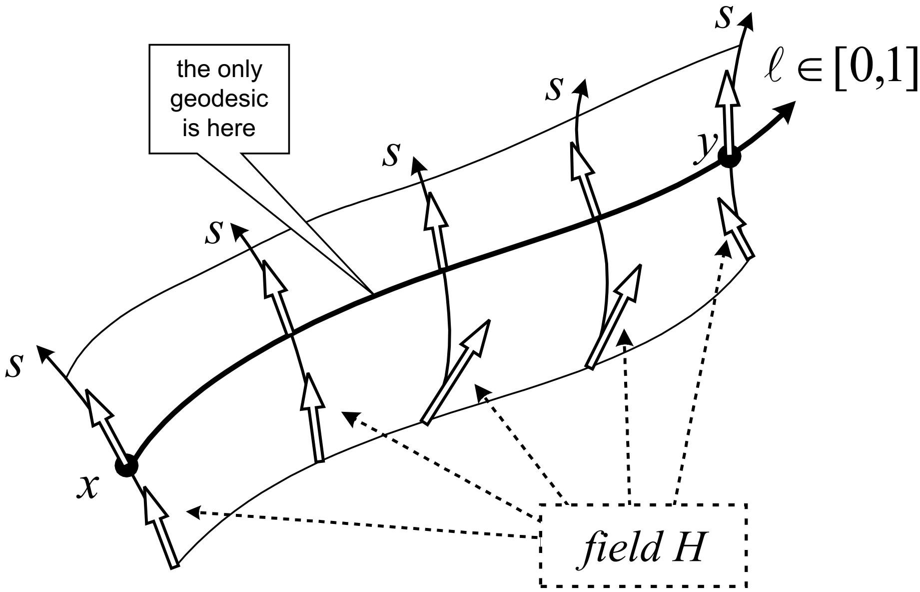

Figure 1: Field H (white vectors) in a vicinity of geodesic from x to y determines a set of paths, parameterized by s. The resulting surface is parameterized by $(\ell, s) \in [0, 1] \times (-\delta, \delta)$. Note that for $s \neq 0$ each path $\gamma(\ell, s)_{s\text{-fixed}}$ should not be geodesic

Step 3. Now let's apply (9) with $N = M \times M$, $X = H^I \otimes H^{II}$ and function $f(z) = \rho(x, y)$ for $z = (x, y)$. Using the minimal property of geodesic, i.e. that the path $\gamma(\ell, s)$ is longer than geodesic from $\gamma(0, s)$ to $\gamma(1, s)$, we can estimate terms with $\pm\varepsilon$ in (9) from above and obtain (*point at which we get rid of implicit representations, see also* (16))

$$\begin{gathered}(H^I + H^{II})\rho^2(x, y) = \lim_{\varepsilon \to 0} \frac{\rho^2(\gamma(1, \varepsilon), \gamma(0, \varepsilon)) - \rho^2(x, y)}{\varepsilon} \\ \leq \lim_{\varepsilon \to 0} \frac{\int_0^1 |\frac{\partial}{\partial \ell}\gamma(\ell, \varepsilon)|^2 d\ell - \int_0^1 |\frac{\partial}{\partial \ell}\gamma(\ell, 0)|^2 d\ell}{\varepsilon} = \int_0^1 \frac{\partial}{\partial s}\Big|_{s=0} \Big|\frac{\partial}{\partial \ell}\gamma(\ell, s)\Big|^2 d\ell.\end{gathered} \tag{11}$$

To find derivative $\frac{\partial}{\partial s}\big|_{s=0}$ in the above expression let us use that by continuity arguments, for any ℓ and sufficiently small $\delta(\ell)$ the path $\{\gamma(\ell, z)\}_{z\in(-\delta(\ell),\delta(\ell))}$ completely lies in some coordinate vicinity (x^i). In this coordinate system relation (10) has integral form

$$\gamma^i(\ell, s) = \gamma^i(\ell) + \int_0^s H^i(\gamma(\ell, z))\, dz \tag{12}$$

