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The Beltrami equation and ring homeomorphisms

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Abstract. With the aid of results by Gehring, we introduce and study plane ring Q-homeomorphisms. This study is then applied in deriving general principles on the existence and uniqueness of homeomorphic ACL solutions to the Beltrami equation extending earlier results. In particular, we obtain new existence criteria which are expressed in terms of finite mean oscillation majorants for tangential dilatations. Moreover, we give a new proof of our generalization of the well-known Lehto existence theorem that has, in turn, a number of other consequences.

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1. Introduction

The existence problem for degenerate Beltrami equations is currently an active area of research. It has intensively been studied and many contributions have been made, see e.g. [7, 10, 21, 25, 32, 34, 36, 41, 43, 59] and [50–54]. Some of those and many other theorems can be derived from the mentioned generalization of the Lehto existence theorem, Theorem 5.5 below, see [54]. A detaled discussion of the above results can be found in the survey [58].

Let D be a domain in the complex plane $\mathbb{C}$, i.e., open and connected subset of $\mathbb{C}$, and let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. The *Beltrami equation* is the equation of the form

$$f_{\overline{z}} = \mu(z) \cdot f_z \tag{1.1}$$

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where $f_{\overline{z}} = \overline{\partial} f = (f_x + if_y)/2$, $f_z = \partial f = (f_x - if_y)/2$, $z = x + iy$, and f_x and f_y are partial derivatives of f in x and y, correspondingly. The function μ is called the *complex coefficient* and

$$K_\mu(z) = \frac{1 + |\mu(z)|}{1 - |\mu(z)|} \tag{1.2}$$

the *maximal dilatation* or in short the *dilatation* of the equation (1.1). The Beltrami equation (1.1) is said to be degenerate if $\operatorname{ess\,sup} K_\mu(z) = 1$. As known, the Beltrami equation plays an important role in the mapping theory. The main goal of this paper is to present general principles which allow to obtain variety of conditions for the existence of homeomorphic ACL solutions in the degenerate case. Our existence theorems are proved by an approximation method.

Given a point z_0 in D, the *tangential dilatation* and the *radial dilatation* of (1.1) with respect to z_0 are respectively defined by

$$K_\mu^T(z, z_0) = \frac{\left|1 - \frac{\overline{z-z_0}}{z-z_0}\mu(z)\right|^2}{1 - |\mu(z)|^2} \tag{1.3}$$

and

$$K_\mu^r(z, z_0) = \frac{1 - |\mu(z)|^2}{\left|1 + \frac{\overline{z-z_0}}{z-z_0}\mu(z)\right|^2}\ , \tag{1.4}$$

cf. [34, 55] and [21]. Reasons for the names will be given in Section 3.

Recall that a mapping $f : D \to \mathbb{C}$ is *absolutely continuous on lines*, abbr. $f \in ACL$, if, for every closed rectangle R in D whose sides are parallel to the coordinate axes, $f|R$ is absolutely continuous on almost all line segments in R which are parallel to the sides of R. In particular, f is ACL if it belongs to the Sobolev class $W^{1,1}_{loc}$, see e.g. [35, p. 8]. Note that, if $f \in$ ACL, then f has partial derivatives f_x and f_y a.e. and, thus, by the well-known Gehring–Lehto theorem every ACL homeomorphism $f : D \to \mathbb{C}$ is totally differentiable a.e., see [20] or [33, p. 128]. For a sense-preserving ACL homeomorphism $f : D \to \mathbb{C}$, the Jacobian $J_f(z) = |f_z|^2 - |f_{\overline{z}}|^2$ is nonnegative a.e., see [33, p. 10]. In this case, the *complex dilatation* μ_f of f is the ratio $\mu(z) = f_{\overline{z}}/f_z$, if $f_z \neq 0$ and $\mu(z) = 0$ otherwise, and the *dilatation* K_f of f is $K_\mu(z)$, see (1.2). Note that $|\mu(z)| \leq 1$ a.e. and $K_\mu(z) \geq 1$ a.e.

Basically, there are three definitions of quasiconformality: analytic, geometric and metric. They are equivalent with the same parameter of quasiconformality K. By the analytic definition, a homeomorphism $f : D \to \mathbb{C}$, $D \subset \mathbb{C}$, is K-quasiconformal, abbr. K-qc, if f is ACL and

$$\operatorname{ess\,sup} K_\mu(z) = K < \infty \tag{1.5}$$

where μ is the complex dilatation of f. According to the geometric definition, f is K-quasiconformal if

$$\sup \frac{M(f\Gamma)}{M(\Gamma)} = K < \infty \tag{1.6}$$

where the supremum is taken over all path families Γ in D with modulus $M(\Gamma) \neq 0$. It was noted by Ahlfors and Gehring that the supremum in (1.6) can be taken over special families yielding the same bound K. In particular, by [16] one may restrict to families of paths connecting the boundary components of rings in D.

Given a measurable function $K : D \to [1, \infty]$, we say that a sense-preserving ACL homeomorphism $f : D \to \overline{\mathbb{C}}$ is *K(z)-quasiconformal*, abbr. $K(z)$-qc, if

$$K_f(z) \leq K(z) \quad \text{a.e.} \tag{1.7}$$

Given a measurable function $Q : D \to [1, \infty]$, we say that a homeomorphism $f : D \to \overline{\mathbb{C}}$ is a *Q-homeomorphism* if

$$M(f\Gamma) \leq \int\limits_D Q(z) \cdot \rho^2(z)\, dx\, dy \tag{1.8}$$

holds for every path family Γ in D and each $\rho \in adm\, \Gamma$. This term was introduced in [37], see also [38, 39] and [26, 27], and the inequality was first used in [50] and [51] as a basic tool in studying BMO-qc mappings.

Recall that, given a family of paths Γ in $\overline{\mathbb{C}}$, a Borel function $\rho : \overline{\mathbb{C}} \to [0, \infty]$ is called *admissible* for Γ, abbr. $\rho \in adm\, \Gamma$, if

$$\int\limits_\gamma \rho(z)\, |dz| \ \geq \ 1 \tag{1.9}$$

for each $\gamma \in \Gamma$. The *modulus* of Γ is defined by

$$M(\Gamma) = \inf_{\rho \in adm\, \Gamma} \int\limits_{\mathbb{C}} \rho^2(z)\, dx\, dy. \tag{1.10}$$

We say that a property P holds for *almost every (a.e.)* path γ in a family Γ if the subfamily of all paths in Γ for which P fails has modulus zero. In particular, almost all paths in $\mathbb{C}$ are rectifiable.

The inequality

$$M(f\Gamma) \leq \int\limits_D K_\mu(z) \cdot \rho^2(z)\, dx\, dy \tag{1.11}$$

for $\mu = \mu_f$ was obtained in [33, p. 221] for quasiconformal mappings. Note that K_μ cannot be replaced by a smaller function in (1.11) unless one restricts either to special families Γ or to special $\rho \in \operatorname{adm}\Gamma$. In Section 3 (1.11) is improved for special Γ and ρ and then used for deriving new criteria for the existence of homeomorphic solutions of the Beltrami equation (1.1).

Given a domain D and two sets E and F in $\overline{\mathbb{C}}$, $\Gamma(E,F,D)$ denotes the family of all paths $\gamma : [a,b] \to \overline{\mathbb{C}}$ which join E and F in D, i.e., $\gamma(a) \in E$, $\gamma(b) \in F$ and $\gamma(t) \in D$ for $a < t < b$. We set $\Gamma(E,F) = \Gamma(E,F,\overline{\mathbb{C}})$ if $D = \overline{\mathbb{C}}$. A *ring domain*, or shortly a *ring* in $\overline{\mathbb{C}}$ is a doubly connected domain R in $\overline{\mathbb{C}}$. Let R be a ring in $\overline{\mathbb{C}}$. If C_1 and C_2 are the connected components of $\overline{\mathbb{C}} \setminus R$, we write $R = R(C_1, C_2)$. The *capacity* of R can be defined by

$$\operatorname{cap} R(C_1,C_2) = M(\Gamma(C_1,C_2,R)), \tag{1.12}$$

see e.g. [24]. Note that

$$M(\Gamma(C_1,C_2,R)) \;=\; M(\Gamma(C_1,C_2))\ , \tag{1.13}$$

see e.g. Theorem 11.3 in [60].

Motivated by the ring definition of quasiconformality in [16], we introduce the following notion that localizes and extends the notion of a Q-homeomorphism. Let D be a domain in $\mathbb{C}$, $z_0 \in D$, $r_0 \le \operatorname{dist}(z_0, \partial D)$ and $Q : D(z_0,r_0) \to [0,\infty]$ a measurable function in the disk

$$D(z_0,r_0) = \{z \in \mathbb{C} : |z - z_0| < r_0\}. \tag{1.14}$$

Set

$$A(r_1,r_2,z_0) = \{z \in \mathbb{C} : r_1 < |z-z_0| < r_2\}, \tag{1.15}$$

$$C(z_0,r_i) = \{z \in \mathbb{C} : |z-z_0| = r_i\}, \quad i = 1,2. \tag{1.16}$$

We say that a homeomorphism $f : D \to \overline{\mathbb{C}}$ is a *ring Q-homeomorphism* at the point z_0 if

$$M \to \Gamma(fC_1, fC_2) \;\le \int\limits_A Q(z)\cdot \eta^2(|z-z_0|)\, dx\, dy \tag{1.17}$$

for every annulus $A = A(r_1,r_2,z_0)$, $0 < r_1 < r_2 < r_0$, and for every measurable function $\eta : (r_1,r_2) \to [0,\infty]$ such that

$$\int\limits_{r_1}^{r_2} \eta(r)\, dr = 1. \tag{1.18}$$

Note that every Q-homeomorphism $f : D \to \overline{\mathbb{C}}$ is a ring Q-homeomorphism at each point $z_0 \in D$. Later on, we give other conditions on f which force it to be a ring Q-homeomorphism.