with point $\gamma(\ell)$ on initial geodesic. Therefore

$$\dot\gamma^i(\ell, s) = \dot\gamma^i(\ell) + \int_0^s \partial_k H^i(\gamma(\ell, z))\dot\gamma^k(\ell, z)\, dz$$

and

$$\frac{\partial}{\partial s}\dot\gamma^i(\ell, s) = \partial_k H^i(\gamma(\ell, s))\dot\gamma^k(\ell, s) = (\nabla_k H^i - \Gamma_{k\ h}^{\ i})\dot\gamma^k(\ell, s), \tag{13}$$

where we changed to the covariant derivatives and introduced notation $\dot\gamma = \frac{\partial}{\partial \ell}\gamma$. In particular, from above formula and (10) we conclude commutation

$$\frac{\partial}{\partial s}\frac{\partial}{\partial \ell}\gamma^i(\ell, s) = \frac{\partial}{\partial \ell}\frac{\partial}{\partial s}\gamma^i(\ell, s).$$

Relation (13) and autoparallel property of Riemannian connection

$$\partial_k g_{mn}(x) = g_{hn}\Gamma_{k\ m}^{\ h} + g_{mh}\Gamma_{k\ n}^{\ h} \tag{14}$$

lead to

$$\begin{aligned}\frac{\partial}{\partial s}|\dot\gamma(\ell, s)|^2 = \frac{\partial}{\partial s}&[g_{ij}(\gamma(\ell, s))\dot\gamma^i(\ell, s)\dot\gamma^j(\ell, s)] \\ &= \partial_k g_{ij}\frac{\partial}{\partial s}\gamma^k \cdot \dot\gamma^i\dot\gamma^j + 2g_{ij}\dot\gamma^i\frac{\partial}{\partial s}\dot\gamma^j \\ &= 2g_{ij}\dot\gamma^i(\nabla_k H^j)\dot\gamma^k = 2\langle\dot\gamma, \nabla H[\dot\gamma]\rangle. \end{aligned}\tag{15}$$

Therefore estimate (11) transforms to

$$(H^I + H^{II})\rho^2(x, y) \le 2\int_0^1 \langle \nabla H[\dot\gamma], \dot\gamma\rangle\, d\ell.$$

Step 4. In a similar to (11) way,

$$
\begin{aligned}
&(H^I + H^{II})(H^I + H^{II})\rho^2(x,y) \\
&\quad = \lim_{\varepsilon\to 0} \frac{\rho^2(\gamma(1,\varepsilon),\gamma(0,\varepsilon)) + \rho^2(\gamma(1,-\varepsilon),\gamma(0,\varepsilon)) - 2\rho^2(x,y)}{\varepsilon^2} \\
&\quad \le \lim_{\varepsilon\to 0} \frac{\int_0^1 |\frac{\partial}{\partial\ell}\gamma(\ell,\varepsilon)|^2 d\ell + \int_0^1 |\frac{\partial}{\partial\ell}\gamma(\ell,-\varepsilon)|^2 d\ell - 2\int_0^1 |\frac{\partial}{\partial\ell}\gamma(\ell,0)|^2 d\ell}{\varepsilon^2} \\
&\qquad\qquad\qquad\qquad = \int_0^1 \frac{\partial^2}{\partial s^2}\Big|_{s=0} \Big|\frac{\partial}{\partial\ell}\gamma(\ell,s)\Big|^2 d\ell. \qquad (16)
\end{aligned}
$$

Using relation (15) we find

$$
\begin{aligned}
&\frac{1}{2}\frac{\partial^2}{\partial s^2}\Big|\frac{\partial}{\partial\ell}\gamma(\ell,\varepsilon)\Big|^2 = \frac{\partial}{\partial s}\langle\dot\gamma(\ell,s), \nabla H[\dot\gamma(\ell,s)]\rangle \\
&\quad = \frac{\partial}{\partial s}\{\, g_{ij}(\gamma)\dot\gamma^i[\nabla_k H^j(\gamma)]\dot\gamma^k \,\} = \partial_m g_{ij}(\gamma)H^m\dot\gamma^i[\nabla_k H^j(\gamma)]\dot\gamma^k \\
&\qquad + g_{ij}\{\, (\nabla_m H^i - \Gamma_{m\ h}^{\ i})\dot\gamma^m \,\}[\nabla_k H^j(\gamma)]\dot\gamma^k \\
&\qquad\quad + g_{ij}\dot\gamma^i[\partial_m\nabla_k H^j(\gamma)\cdot H^m(\gamma)]\dot\gamma^k \\
&\qquad\qquad + g_{ij}(\gamma)\dot\gamma^i[\nabla_k H^j(\gamma)]\{\, (\nabla_m H^k - \Gamma_{m\ h}^{\ k})\dot\gamma^m \,\},
\end{aligned}
$$

where, after the differentiation of product, we substituted relations (10) and (13).

Using property (14), transforming partial derivative $\partial_m\nabla_k H^j$ to covariant $\nabla_m\nabla_k H^j$ and contracting the terms with connection Γ we have

$$
\begin{aligned}
&\frac{1}{2}\frac{\partial^2}{\partial s^2}|\dot\gamma(\ell,\varepsilon)|^2 = g_{ij}(\nabla_m H^i)\dot\gamma^m(\nabla_k H^j)\dot\gamma^k \\
&\qquad + g_{ij}\dot\gamma^i(\nabla_m\nabla_k H^j)H^m\dot\gamma^k + g_{ij}(\gamma)\dot\gamma^i(\nabla_k H^j)(\nabla_m H^k)\dot\gamma^m.
\end{aligned}
$$

Now we commute the covariant derivatives in the second term

$$
\nabla_m\nabla_k H^j = \nabla_k\nabla_m H^j + R_{h\ km}^{\ j} H^h
$$

to obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{\partial^2}{\partial s^2}|\dot\gamma(\ell,\varepsilon)|^2 = |\nabla H[\dot\gamma]\,|^2 + g_{ij}\dot\gamma^i(\nabla_k\nabla_m H^j + R_{h\ km}^{\ j}H^h)H^m\dot\gamma^k \\
&\qquad + g_{ij}\dot\gamma^i(\nabla_k H^j)(\nabla_m H^k)\dot\gamma^m = |\nabla H[\dot\gamma]\,|^2 - \langle\dot\gamma, R(H,\dot\gamma)H\rangle \\
&\qquad\qquad + g_{ij}\dot\gamma^i(\nabla_k\nabla_m H^j)H^m\dot\gamma^k + g_{ij}\dot\gamma^i(\nabla_k H^j)(\nabla_m H^k)\dot\gamma^m
\end{aligned}
$$

with curvature operator.

Redenoting indexes $m \leftrightarrow k$ in the third term we have

$$3^{rd} + 4^{th} \text{ terms} = g_{ij}\dot\gamma^i(\nabla_m\nabla_k H^j)H^k\dot\gamma^m \\ + g_{ij}\dot\gamma^i(\nabla_k H^j)(\nabla_m H^k)\dot\gamma^m = g_{ij}\dot\gamma^i(\nabla_m\{H^k\nabla_k H^j\})\dot\gamma^m$$

which leads to the final representation

$$\frac{1}{2}\frac{\partial^2}{\partial s^2}|\dot\gamma(\ell,\varepsilon)|^2 = |\nabla H[\dot\gamma]\,|^2 - \langle\dot\gamma, R(H,\dot\gamma)H\rangle + \langle\dot\gamma, \nabla(\nabla_H H)[\dot\gamma]\rangle.$$

Taking now $H = A_0$ for the first order estimate and $H = A_\alpha$ for the second order estimate we find

$$\Big\{A_0^I + A_0^{II} + \frac{1}{2}\sum_{\alpha=1}^d (A_\alpha^I + A_\alpha^{II})^2\Big\}\rho^2(x,y) \\ \le \int_0^1 \Big(2\langle\nabla\widetilde{A_0}[\dot\gamma],\dot\gamma\rangle + \sum_{\alpha=1}^d\{|\nabla A_\alpha[\dot\gamma]\,|^2 - \langle R(A_\alpha,\dot\gamma)A_\alpha,\dot\gamma\rangle\}\Big)\,d\ell. \quad (17)$$

Dissipativity condition (4), in view of (2) leads to the statement (7).