An ACL homeomorphism $f : D \to \mathbb{C}$ is called a *ring solution* of the Beltrami equation (1.1) with a complex coefficient μ if f satisfies (1.1) a.e., $f^{-1} \in W^{1,2}_{loc}$ and f is a ring Q-homeomorphism at every point $z_0 \in D$ with $Q_{z_0}(z) = K^T_\mu(z, z_0)$, see (1.3). We show that ring solutions exist for wide classes of the degenerate Beltrami equations.

The condition $f^{-1} \in W^{1,2}_{loc}$ given in the definition of a ring solution implies that a.e. point z is a *regular point* for the mapping f, i.e., f is differentiable at z and $J_f(z) \neq 0$. Note that the condition $K_\mu \in L^1_{loc}$ is necessary for a homeomorphic ACL solution f of (1.1) to have the property $g = f^{-1} \in W^{1,2}_{loc}$ because this property implies that

$$\int_C K_\mu(z)\, dx\, dy \leq 4 \int_C \frac{dxdy}{1 - |\mu(z)|^2} = 4 \int_{f(C)} |\partial g|^2\, du\, dv < \infty$$

for every compact set $C \subset D$.

Note that every homeomorphic ACL solution f of the Beltrami equation with $K_\mu \in L^1_{loc}$ belongs to the class $W^{1,1}_{loc}$ as in all our theorems. Note also that if, in addition, $K_\mu \in L^p_{loc}$, $p \in [1, \infty]$, then $f \in W^{1,s}_{loc}$ where $s = 2p/(1 + p) \in [1, 2]$. Indeed,

$$|\partial f| + |\overline{\partial} f| \;=\; K^{1/2}_\mu(z) \cdot J^{1/2}_f(z),$$

and by Hölder's inequality, on every compact set $C \subset D$,

$$\begin{aligned} \|\overline{\partial} f\|_s \leq \|\partial f|_s \leq \|K^{1/2}_\mu\|_p \cdot \|J^{1/2}_f\|_2 & \\ = \|K_\mu\|^{1/2}_q \cdot \|J_f\|^{1/2}_1 \leq \|K_\mu\|^{1/2}_q \cdot A(f(C))^{1/2}, \end{aligned}$$

see e.g. [33, p. 131] where $A(f(C))$ is the area of the set $f(C)$ and $\frac{1}{p} + \frac{1}{2} = \frac{1}{s}$ and $q = p/2$. Hence $f \in W^{1,s}_{loc}$, see e.g. [35, p. 8].

In the classical case when $\|\mu\|_\infty < 1$, equivalently, when $K_\mu \in L^\infty$, every ACL homeomorphic solution f of the Beltrami equation (1.1) is in the class $W^{1,2}_{loc}$ together with its inverse mapping f^{-1}, and hence f is a ring solution of (1.1) by Theorem 3.1 below. In the case $\|\mu\|_\infty = 1$ with $K_\mu \leq Q \in \mathrm{BMO}$, again $f^{-1} \in W^{1,2}_{loc}$ and f belongs to $W^{1,s}_{loc}$ for all $1 \leq s < 2$ but not necessarily to $W^{1,2}_{loc}$, see [50] and [51]. However, there is a varity of degenerate Beltrami equations for which ring solutions exist as shown below. The inequality (1.17), which ring solutions satisfy, is an important tool in deriving various properties of the solutions.

Recall that a real valued function $\varphi \in \mathrm{L}^1_{loc}(D)$ is said to be of *bounded mean oscillation* in D, abbr. $\varphi \in$ BMO(D) or simply $\varphi \in BMO$, if

$$\|\varphi\|_* = \sup_{B \subset D} \fint_B |\varphi(z) - \varphi_B| \, dx \, dy < \infty \tag{1.19}$$

where the supremum is taken over all disks B in D and

$$\varphi_B = \fint_B \varphi(z) \, dx \, dy = \frac{1}{|B|} \int_B \varphi(z) \, dx \, dy \tag{1.20}$$

is the mean value of the function φ over B. It is well-known that L^∞(D) $\subset$ BMO(D) $\subset \mathrm{L}^p_{loc}$(D) for all $1 \le p < \infty$, see e.g. [48]. A function φ in BMO is said to have *vanishing mean oscillation*, abbr. $\varphi \in VMO$, if the supremum in (1.19) taken over all disks B in D with $|B| < \varepsilon$ converges to 0 as $\varepsilon \to 0$.

The BMO space was introduced by John and Nirenberg, see [31], and soon became one of the main concepts in harmonic analysis, complex analysis and partial differential equations. BMO functions are related in many ways to quasiconformal and quasiregular mappings, see e.g. [2, 3, 14, 30, 40] and [47], as well as to the modern classes of mappings with finite distortion, see e.g. [4] and [25]. VMO has been introduced by Sarason, see [57]. It is known a large number of papers devoted to the existence, uniqueness and properties of solutions for various kind of differential equations and, in particular, of elliptic type with coefficients of the class VMO, see e.g. [8, 28, 42, 46]. In this connection, it should be noted that by the recent result of Brezis and Nirenberg in [6] the Sobolev class $W^{1,2}_{loc}$ is a subclass of VMO, see also [9].

Conditions for the existence and uniqueness of ACL homeomorphic solutions for the Beltrami equation can be given in terms of the maximal dilatation $K_\mu(z)$. In particular, it was proved that, if $K_\mu(z)$ has a BMO majorant, then the Beltrami equation (1.1) has a homeomorphic ACL solution, see e.g. [51]. Various conditions for the existence of solutions for the Beltrami equation have been formulated in terms of integral and measure constraints on K_μ, see e.g. [7, 10, 25, 32, 36, 41, 43, 59]. These conditions assume either exponential integrability or at least high local integrability of the dilatation. As in [21,34] and [54], the existence criteria which are established in the present paper are expressed in terms of the tangential dilatations $K^T_\mu(z, z_0)$ with the assumption that $K_\mu \in L^1_{loc}$.

In [52, 53], we proved that if $K_\mu(z)$ has a majorant $Q(z)$ in D which belongs to the class FMO (functions of finite mean oscillation) described in Section 2, then (1.1) has a homeomorphic ACL solution. Here we prove, in particular, a stronger result on the existence of a ring solution of (1.1) where the assumption on a FMO majorant for K_μ in D is replaced by the condition that every point $z_0 \in D$ has a neighborhood U_{z_0} and a function $Q_{z_0} : U_{z_0} \to [0, \infty]$ which is of finite mean oscillation at z_0 such that $K_\mu^T(z, z_0) \le Q_{z_0}(z)$ for all $z \in U_{z_0}$, see Theorem 5.1 in Section 5 below. This as well as other new existence theorems here are based on a general existence principle, Lemma 5.1. Some of these existence theorems are expressed in terms of mean and logarithmic mean of the tangential dilatation $K_\mu^T(z, z_0)$ over infinitesimal disks and annuli centered at z_0, see e.g. Theorems 5.2 and 5.4. We also use Lemma 5.1 in a new proof of an extension of Lehto's theorem that we established [54]. Finally, we prove the corresponding uniqueness theorems for the Beltrami equation in Section 6.

2. Finite mean oscillation

Let D be a domain in the complex plane $\mathbb{C}$. We say that a function $\varphi : D \to \mathbb{R}$ has *finite mean oscillation* at a point $z_0 \in D$ if

$$d_\varphi(z_0) = \overline{\lim_{\varepsilon \to 0}} \ \ \fint_{D(z_0,\varepsilon)} |\varphi(z) - \overline{\varphi}_\varepsilon(z_0)| \, dx \, dy < \infty \tag{2.1}$$

where

$$\overline{\varphi}_\varepsilon(z_0) = \fint_{D(z_0,\varepsilon)} \varphi(z) \, dx \, dy < \infty \tag{2.2}$$

is the mean value of the function $\varphi(z)$ over the disk $D(z_0, \varepsilon)$ with small $\varepsilon > 0$. Thus, the notion includes the assumption that φ is integrable in some neighborhood of the point z_0. We call $d_\varphi(z_0)$ the *dispersion* of the function φ at the point z_0. We say that a function $\varphi : D \to \mathbb{R}$ is of *finite mean oscillation* in D, abbr. $\varphi \in$ FMO(D) or simply $\varphi \in FMO$, if φ has a finite dispersion at every point $z \in D$.