Step 5. To get estimate (8), one proceeds like above with a choice $y = o$ and

$$\frac{\partial}{\partial s}\gamma(\ell,s) = c(\ell,s)H(\gamma(\ell,s))$$

instead of (10), taking $c(\ell,s_0) = a(\ell)$ and $c(\ell,s_\alpha) = b(\ell)$ for the first and second order operators A_0 and $(A_\alpha)^2$ correspondingly. One has from (17)

$$\Big\{A_0^I + \frac{1}{2}\sum_{\alpha=1}^d (A_\alpha^I)^2\Big\}\rho^2(x,o) \le \int_0^1 \Big(2\langle\nabla(aA_0 + \frac{1}{2}\sum_{\alpha=1}^d \nabla_{bA_\alpha}[bA_\alpha]\,)[\dot\gamma],\dot\gamma\rangle \\ + \sum_{\alpha=1}^d\{|\nabla(bA_\alpha)[\dot\gamma]\,|^2 - \langle R(bA_\alpha,\dot\gamma)bA_\alpha,\dot\gamma\rangle\}\Big)\,d\ell.$$

Using that $\nabla c(\ell)[\dot\gamma] = \frac{\partial c(\ell)}{\partial\ell}$ we can further rewrite the last inequality

$$\Big\{A_0^I + \frac{1}{2}\sum_{\alpha=1}^d (A_\alpha^I)^2\Big\}\rho^2(x,o) \le \int_0^1 (2a\langle\nabla A_0[\dot\gamma],\dot\gamma\rangle + 2\frac{\partial a}{\partial\ell}\langle A_0,\dot\gamma\rangle \\ + \sum_{\alpha=1}^d\{b^2|\nabla A_\alpha[\dot\gamma]\,|^2 + \frac{\partial b^2}{\partial\ell}\langle A_\alpha,\nabla A_\alpha[\dot\gamma]\rangle + \Big(\frac{\partial b}{\partial\ell}\Big)^2|A_\alpha|^2 - b^2\langle R(A_\alpha,\dot\gamma)A_\alpha,\dot\gamma\rangle \\ + b^2\langle\nabla(\nabla_{A_\alpha}A_\alpha)[\dot\gamma],\dot\gamma\rangle + \frac{\partial b^2}{\partial\ell}\langle\nabla_{A_\alpha}A_\alpha,\dot\gamma\rangle\})\,d\ell. \quad (18)$$

To get the last line we also applied $\nabla_{A_\alpha} b(\ell) = \frac{\partial b(\ell)}{\partial s} = 0$, leading to calculation

$$\nabla(\nabla_{bA_\alpha}[bA_\alpha])[\dot\gamma] = \nabla_{\dot\gamma}(b^2\nabla_{A_\alpha}A_\alpha) = b^2\nabla(\nabla_{A_\alpha}A_\alpha)[\dot\gamma] + \frac{\partial b^2}{\partial l}\nabla_{A_\alpha}A_\alpha.$$

Taking further $a(\ell) = b^2(\ell),\ b(\ell) = 1-\ell$ and using estimate

$$|\langle\nabla A_\alpha[\dot\gamma], A_\alpha\rangle| \le \frac{(1-\ell)}{2}|\nabla A_\alpha[\dot\gamma]\,|^2 + \frac{1}{2(1-\ell)}|A_\alpha|^2$$

we find

$$\mathcal{L}^I\rho^2(x,o) \le \int_0^1 \Big\{(1-\ell)^2(2\langle\nabla\widetilde{A_0}[\dot\gamma],\dot\gamma\rangle$$

$$+2\sum_{\alpha=1}^d|\nabla A_\alpha[\dot\gamma]\,|^2 - \sum_{\alpha=1}^d\langle R(A_\alpha,\dot\gamma)A_\alpha,\dot\gamma\rangle) \tag{19}$$

$$+\,4(\ell-1)\langle\widetilde{A_0}(\gamma),\dot\gamma\rangle + 2\sum_{\alpha=1}^d|A_\alpha|^2\Big\}d\ell. \tag{20}$$