Remark 2.1. Note that, if a function $\varphi : D \to \mathbb{R}$ is integrable over $D(z_0, \varepsilon_0) \subset D$, then

$$\fint_{D(z_0,\varepsilon)} |\varphi(z) - \overline{\varphi}_\varepsilon(z_0)| \, dx \, dy \le 2 \cdot \overline{\varphi}_\varepsilon(z_0) \tag{2.3}$$

and the right side in (2.3) is continuous in the parameter $\varepsilon \in (0, \varepsilon_0]$ by the absolute continuity of the indefinite integral. Thus, for every $\delta_0 \in (0, \varepsilon_0)$,

$$\sup_{\varepsilon \in [\delta_0, \varepsilon_0]} \ \fint_{D(z_0,\varepsilon)} |\varphi(z) - \overline{\varphi}_\varepsilon(z_0)| \, dx \, dy < \infty. \tag{2.4}$$

If (2.1) holds, then

$$\sup_{\varepsilon\in(0,\varepsilon_0]} \quad \fint_{D(z_0,\varepsilon)} |\varphi(z)-\overline{\varphi}_\varepsilon(z_0)|\,dx\,dy<\infty. \tag{2.5}$$

The number in (2.5) is called the *maximal dispersion* of the function φ in the disk $D(z_0,\varepsilon_0)$.

Proposition 2.1. *If, for some collection of numbers $\varphi_\varepsilon\in\mathbb{R}$, $\varepsilon\in(0,\varepsilon_0]$,*

$$\overline{\lim_{\varepsilon\to 0}} \quad \fint_{D(z_0,\varepsilon)} |\varphi(z)-\varphi_\varepsilon|\,dx\,dy<\infty, \tag{2.6}$$

then φ is of finite mean oscillation at z_0.

Proof. Indeed, by the triangle inequality,

$$\begin{aligned}\fint_{D(z_0,\varepsilon)} |\varphi(z)-\overline{\varphi}_\varepsilon(z_0)|\,dx\,dy \\ &\le \fint_{D(z_0,\varepsilon)} |\varphi(z)-\varphi_\varepsilon|\,dx\,dy+|\varphi_\varepsilon-\overline{\varphi}_\varepsilon(z_0)| \\ &\le 2\fint_{D(z_0,\varepsilon)} |\varphi(z)-\varphi_\varepsilon|\,dx\,dy.\end{aligned}$$

□

Corollary 2.1. *If, for a point $z_0\in D$,*

$$\overline{\lim_{\varepsilon\to 0}} \quad \fint_{D(z_0,\varepsilon)} |\varphi(z)|\,dx\,dy<\infty, \tag{2.7}$$

then φ has finite mean oscillation at z_0.

Remark 2.2. Clearly BMO $\subset$ FMO. The example given in the end of this section shows that the inclusion is proper. Note that the function $\varphi(z)=\log\frac{1}{|z|}$ belongs to BMO in the unit disk Δ, see e.g. [48, p. 5], and hence also to FMO. However, $\overline{\varphi}_\varepsilon(0)\to\infty$ as $\varepsilon\to 0$, showing that the condition (2.7) is only sufficient but not necessary for a function φ to be of finite mean oscillation at z_0.

A point $z_0 \in D$ is called a *Lebesgue point* of a function $\varphi : D \to \mathbb{R}$ if φ is integrable in a neighborhood of z_0 and

$$\lim_{\varepsilon \to 0} \quad \fint_{D(z_0,\varepsilon)} |\varphi(z) - \varphi(z_0)| \, dx \, dy = 0. \tag{2.8}$$

It is known that, for every function $\varphi \in L^1(D)$, almost every point in D is a Lebesgue point.

Corollary 2.2. *Every function $\varphi : D \to \mathbb{R}$, which is locally integrable, has a finite mean oscillation at almost every point in D.*

Below we use the notations $D(r) = D(0, r) = \{z \in \mathbb{C} : |z| < r\}$ and

$$A(\varepsilon, \varepsilon_0) = \{z \in \mathbb{C} : \varepsilon < |z| < \varepsilon_0\}. \tag{2.9}$$

Lemma 2.1. *Let $D \subset \mathbb{C}$ be a domain such that $D(1/2) \subset D$, and let $\varphi : D \to \mathbb{R}$ be a nonnegative function. If φ is integrable in $D(1/2)$ and of FMO at* 0, *then*

$$\int_{A(\varepsilon,1/2)} \frac{\varphi(z) \, dx \, dy}{\left(|z| \log_2 \frac{1}{|z|}\right)^2} \leq C \cdot \log_2 \log_2 \frac{1}{\varepsilon} \tag{2.10}$$

for $\varepsilon \in (0, 1/4)$, where

$$C = 4\pi \ [\varphi_0 + 6d_0], \tag{2.11}$$

φ_0 is the mean value of φ over the disk $D(1/2)$ and and d_0 is the maximal dispersion of φ in $D(1/2)$.

Versions of this lemma have been first established for BMO functions and $n = 2$ in [50] and [51] and then for FMO functions in [26] and [53]. An n-dimensional version of the lemma for BMO functions was established in [38].

Proof. Let $0 < \varepsilon < 2^{-2}$, $\varepsilon_k = 2^{-k}$, $A_k = \{z \in D : \varepsilon_{k+1} \leq |z| < \varepsilon_k\}$, $D_k = D(\varepsilon_k)$ and let φ_k be the mean value of $\varphi(z)$ over D_k, $k = 1, 2 \ldots$. Take a natural number N such that $\varepsilon \in [\varepsilon_{N+1}, \varepsilon_N)$ and denote $\alpha(t) = (t \log 1/t)^{-2}$. Then $A(\varepsilon, 2^{-1}) \subset A(\varepsilon) = \bigcup_{k=1}^{N} A_k$ and

$$\eta(\varepsilon) = \int_{A(\varepsilon)} \varphi(z) \, \alpha(|z|) \, dx \, dy \leq |S_1| + S_2$$

where

$$S_1(\varepsilon) \;=\; \sum_{k=1}^{N} \int\limits_{A_k} (\varphi(z)-\varphi_k)\,\alpha(|z|)\,dx\,dy,$$

$$S_2(\varepsilon) = \sum_{k=1}^{N} \;\varphi_k \int\limits_{A_k} \alpha(|z|)\,dx\,dy.$$

Since $A_k \subset D_k$, $|z|^{-2} \le 4\pi/|D_k|$ for $z \in A_k$ and $\log\frac{1}{|z|} > k$ in A_k, then

$$|S_1| \le 4\pi d_0 \sum_{k=1}^{N} \frac{1}{k^2} < 8\pi d_0$$

because

$$\sum_{k=2}^{\infty} \frac{1}{k^2} < \int\limits_{1}^{\infty} \frac{dt}{t^2} = 1.$$

Now,

$$\int\limits_{A_k} \alpha(|z|)\,dx\,dy \le \frac{1}{k^2} \int\limits_{A_k} \frac{dxdy}{|z|^2} = \frac{2\pi}{k^2}.$$

Moreover,

$$|\varphi_k - \varphi_{k-1}| = \frac{1}{|D_k|} \left| \int\limits_{D_k} (\varphi(z)-\varphi_{k-1})\,dx\,dy \right|$$

$$\le \frac{4}{|D_{k-1}|} \int\limits_{D_{k-1}} |\varphi(z)-\varphi_{k-1}|\,dx\,dy \le 4d_0$$

and by the triangle inequality, for $k \ge 2$,

$$\varphi_k = |\varphi_k| \le \varphi_1 + \sum_{l=2}^{k} |\varphi_l - \varphi_{l-1}| \le \varphi_1 + 4kd_0 = \varphi_0 + 4kd_0.$$

Hence

$$S_2 = |S_2| \le 2\pi \sum_{k=1}^{N} \frac{\varphi_k}{k^2} \;\le\; 4\pi\varphi_0 + 8\pi d_0 \sum_{k=1}^{N} \frac{1}{k}.$$

But

$$\sum_{k=2}^{N} \frac{1}{k} < \int\limits_{1}^{N} \frac{dt}{t} = \log N < \log_2 N$$

and, for $\varepsilon < \varepsilon_N$,

$$N = \log_2 \frac{1}{\varepsilon_N} < \log_2 \frac{1}{\varepsilon}.$$

Consequently,

$$\sum_{k=1}^{N} \frac{1}{k} < 1 + \log_2 \log_2 \frac{1}{\varepsilon},$$

and, thus, for $\varepsilon \in (0, 2^{-2})$,

$$\eta(\varepsilon) \leq 4\pi\Big(2d_0 + \frac{4d_0 + \varphi_0}{\log_2 \log_2 \frac{1}{\varepsilon}}\Big) \cdot \log_2 \log_2 \frac{1}{\varepsilon} \leq C \cdot \log_2 \log_2 \frac{1}{\varepsilon}.$$

$\square$

We complete this section by constructing a function $\varphi : \mathbb{C} \to \mathbb{R}$ which belongs to FMO but not to L^p_{loc} for any $p > 1$, and in particular it does not belong to BMO_{loc}.

Example. Fix $p > 1$. For $k = 1, 2, \ldots$, set $z_k = 2^{-k}$, $r_k = 2^{-pk^2}$ and $D_k = D(z_k, r_k)$. Define $\varphi(z) = \sum_{k=2}^{\infty} \varphi_k(z)$ where $\varphi_k(z) = 2^{2k^2}$ if $z \in D_k$ and 0 otherwise. Then φ is locally bounded in $\mathbb{C} \setminus \{0\}$ and hence belongs to $\mathrm{BMO}_{loc}(\mathbb{C} \setminus \{0\})$ and therefore to $\mathrm{FMO}(\mathbb{C} \setminus \{0\})$. To show that φ is of FMO at $z = 0$, calculate

$$\int\limits_{D_k} \varphi_k(z)\, dx\, dy \ = \ \pi 2^{-2(p-1)k^2}. \tag{2.12}$$

Consequently,

$$\overline{\lim_{\varepsilon \to 0}} \ \ \fint\limits_{D(\varepsilon)} \varphi(z)\, dx\, dy < \infty. \tag{2.13}$$

Indeed, setting

$$K = K(\varepsilon) = \Big[\log_2 \frac{1}{\varepsilon}\Big] \leq \log_2 \frac{1}{\varepsilon} \tag{2.14}$$

where $[A]$ is the integral part of a number A, we have that

$$J = \fint\limits_{D(\varepsilon)} \varphi(z)\, dx\, dy \leq \sum_{k=K}^{\infty} 2^{-2(p-1)k^2} / 2^{-2(K+1)}. \tag{2.15}$$

If $(p-1)K > 1$, i.e. $K > 1/(p-1)$, then

$$\sum_{k=K}^{\infty} 2^{-2(p-1)k^2} \leq \sum_{k=K}^{\infty} 2^{-2k} = 2^{-2K} \sum_{k=0}^{\infty} \Big(\frac{1}{4}\Big)^k = \frac{4}{3} \cdot 2^{-2K}, \tag{2.16}$$

i.e., $J \le 16/3$. Thus, by Corollary 2.1 $\varphi \in \mathrm{FMO}$.

Finally, note that

$$\int\limits_{D_k} \varphi_k^p(z)\, dx\, dy = \pi \tag{2.17}$$

and hence $\varphi \notin L^p(U)$ for any neighborhood U of 0.

3. Ring Q-homeomorphisms in the plane

The notation of ring Q-homeomorphisms which was introduced in (1.17) appears in Theorems 3.1 and 3.2 below.

Let z be a regular point for a mapping $f : D \to \mathbb{C}$. Given $\omega \in \mathbb{C}$, $|\omega| = 1$, the *derivative in the direction* ω of the mapping f at the point z is

$$\partial_\omega f(z) \;=\; \lim_{t \to +0} \frac{f(z + t\cdot\omega) - f(z)}{t}\,. \tag{3.1}$$

The *radial direction* at a point $z \in D$ with respect to the center $z_0 \in \mathbb{C}$, $z_0 \ne z$, is

$$\omega_0 \;=\; \omega_0(z, z_0) \;=\; \frac{z - z_0}{|z - z_0|}\,. \tag{3.2}$$

The *radial dilatation* of f at z with respect to z_0 is defined by

$$K^r(z, z_0, f) \;=\; \frac{|J_f(z)|}{|\partial_r^{z_0} f(z)|^2} \tag{3.3}$$

and the *tangential dilatation* by

$$K^T(z, z_0, f) \;=\; \frac{|\partial_T^{z_0} f(z)|^2}{|J_f(z)|} \tag{3.4}$$

where $\partial_r^{z_0} f(z)$ is the derivative of f at z in the direction ω_0 and $\partial_T^{z_0} f(z)$ in $\tau = i\omega_0$, correspondingly.

Note that if z is a regular point of f and $|\mu(z)| < 1$, $\mu(z) = f_{\overline{z}}/f_z$, then

$$K^r(z, z_0, f) = K_\mu^r(z, z_0) \tag{3.5}$$

and

$$K^T(z, z_0, f) = K_\mu^T(z, z_0) \tag{3.6}$$

where $K_\mu^r(z, z_0)$ and $K_\mu^T(z, z_0)$ are defined by (1.4) and (1.3), respectively. Indeed, the equalities (3.5) and (3.6) follow directly from the calculations

$$\partial_r f = \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial r} + \frac{\partial f}{\partial \overline{z}}\cdot\frac{\partial \overline{z}}{\partial r} = \frac{z - z_0}{|z - z_0|}\cdot f_z + \frac{\overline{z - z_0}}{|z - z_0|}\cdot f_{\overline{z}} \tag{3.7}$$

where $r = |z - z_0|$ and

$$\partial_T f = \frac{1}{r}\Big(\frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial \vartheta} + \frac{\partial f}{\partial \overline{z}}\cdot\frac{\partial \overline{z}}{\partial \vartheta}\Big) = i\cdot\Big(\frac{z-z_0}{|z-z_0|}\cdot f_z - \frac{\overline{z-z_0}}{|z-z_0|}\cdot f_{\overline{z}}\Big) \quad (3.8)$$

where $\vartheta = \arg(z - z_0)$ because $J_f(z) = |f_z|^2 - |f_{\overline{z}}|^2$.

The *big radial dilatation* of f at z with respect to z_0 is defined by

$$K^R(z, z_0, f) = \frac{|J_f(z)|}{|\partial_R^{z_0} f(z)|^2} \quad (3.9)$$

where

$$|\partial_R^{z_0} f(z)| = \min_{\substack{\omega\in\mathbb{C},\\ |\omega|=1}} \frac{|\partial_\omega f(z)|}{|\mathrm{Re}\,\omega\overline{\omega_0}|}. \quad (3.10)$$

Here $\mathrm{Re}\,\omega\overline{\omega_0}$ is the scalar product of vectors ω and ω_0. In the other words, $\mathrm{Re}\,\omega\overline{\omega_0}$ is the projection of the vector ω onto the radial direction ω_0. Obviously, there is a unit vector ω_* such that

$$|\partial_R^{z_0} f(z)| = \frac{|\partial_{\omega_*} f(z)|}{|\mathrm{Re}\,\omega_*\overline{\omega_0}|}\ . \quad (3.11)$$

It is clear that

$$|\partial_r^{z_0} f(z)| \geq |\partial_R^{z_0} f(z)| \geq \min_{\substack{\omega\in\mathbb{C},\\ |\omega|=1}} |\partial_\omega f(z)| \quad (3.12)$$

and hence

$$K^r(z, z_0, f) \leq K^R(z, z_0, f) \leq K_\mu(z) \quad (3.13)$$

and the equalities hold in (3.13) if and only if the minimum in the right hand side of (3.12) is realized for the radial direction $\omega = \omega_0$.

Note that $\partial_r^{z_0} f(z) \neq 0$, $|\partial_R^{z_0} f(z)| \neq 0$ and $\partial_T^{z_0} f(z) \neq 0$ at every regular point $z \neq z_0$ of f, see e.g. 1.2.1 in [49]. In view of (3.4), (3.6) and (1.3), the following lemma shows that the big radial dilatation coincides with the tangential dilatation at every regular point.

Lemma 3.1. *Let $z \in D$ be a regular point of a mapping $f : D \to \mathbb{C}$ with the complex dilatation $\mu(z) = f_{\overline{z}}/f_z$ such that $|\mu(z)| < 1$. Then*

$$K^R(z, z_0, f) = \frac{\left|1 - \frac{\overline{z-z_0}}{z-z_0}\mu(z)\right|^2}{1 - |\mu(z)|^2}. \quad (3.14)$$

Proof. The derivative of f at the regular point z in the arbitrary direction $\omega = e^{i\alpha}$ is the quantity $\partial_\omega f(z) = f_z + f_{\overline{z}}\cdot e^{-2i\alpha}$, see e.g. [33, p. 17 and 182]. Consequently,

$$
\begin{aligned}
X := \frac{|\partial_R^{z_0} f(z)|^2}{|f_z|^2} = \min_{\alpha\in[0,2\pi]} \frac{|\mu(z)+e^{2i\alpha}|^2}{\cos^2(\alpha-\vartheta)} &= \min_{\beta\in[0,2\pi]} \frac{|\nu-e^{2i\beta}|^2}{\sin^2\beta} \\
&= \min_{\beta\in[0,2\pi]} \frac{1+|\nu|^2-2|\nu|\cos(\kappa-2\beta)}{\sin^2\beta} \\
= \min_{t\in[-1,1]} \frac{1+|\nu|^2-2|\nu|\cdot[(1-2t^2)\cos\,\kappa \pm 2t(1-t^2)^{1/2}\sin\,\kappa]}{t^2}
\end{aligned}
$$

where $t=\sin\,\beta$, $\beta=\alpha+\frac{\pi}{2}-\vartheta$, $\nu=\mu(z)e^{-2i\vartheta}$ and $\kappa=\arg\nu=\arg\mu-2\vartheta$. Hence $X=\min_{\tau\in[1,\infty]}\,\varphi_\pm(\tau)$ where $\tau=1/\sin^2\beta$,

$$
\begin{gathered}
\varphi_\pm(\tau)=b+a\tau\pm c(\tau-1)^{1/2}, \\
a=1+|\nu|^2-2|\nu|\cos\kappa,\quad b=4|\nu|\cos\kappa,\quad c=4|\nu|\sin\kappa.
\end{gathered}
$$

Since $\varphi'_\pm(\tau)=a\pm(\tau-1)^{-1/2}c/2$ the minimum is realized for $\tau=1+c^2/4a^2$ under $(\tau-1)^{1/2}=\mp c/2a$, correspondingly, where the signs are agreed. Thus,

$$
X=b+\left(a+\frac{1}{4}\frac{c^2}{a}\right)-\frac{1}{2}\frac{c^2}{a}=\frac{(1-|\nu|^2)^2}{1+|\nu|^2-2|\nu|\cos\kappa}
$$

that implies (3.14). □

Next, we recall some general properties of homeomorphisms in the Sobolev class $W^{1,2}_{loc}$.