Using that

$$\nabla^{\gamma(\ell)}\rho^2(\gamma(\ell),o) = 2\rho(\gamma(\ell),o)\nabla^{\gamma(\ell)}\rho(\gamma(\ell),o)$$
$$= 2(\ell-1)\rho(x,o)\frac{\dot\gamma(\ell)}{\rho(x,o)} = 2(\ell-1)\dot\gamma,$$

the first term in (20) gives

$$2(\ell-1)\langle\widetilde{A_0}(\gamma),\dot\gamma\rangle = \langle\widetilde{A_0}(\gamma),\nabla^{\gamma(\ell)}\rho^2(\gamma(\ell),o)\rangle.$$

Finally, using the coercitivity and dissipativity assumptions (3)–(4) for lines (19) and (20) correspondingly, we conclude

$$\mathcal{L}^I\rho^2(x,o) \le \int_0^1 \left\{2K_C(1-\ell)^2|\dot\gamma|^2 + K_{C'}(1+\rho^2(\gamma(\ell),o))\right\}d\ell$$
$$\le K(1+\rho^2(x,o))$$

where we applied $(1-\ell)\le 1$ for $\ell\in[0,1]$ and that path $\gamma(\ell,0)=\gamma(\ell)$ realizes the geodesic distance.

Term $\frac{(A^I_\alpha \rho^2(x,o))^2}{\rho^2(x,o)}$ in (8) is treated like the first order term in (11) with choice of coefficient $c(\ell) = b(\ell) = 1 - \ell$. We get

$$A^I_\alpha \rho^2(x,o) \leq 2 \int_0^1 \langle \nabla(bA_\alpha)[\dot\gamma], \dot\gamma\rangle \, d\ell$$

$$= \int_0^1 \Big\{ \frac{\partial b}{\partial \ell} \langle A_\alpha(\gamma), \dot\gamma\rangle + b\langle \nabla A_\alpha[\dot\gamma], \dot\gamma\rangle \Big\} \, d\ell.$$

Therefore

$$(A^I_\alpha \rho^2(x,o))^2 \leq \int_0^1 \Big\{ \Big(\frac{\partial b}{\partial \ell}\Big)^2 \|A_\alpha\|^2 + b^2 \|\nabla A_\alpha[\dot\gamma]\,\|^2 \Big\} \, d\ell + 2 \int_0^1 |\dot\gamma|^2 \, d\ell.$$

The last integral gives $\rho^2(x,o)$ by (2). Moreover, we can add the first and second terms to line (18) to apply, like before, the coercitivity and dissipativity conditions and finish estimate (8). □

Remark 1. Let us note, that the upper bounds in (11) and (16) gave us more freedom in a choice of paths $\{\gamma(\ell,s)\}_{\ell\in[0,1]}$ for $s \neq 0$. Otherwise they all should be geodesics (compare with picture 1) and we would have to work with curvature, arising in Jacobi equation on geodesic deviations

$$\nabla_{\dot\gamma} \nabla_{\dot\gamma} \frac{\partial \gamma(\ell,s)}{\partial s} + R\Big(\dot\gamma, \frac{\partial \gamma}{\partial s}\Big)\dot\gamma = 0. \tag{21}$$

The work with Jacobi equation, as a second order Sturm-Liuville equations which depends on curvature, would require the knowledge of global geometry instead of pointwise conditions (3)–(4), i.e., for example, more precise information about the structure of harmonic tensors and Betti numbers, etc, e.g. [5, 6].

Remark 2. The applications of upper bounds (7)–(8) to the smooth properties of parabolic equations on manifolds are discussed in [1–4].

Here we show that under conditions (3)–(4) plus some additional assumption on the behaviour of coefficients on the infinity, one has C^∞–regularity of process y^x_t with respect to the initial data x and regularity properties of corresponding diffusion semigroups. In particular, dissipativity condition (4) actually represents the coercitivity condition for the high order variational systems for process y^x_t.

Finally, in [4] we apply the technique of upper bounds on operators, acting on metric function, to state the existence and uniqueness of solutions to the Stratonovich diffusions on noncompact manifolds with globally non-Lipschitz coefficients.

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