Proposition 3.1. *Let $f:D\to\mathbb{C}$ be a homeomorphism of the class $W^{1,2}_{loc}$. Then f is differentiable a.e. and satisfies Lusin's property (N). If, in addition, f^{-1} belongs to the class $W^{1,2}_{loc}$, then*

$$
J_f(z)\neq 0 \qquad a.e. \tag{3.15}
$$

The statement follows from the well-known results for $W^{1,2}_{loc}$ homeomorphisms, see e.g. [33, p. 121, 128–130 and 150], and the equivalence of the (N^{-1})-property and the property (3.15) for mappings f which are differentiable a.e., see Theorem 1 in [44]. Recall that a mapping $f:X\to Y$ between measurable spaces (X,Σ,μ) and (X',Σ',μ') is said to have the *Lusin (N)-property* if $\mu'(f(S))=0$ whenever $\mu(S)=0$. Similarly, f has the *(N^{-1})-property* if $\mu(S)=0$ whenever $\mu'(f(S))=0$.

Some prototypes of the following theorem can be found in [21, 34, 55]. In these theorems, both $|\mu|$ and $\arg\mu$ are incorporated in modulus estimations.

Theorem 3.1. *Let $f : D \to \mathbb{C}$ be a sense-preserving homeomorphism of the class $W^{1,2}_{loc}$ such that $f^{-1} \in W^{1,2}_{loc}$. Then at every point $z_0 \in D$ the mapping f is a ring Q-homeomorphism with $Q(z) = K^T_\mu(z, z_0)$ where $\mu(z) = \mu_f(z)$.*

Proof. Fix $z_0 \in D$, and r_1 and r_2 such that $0 < r_1 < r_2 < r_0 \le \operatorname{dist}(z_0, \partial D)$ and let $C_1 = \{z \in \mathbb{C} : |z - z_0| = r_1\}$ and $C_2 = \{z \in \mathbb{C} : |z - z_0| = r_2\}$. Set $\Gamma = \Gamma(C_1, C_2, D)$ and denote by Γ_* the family of all rectifiable paths $\gamma_* \in f\Gamma$ for which f^{-1} is absolutely continuous on every closed subpath of γ_*. Then $M(f\Gamma) = M(\Gamma_*)$ by the Fuglede theorem, see [13] and [60], because $f^{-1} \in ACL^2$, see e.g. [35, p. 8].

Fix $\gamma_* \in \Gamma_*$. Set $\gamma = f^{-1} \circ \gamma_*$ and denote by s and s_* natural (length) parameters of γ and γ_*, correspondingly. Note that the correspondence $s_*(s)$ between the natural parameters of γ_* and γ is a strictly monotone function and we may assume that $s_*(s)$ is increasing. For $\gamma_* \in \Gamma_*$, the inverse function $s(s_*)$ has the (N)-property and $s_*(s)$ is differentiable a.e. as a monotone function. Thus, $\frac{ds_*}{ds} \ne 0$ a.e. on γ by [44]. Let s be such that $z = \gamma(s)$ is a regular point for f and suppose that γ is differentiable at s with $\frac{ds_*}{ds} \ne 0$. Let $r = |z - z_0|$ and let ω be a unit tangential vector to the path γ at the point $z = \gamma(s)$. Then

$$\left|\frac{dr}{ds_*}\right| = \frac{\frac{dr}{ds}}{\frac{ds_*}{ds}} = \frac{|\operatorname{Re}\omega\overline{\omega_0}|}{|\partial_\omega f(z)|} \le \frac{1}{|\partial^{z_0}_R f(z)|} \tag{3.16}$$

where $|\partial^{z_0}_R f(z)|$ is defined by (3.10).

Now, let $\eta : (r_1, r_2) \to [0, \infty]$ be an arbitrary measurable function such that

$$\int\limits_{r_1}^{r_2} \eta(r)\, dr = 1. \tag{3.17}$$

By the Lusin theorem, there is a Borel function $\eta_* : (r_1, r_2) \to [0, \infty]$ such that $\eta_*(r) = \eta(r)$ a.e., see e.g. 2.3.5 in [13] and [56, p. 69]. Let

$$\rho(z) = \eta_*(|z - z_0|)$$

in the annulus $A = \{z \in \mathbb{C} : r_1 < |z - z_0| < r_2\}$ and $\rho(z) = 0$ outside of A. Set

$$\rho_*(w) = \{\rho/|\partial^{z_0}_R f|\} \circ f^{-1}(w)$$

if $z = f^{-1}(w)$ is a regular point of f, $\rho_*(w) = \infty$ at the rest points of $f(D)$ and $\rho_*(w) = 0$ outside $f(D)$. Then by (3.16) and (3.17), for $\gamma_* \in \Gamma_*$,

$$\int\limits_{\gamma_*} \rho_* ds_* \ge \int\limits_{\gamma_*} \eta(r)\left|\frac{dr}{ds_*}\right| ds_* \ge \int\limits_{\gamma_*} \eta(r)\frac{dr}{ds_*}\, ds_* = \int\limits_{r_1}^{r_2} \eta(r)\, dr = 1$$

because the function $z = \gamma(s(s_*))$ is absolutely continuous and hence so is $r = |z - z_0|$ as a function of the parameter s_*. Consequently, ρ_* is admissible for all $\gamma_* \in \Gamma_*$.

By Proposition 3.1 f and f^{-1} are regular a.e. and have the property (N). Thus, by change of variables, see e.g. Theorem 6.4 in [59], we have in view of Lemma 3.1 that

$$M(f\Gamma) \leq \int\limits_{f(A)} \rho_*(w)^2 \, du \, dv = \int\limits_{A} \rho(z)^2 \, K_\mu^T(z, z_0) \, dx \, dy$$
$$= \int\limits_{A} K_\mu^T(z, z_0) \cdot \eta^2(|z - z_0|) \, dx \, dy,$$

i.e., f is a ring Q-homeomorphism with $Q(z) = K_\mu^T(z, z_0)$.

If f is a plane $W^{1,2}_{loc}$ homeomorphism with a locally integrable $K_f(z)$, then $f^{-1} \in W^{1,2}_{loc}$, see e.g. [23]. Hence we obtain the following consequences of Theorem 3.1 which will be quoted below.

Corollary 3.1. *Let $f : D \to \mathbb{C}$ be a sense-preserving homeomorphism of the class $W^{1,2}_{loc}$ and suppose that $K_f(z)$ is integrable in a disk $D(z_0, r_0) \subset D$ for some $z_0 \in D$ and $r_0 > 0$. Then f is a ring Q-homeomorphism at the point $z_0 \in D$ with $Q(z) = K_\mu^T(z, z_0)$ where $\mu(z) = \mu_f(z)$.*

Corollary 3.2. *Let $f : D \to \mathbb{C}$ be a sense-preserving homeomorphism of the class $W^{1,2}_{loc}$ with $K_\mu \in L^1_{loc}$. Then f is a ring Q-homeomorphism at the every point $z_0 \in D$ with $Q(z) = K_\mu(z)$ where $\mu(z) = \mu_f(z)$.*

We close this section with a convergence theorem which plays an important role in our scheme for deriving the existence theorems of the Beltrami equation.

Theorem 3.2. *Let $f_n : D \to \overline{\mathbb{C}}$, $n = 1, 2, \ldots,$ be a sequence of ring Q-homeomorphisms at a point $z_0 \in D$. If f_n converge locally uniformly to a homeomorphism $f : D \to \overline{\mathbb{C}}$, then f is also a ring Q-homeomorphism at the point z_0.*

Indeed, it follows from the uniform convergence of the rings $f_n R(C_1, C_2)$ to the ring $fR(C_1, C_2)$, see [16–18].

4. Distortion estimates

Below we use the standard conventions $a/\infty = 0$ for $a \neq \infty$ and $a/0 = \infty$ if $a > 0$ and $0 \cdot \infty = 0$, see e.g. [56, p. 6].

For points $z, \zeta \in \overline{\mathbb{C}}$, the *spherical* (*chordal*) *distance* $s(z,\zeta)$ between z and ζ is given by

$$s(z,\zeta) = \frac{|z-\zeta|}{(1+|z|^2)^{\frac{1}{2}}(1+|\zeta|^2)^{\frac{1}{2}}} \quad \text{if} \quad z \neq \infty \neq \zeta, \tag{4.1}$$

$$s(z,\infty) = \frac{1}{(1+|z|^2)^{\frac{1}{2}}} \quad \text{if} \quad z \neq \infty.$$

Given a set $E \subset \mathbb{C}$, $\delta(E)$ denotes the *spherical diameter* of E, i.e.,

$$\delta(E) = \sup_{z_1, z_2 \in E} s(z_1, z_2). \tag{4.2}$$

The following lemma is based on well known capacity estimates by Gehring.

Lemma 4.1. *Let $f : D \to \mathbb{C}$ be a homeomorphism with $\delta(\overline{\mathbb{C}} \setminus f(D)) \geq \Delta > 0$ and let z_0 be a point in D, $\zeta \in D(z_0, r_0)$, $r_0 \leq \operatorname{dist}(z_0, \partial D)$, $C_0 = \{z \in \mathbb{C} : |z - z_0| = r_0\}$ and $C = \{z \in \mathbb{C} : |z - z_0| = |\zeta - z_0|\}$. Then*

$$s(f(\zeta), f(z_0)) \leq \frac{32}{\Delta} \cdot \exp\Big(-\frac{2\pi}{\operatorname{cap} R(fC, fC_0)}\Big). \tag{4.3}$$

Proof. Let E denote the component of $\overline{\mathbb{C}} \setminus fA$ containing $f(z_0)$ and F the component containing ∞ where $A = \{z \in \mathbb{C} : |\zeta - z_0| < |z - z_0| < r_0\}$. By the known Gehring lemma

$$\operatorname{cap} R(E,F) \geq \operatorname{cap} R_T\Big(\frac{1}{\delta(E)\delta(F)}\Big) \tag{4.4}$$

where $\delta(E)$ and $\delta(F)$ denote the spherical diameters of the continua E and F, correspondingly, and $R_T(t)$ is the Teichmüller ring

$$R_T(t) = \overline{\mathbb{C}} \setminus ([-1,0] \cup [t,\infty]), \qquad t > 1, \tag{4.5}$$

see e.g. 7.37 in [62] or [18]. It is also known that

$$\operatorname{cap} R_T(t) = \frac{2\pi}{\log \Phi(t)} \tag{4.6}$$

where the function Φ admits the good estimates:

$$t + 1 \leq \Phi(t) \leq 16 \cdot (t+1) < 32 \cdot t, \qquad t > 1, \tag{4.7}$$

see e.g. [18, p. 225–226], and (7.19) and (7.22) in [62]. Hence the inequality (4.4) implies that

$$\operatorname{cap} R(E,F) \geq \frac{2\pi}{\log \frac{32}{\delta(E)\delta(F)}}. \tag{4.8}$$

Thus,

$$\delta(E) \leq \frac{32}{\delta(F)} \exp\Big(- \frac{2\pi}{\operatorname{cap} R(E,F)}\Big) \tag{4.9}$$

that implies the desired statement.

Lemma 4.2. *Let $f : D \to \mathbb{C}$ be a ring Q-homeomorphism at a point $z_0 \in D$ for a given measurable function $Q : D(z_0, r_0) \to [0,\infty]$, $r_0 \leq \operatorname{dist}(z_0, \partial D)$. Let $\psi_\varepsilon : [0,\infty] \to [0,\infty]$, $0 < \varepsilon < \varepsilon_0 < r_0$, be a given one parameter family of measurable functions such that*

$$0 < I(\varepsilon) = \int\limits_{\varepsilon}^{\varepsilon_0} \psi_\varepsilon(t)\, dt < \infty, \qquad \varepsilon \in (0, \varepsilon_0). \tag{4.10}$$

Set $C = \{z \in \mathbb{C} : |z - z_0| = \varepsilon\}$, $C_0 = \{z \in \mathbb{C} : |z - z_0| = \varepsilon_0\}$ and

$$A(\varepsilon) = A(\varepsilon, \varepsilon_0, z_0) = \{z \in \mathbb{C} : \varepsilon < |z - z_0| < \varepsilon_0\}. \tag{4.11}$$

Then

$$\operatorname{cap} R(fC, fC_0) \leq \omega(\varepsilon) \tag{4.12}$$

where

$$\omega(\varepsilon) = \frac{1}{I^2(\varepsilon)} \int\limits_{A(\varepsilon)} Q(z) \cdot \psi_\varepsilon^2(|z - z_0|)\, dx\, dy. \tag{4.13}$$

Proof. Given a ring Q-homeomorphism f at z_0, then (4.12) follows by the definition with choosing $\eta(r) = \psi_\varepsilon(r)/I(\varepsilon)$, $r \in (\varepsilon, \varepsilon_0)$, in (1.17). $\square$

Using Lemma 4.2, we present now a sharp capacity estimate for ring Q-homeomorphisms $f : D \to \mathbb{C}$ at a point $z_0 \in D$. This estimate depends only on Q and it implies as a special case a known inequality which was proved for qc mappings by Reich and Walczak in [55] and that has later been applied by several authors.

Lemma 4.3. *Let D be a domain in $\mathbb{C}$, z_0 a point in D, $r_0 \leq \operatorname{dist}(z_0, \partial D)$, $Q : D(z_0, r_0) \to [0,\infty]$ a measurable function and $q(r)$ the mean of $Q(z)$ over the circle $|z - z_0| = r$, r, r_0. For $0 < r_1 < r_2 < r_0$, set*

$$I = I(r_1, r_2) = \int\limits_{r_1}^{r_2} \frac{dr}{rq(r)} \tag{4.14}$$

and $C_j = \{z \in \mathbb{C} : |z - z_0| = r_j\}$, $j = 1, 2$. Then

$$\operatorname{cap} R(fC_1, fC_2) \leq \frac{2\pi}{I}, \tag{4.15}$$

whenever $f : D \to \mathbb{C}$ is a ring Q-homeomorphism at z_0.

Proof. With no loss of generality we may assume that $I \neq 0$ because otherwise (4.15) is trivial and that $I \neq \infty$ because otherwise we can replace $Q(z)$ by $Q(z) + \delta$ with arbitrarily small $\delta > 0$ and then take the limit as $\delta \to 0$ in (4.15). The condition $I \neq \infty$ implies, in particular, that $q(r) \neq 0$ a.e. in (r_1, r_2).

For $I \neq 0, \infty$, Lemma 4.3 follows from Lemma 4.2 by choosing the functional parameter

$$\psi_\varepsilon(t) \equiv \psi(t) \colon = \begin{cases} 1/[tq(t)], & t \in (0, \varepsilon_0), \\ 0, & \text{otherwise} \end{cases} \tag{4.16}$$

with $\varepsilon = r_1$ and $\varepsilon_0 = r_2$, since

$$\int\limits_A Q(z) \cdot \psi^2(|z - z_0|)\, dx\, dy = 2\pi I \tag{4.17}$$

where

$$A = A(r_1, r_2, z_0) = \{z \in \mathbb{C} : r_1 < |z - z_0| < r_2\}. \tag{4.18}$$

$\square$

Corollary 4.1. *For every ring Q-homeomorphism $f : D \to \mathbb{C}$ at $z_0 \in D$ and $0 < r_1 < r_2 < r_0$,*

$$\int\limits_{r_1}^{r_2} \frac{dr}{rq(r)} < \infty \tag{4.19}$$

where $q(r)$ is the mean of $Q(z)$ over the circle $|z - z_0| = r$.

Indeed, by (4.8) with $E = fC_1$ and $F = fC_2$, $C_1 = \{z \in \mathbb{C} : |z - z_0| = r_1\}$ and $C_2 = \{z \in \mathbb{C} : |z - z_0| = r_2\}$

$$\operatorname{cap} R(fC_1, fC_2) \geq \frac{2\pi}{\log \frac{32}{\delta(fC_1)\delta(fC_2)}}. \tag{4.20}$$

The right side in (4.20) is positive since f is injective. Thus, Corollary 4.1 follows from (4.15) in Lemma 4.3.

Corollary 4.2. *Let $f : D \to \mathbb{C}$ be a $W^{1,2}_{loc}$ homeomorphism in a domain $D \subset \mathbb{C}$ such that*

$$K_\mu(z) \;=\; \frac{1 + |\mu(z)|}{1 - |\mu(z)|} \in L^1_{loc}(D) \tag{4.21}$$

where $\mu(z) = \mu_f(z)$. Set

$$q^T_{z_0}(r) = \frac{1}{2\pi} \int\limits_0^{2\pi} \frac{|1 - e^{-2i\vartheta}\mu(z_0 + re^{i\vartheta})|^2}{1 - |\mu(z_0 + re^{i\vartheta})|^2}\, d\vartheta. \tag{4.22}$$

Then

$$\int_{r_1}^{r_2} \frac{dr}{rq_{z_0}^T(r)} < \infty \tag{4.23}$$

for every $z_0 \in D$ *and* $0 < r_1 < r_2 < d_0$ *where* $d_0 = \operatorname{dist}(z_0, \partial D)$.

Corollary 4.2 follows from Corollaries 4.1 and 3.1 and from the definition of the tangential dilatation $K_\mu^T(z, z_0)$, see (1.3).

Corollary 4.3. *Let* $f : D \to \mathbb{C}$ *be a* $W_{loc}^{1,2}$ *homeomorphism with* $K_\mu(z) \in L_{loc}^1$ *where* $\mu(z) = \mu_f(z)$. *Then*

$$\operatorname{cap} R(fC_1, fC_2) \leq \left[\int_{r_1}^{r_2} \frac{dr}{r \int_0^{2\pi} \frac{|1-e^{-2i\vartheta}\mu(z_0+re^{i\vartheta})|^2}{1-|\mu(z_0+re^{i\vartheta})|^2}\, d\vartheta} \right]^{-1}. \tag{4.24}$$

Indeed, by Corollary 3.1 f is a ring Q-homeomorphism at z_0 with $Q(z) = K_\mu^T(z, z_0)$. The tangential dilatation $K_\mu^T(z, z_0)$ is given by (1.3) and thus (4.24) follows from Lemma 4.3.

Remark 4.1. (4.24) was first derived by Reich and Walczak [55] for quasiconformal mappings and then by Lehto [34] for certain μ-homeomorphisms, and was later applied by Brakalova and Jenkins [7] and Gutlyanskii, Martio, Sugawa and Vuorinen [21] in the study of degenerate Beltrami equations.

The following lemma shows that the estimate (4.15), which implies (4.24), cannot be improved in the class of all ring Q-homeomorphisms. Note that the additional condition (4.25) which appears in the following lemma holds automatically for every ring Q-homeomorphism by Corollary 4.1.

Lemma 4.4. *Fix* $0 < r_1 < r_2 < r_0$, $A = \{z \in \mathbb{C} : r_1 < |z - z_0| < r_2\}$, *and suppose that* $Q : D(z_0, r_0) \to [0, \infty]$ *is a measurable function such that*

$$c_0 = \int_{r_1}^{r_2} \frac{dr}{rq(r)} < \infty \tag{4.25}$$

where $q(r)$ *is the mean of* $Q(z)$ *over the circle* $|z - z_0| = r$ *and let*

$$\eta_0(r) = \frac{1}{c_0 rq(r)}. \tag{4.26}$$

Then

$$\frac{2\pi}{c_0} = \int_A Q(z) \cdot \eta_0^2(|z - z_0|)\, dx\, dy \leq \int_A Q(z) \cdot \eta^2(|z - z_0|)\, dx\, dy \tag{4.27}$$

for any $\eta : (r_1, r_2) \to [0, \infty]$ *such that*

$$\int\limits_{r_1}^{r_2} \eta(r)\, dr = 1. \tag{4.28}$$

Proof. If $c_0 = 0$, then $q(r) = \infty$ for a.e. $r \in (r_1, r_2)$ and the both sides in (4.27) are equal to ∞. Hence we may assume below that $0 < c_0 < \infty$.

Now, by (4.25) and (4.28) $q(r) \neq 0$ and $\eta(r) \neq \infty$ a.e. in (r_1, r_2). Set $\alpha(r) = rq(r)\eta(r)$ and $w(r) = 1/rq(r)$. Then by the standard conventions $\eta(r) = \alpha(r)w(r)$ a.e. in (r_1, r_2) and

$$C := \int\limits_{A} Q(z) \cdot \eta^2(|z - z_0|)\, dx\, dy = 2\pi \int\limits_{r_1}^{r_2} \alpha^2(r) \cdot w(r)\, dr. \tag{4.29}$$

By Jensen's inequality with weights, see e.g. Theorem 2.6.2 in [46], applied to the convex function $\varphi(t) = t^2$ in the interval $\Omega = (r_1, r_2)$ with the probability measure

$$\nu(E) = \frac{1}{c_0} \int\limits_{E} w(r)\, dr \tag{4.30}$$

we obtain that

$$\left(\fint \alpha^2(r) w(r)\, dr \right)^{1/2} \geq \fint \alpha(r) w(r)\, dr = \frac{1}{c_0} \tag{4.31}$$

where we also used the fact that $\eta(r) = \alpha(r)w(r)$ satisfies (4.28). Thus,

$$C \geq \frac{2\pi}{c_0} \tag{4.32}$$

and the proof is complete. $\square$

Given a number $\Delta \in (0, 1)$, a domain $D \subset \mathbb{C}$, a point $z_0 \in D$, a number $r_0 \leq \operatorname{dist}(z_0, \partial D)$, and a measurable function $Q : D(z_0, r_0) \to [0, \infty]$, let $\mathfrak{R}_Q^{\Delta}$ denote the class of all ring Q-homeomorphisms $f : D \to \overline{\mathbb{C}}$ at z_0 such that

$$\delta(\overline{\mathbb{C}} \setminus f(D)) \geq \Delta. \tag{4.33}$$

Next, we introduce the classes $\mathfrak{B}_Q^{\Delta}$ and $\mathfrak{F}_Q^{\Delta}$ of certain qc mappings. Let $\mathfrak{B}_Q^{\Delta}$ denote the class of all quasiconformal mappings $f : D \to \overline{\mathbb{C}}$ satisfying (4.33) such that

$$K_{\mu}^{T}(z, z_0) = \frac{\left|1 - \frac{\overline{z - z_0}}{z - z_0} \mu(z)\right|^2}{1 - |\mu(z)|^2} \;\leq\; Q(z) \quad \text{a.e. in} \quad D(z_0, r_0) \tag{4.34}$$

where $\mu = \mu_f$. Similarly, let $\mathfrak{F}_Q^\Delta$ denote the class of all quasiconformal mappings $f : D \to \overline{\mathbb{C}}$ satisfying (4.33) such that

$$K_\mu(z) = \frac{1+|\mu(z)|}{1-|\mu(z)|} \le Q(z) \quad \text{a.e. in} \quad D(z_0, r_0). \tag{4.35}$$

Remark 4.2. By Corollaries 3.1 and the relations (3.13) and (3.6)

$$\mathfrak{F}_Q^\Delta \ \subset \ \mathfrak{B}_Q^\Delta \subset \mathfrak{R}_Q^\Delta. \tag{4.36}$$

Combining Lemmas 4.2 and 4.1 we obtain the following distortion estimates in the class $\mathfrak{R}_Q^\Delta$ which allow us to obtain further basic distortion bounds, see theorems and corollaries following Theorem 4.1.

Corollary 4.4. *Let $f \in \mathfrak{R}_Q^\Delta$, and let $\omega(\varepsilon)$ be as in Lemma 4.2. Then*

$$s(f(\zeta), f(z_0)) \le \frac{32}{\Delta} \cdot \exp\Big(-\frac{2\pi}{\omega(|\zeta - z_0|)}\Big) \tag{4.37}$$

for all $\zeta \in D(z_0, \varepsilon_0)$.

Theorem 4.1. *Let $f \in \mathfrak{R}_Q^\Delta$, and let $\psi : [0,\infty] \to [0,\infty]$ be a measurable function such that*

$$0 < \int\limits_{\varepsilon}^{\varepsilon_0} \psi(t)\, dt < \infty, \qquad \varepsilon \in (0, \varepsilon_0). \tag{4.38}$$

Suppose that

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} Q(z) \cdot \psi^2(|z-z_0|)\, dx\, dy \le C \cdot \int\limits_{\varepsilon}^{\varepsilon_0} \psi(t)\ dt \tag{4.39}$$

for all $\varepsilon \in (0, \varepsilon_0)$. Then

$$s(f(\zeta), f(z_0)) \le \frac{32}{\Delta} \cdot \exp\left(-\frac{2\pi}{C} \cdot \int\limits_{|\zeta-z_0|}^{\varepsilon_0} \psi(t)\, dt\right) \tag{4.40}$$

whenever $\zeta \in D(z_0, \varepsilon_0)$.

Choosing in Theorem 4.1 the special functional parameter $\psi(t)$ given by (4.16) we obtain the following distortion theorem for ring Q-homeomorphisms.

Theorem 4.2. *Let D be a domain in $\mathbb{C}$, z_0 a point in D, $r_0 \leq \operatorname{dist}(z_0, \partial D)$, $Q : D(z_0, r_0) \to [0, \infty]$ a measurable function and let $f \in \mathfrak{R}^{\Delta}_{Q}$. Then*

$$s(f(\zeta), f(z_0)) \leq \frac{32}{\Delta} \cdot \exp\left(- \int\limits_{|\zeta - z_0|}^{r_0} \frac{dr}{rq(r)} \right) \tag{4.41}$$

for all $\zeta \in D(z_0, r_0)$ where $q(r)$ is the mean of $Q(z)$ over the circle $|z - z_0| = r$.

In the following theorem the estimate of distortion is expressed in terms of maximal dispersion, see (2.5).

Theorem 4.3. *Let $f \in \mathfrak{R}^{\Delta}_{Q}$ for $\Delta > 0$ and Q with finite mean oscillation at $z_0 \in D$. If Q is integrable over a disk $D(z_0, \varepsilon_0) \subset D$, then*

$$s(f(\zeta), f(z_0)) \leq \frac{32}{\Delta} \cdot \left(\log \frac{2\varepsilon_0}{|\zeta - z_0|} \right)^{-\beta_0} \tag{4.42}$$

for every point $\zeta \in D(z_0, \varepsilon_0/2)$ where

$$\beta_0 = \frac{1}{2}[q_0 + 6d_0]^{-1}, \tag{4.43}$$

q_0 is the mean and d_0 the maximal dispersion of $Q(z)$ in $D(z_0, \varepsilon_0)$.

Proof. The mean and the dispersion of a function over disks are invariant under linear transformations $w = (z - z_0)/2\varepsilon_0$. Hence, (4.42) follows by Theorem 4.1 and Lemma 2.1. □

The following two corollaries, which are formulated in terms of mean of Q over disks and annuli, are obtained from Corollary 4.4 by setting $\psi_\varepsilon(t) \equiv 1$ for $0 < \varepsilon < \varepsilon_0$ in (4.13) where we choose $\varepsilon = |\zeta - z_0|$ and $\varepsilon_0 = 4|\zeta - z_0|$ in the case of Corollary 4.5 and $\varepsilon = |\zeta - z_0|$ and $\varepsilon_0 = 3|\zeta - z_0|$ in the case of Corollary 4.6.

Corollary 4.5. *Let $Q : D(z_0, r_0) \to [0, \infty]$, $r_0 \leq \mathit{dist}(z_0, \partial D)$, be a measurable function, for $r \leq r_0/4$, let $M_Q(r)$ denote the mean of Q over the disk $D(z_0, 4r)$ and let $\Delta > 0$. If $f \in \mathfrak{R}^{\Delta}_{Q}$, then*

$$s(f(\zeta), f(z_0)) \leq \frac{32}{\Delta} \cdot e^{-1/M_Q(|\zeta - z_0|)} \tag{4.44}$$

for all $\zeta \in D(z_0, r_0/4)$.

Corollary 4.6. *Let* $Q : D(z_0, r_0) \to [0, \infty]$, $r_0 \le \operatorname{dist}(z_0, \partial D)$, *be a measurable function, for* $r \le r_0/3$, *let* $M^Q(r)$ *denote the mean of* Q *over the annulus* $A = \{z \in \mathbb{C} : r < |z - z_0| < 3r\}$ *and let* $\Delta > 0$. *If* $f \in \mathfrak{R}_Q^\Delta$, *then*

$$s(f(\zeta), f(z_0)) \le \frac{32}{\Delta} \cdot e^{-1/M^Q(|\zeta - z_0|)} \tag{4.45}$$

for all $\zeta \in D(z_0, r_0/3)$.

Another consequence of Lemma 4.2, see Corollary 4.4 above, can be formulated in terms of the *logarithmic mean* of Q over an annulus $A(\varepsilon) = A(\varepsilon, \varepsilon_0, z_0) = \{z \in \mathbb{C} : \varepsilon < |z - z_0| < \varepsilon_0\}$ which is defined by

$$M_{\log}^Q(\varepsilon) = \fint_{\varepsilon}^{\varepsilon_0} q(t)\ d\log t := \frac{1}{\log \varepsilon_0/\varepsilon} \int_{\varepsilon}^{\varepsilon_0} q(t) \frac{dt}{t} \tag{4.46}$$

where $q(t)$ denotes the mean value of Q over the circle $|z - z_0| = t$. Choosing in the expression (4.13) $\psi_\varepsilon(t) = 1/t$ for $0 < \varepsilon < \varepsilon_0$, and setting $\varepsilon = |\zeta - z_0|$ we have the following statement.

Corollary 4.7. *Let* $Q : D(z_0, r_0) \to [0, \infty]$, $r_0 \le \operatorname{dist}(z_0, \partial D)$, *be a measurable function,* $\varepsilon_0 \in (0, r_0)$ *and* $\Delta > 0$. *If* $f \in \mathfrak{R}_Q^\Delta$, *then*

$$s(f(\zeta), f(z_0)) \le \frac{32}{\Delta} \cdot \Big(\frac{|\zeta - z_0|}{\varepsilon_0}\Big)^{1/M_{\log}^Q(|\zeta - z_0|)} \tag{4.47}$$

for all $\zeta \in D(z_0, \varepsilon_0)$.

Note that, for $Q \equiv K \in [1, \infty)$, (4.47) is reduced to the following known distortion estimate for qc mappings

$$s(f(\zeta), f(z_0)) \le \frac{32}{\Delta} \cdot \Big(\frac{|\zeta - z_0|}{\varepsilon_0}\Big)^{1/K}. \tag{4.48}$$

The corollaries and theorems presented above show that Lemmas 4.1 and 4.2 are useful tools in deriving various distortion estimates for ring Q-homeomorphisms. These, in turn, are instrumental in the study of properties of ring Q-homeomorphisms and, in particular, of ring solutions of the Beltrami equation (1.1) where $Q(z)$ can be either the maximal dilatation $K_\mu(z)$ or the tangential dilatation $K_\mu^T(z, z_0)$ that are defined in (1.2) and (1.3), respectively.

5. A general existence lemma and its corollaries

The following lemma and corollary serve as the main tool in obtaining many criteria of existence of ring solutions for the Beltrami equation. See Section 1 for the definition of a ring solution. In Theorem 5.1 the existence of a ring solution is established when at every point $z_0 \in D$ the tangential dilatation $K^T_\mu(z, z_0)$ is assumed to be dominated by a function of finite mean oscillation at z_0 in the variable z. In Theorem 5.2 below the condition for existence is formulated in terms of the mean of the tangential dilatation over infinitesimal disks. Since the maximal dilatation dominates the tangential dilatation, these two results obviously imply similar existence theorems in terms of conditions on the maximal dilatation, Theorem 5.3 and Corollary 5.2 below. The results in this theorem and corollary were established earlier in [53] with different proofs. The criterion for the existence in the last theorem in this section is formulated in terms of the logarithmic mean.

Lemma 5.1. *Let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L^1_{loc}$. Suppose that for every $z_0 \in D$ there exist $\varepsilon_0 \leq$* $\mathrm{dist}(z_0, \partial D)$ *and an one parameter family of measurable functions $\psi_{z_0,\varepsilon}$: $(0,\infty) \to (0,\infty)$, $\varepsilon \in (0, \varepsilon_0)$, such that*

$$0 < I_{z_0}(\varepsilon) := \int\limits_{\varepsilon}^{\varepsilon_0} \psi_{z_0,\varepsilon}(t)\, dt < \infty, \tag{5.1}$$

and such that

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} K^T_\mu(z, z_0) \cdot \psi^2_{z_0,\varepsilon}(|z - z_0|)\, dx\, dy \ = o(I^2_{z_0}(\varepsilon)) \tag{5.2}$$

as $\varepsilon \to 0$. Then the Beltrami equation (1.1) *has a ring solution f_μ.*

Proof. Fix z_1 and z_2 in D. For $n \in \mathbb{N}$, define $\mu_n : D \to \mathbb{C}$ by letting $\mu_n(z) = \mu(z)$ if $|\mu(z)| \leq 1 - 1/n$ and 0 otherwise. Let $f_n : D \to \mathbb{C}$ be a homeomorphic ACL solution of (1.1), with μ_n instead of μ, which fixes z_1 and z_2. Such f_n exists by the well-known existence theorem in the nondegenerate case, see e.g. [1, p. 98], cf. [33, p. 185 and 194]. By Theorem 3.1 and Corollary 4.4, in view of (5.2), the sequence f_n is equicontinuous and hence by the Arzela–Ascoli theorem, see e.g. [12, p. 267], and [11, p. 382] it has a subsequence, denoted again by f_n, which converges locally uniformly to some nonconstant mapping f in D. Then, by Theorem 3.1 and Corollary 5.12 in [50] on convergence, f is $K(z)$-qc with $K(z) = K_\mu(z)$ and f satisfies (1.1) a.e. Thus, f is a homeomorphic

ACL solution of (1.1). Moreover, by Theorems 3.1 and 3.2 f is a ring Q-homeomorphism, see (1.17), with $Q(z) = K_\mu^T(z, z_0)$ at every point $z_0 \in D$.

Since the locally uniform convergence $f_n \to f$ of the sequence f_n is equivalent to the continuous convergence, i.e., $f_n(z_n) \to f(z_0)$ if $z_n \to z_0$, see [12, p. 268] and since f is injective, it follows that $g_n = f_n^{-1} \to f^{-1} = g$ continuously, and hence locally uniformly. By a change of variables which is permitted because f_n and g_n are in $W^{1,2}_{loc}$ we obtain that for large n

$$\int\limits_B |\partial g_n|^2 \, du \, dv = \int\limits_{g_n(B)} \frac{dxdy}{1 - |\mu_n(z)|^2} \leq \int\limits_{B^*} Q(z) \, dx \, dy < \infty \tag{5.3}$$

where B^* and B are relatively compact domains in D and in $f(D)$, respectively, such that $g(\bar{B}) \subset B^*$. The relation (5.3) implies that the sequence g_n is bounded in $\mathrm{W}^{1,2}(B)$, and hence $f^{-1} \in \mathrm{W}^{1,2}_{loc}(f(D))$, see e.g. [49, p. 319]. $\square$

Remark 5.1. If f_μ is as in Lemma 5.1, then f_μ^{-1} is locally absolutely continuous and preserves nulls sets, and f_μ is regular a.e., i.e., differentiable with $J_{f_\mu}(z) > 0$ a.e. Indeed, the assertion about f_μ^{-1} follows from the fact that $f_\mu^{-1} \in W^{1,2}_{loc}$, see [33, p. 131 and 150]. As an ACL mapping f_μ has a.e. partial derivatives and hence by [20] it has a total differential a.e. Let E denote the set of points of D where f_μ is differentiable and $J_{f_\mu}(z) = 0$, and suppose that $|E| > 0$. Then $|f_\mu(E)| > 0$, since $E = f_\mu^{-1}(f_\mu(E))$ and f_μ^{-1} preserves null sets. Clearly f_μ^{-1} is not differentiable at any point of $f_\mu(E)$, contradicting the fact that f_μ^{-1} is differentiable a.e.

Corollary 5.1. *Let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e., $K_\mu \in L^1_{loc}$, and let $\psi : (0, \infty) \to (0, \infty)$ be a measurable function such that for all $0 < t_1 < t_2 < \infty$*

$$0 < \int\limits_{t_1}^{t_2} \psi(t) \, dt < \infty, \qquad \int\limits_{0}^{t_2} \psi(t) \, dt = \infty. \tag{5.4}$$

Suppose that for every $z_0 \in D$ there is $\varepsilon_0 < dist\,(z_0, \partial D)$ such that

$$\int\limits_{\varepsilon < |z - z_0| < \varepsilon_0} K_\mu(z) \cdot \psi^2(|z - z_0|) \, dx \, dy \leq O\left(\int\limits_{\varepsilon}^{\varepsilon_0} \psi(t) \, dt \right) \tag{5.5}$$

as $\varepsilon \to 0$. Then (1.1) has a ring solution.

Lemma 5.1 yields the following theorem by choosing

$$\psi_{z_0,\varepsilon}(t) = \frac{1}{t\log\frac{1}{t}}, \tag{5.6}$$

see also Lemma 2.1.

Theorem 5.1. *Let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L^1_{loc}$. Suppose that every point $z_0 \in D$ has a neighborhood U_{z_0} such that*

$$K^T_\mu(z, z_0) \leq \; Q_{z_0}(z) \qquad a.e. \tag{5.7}$$

for some function $Q_{z_0}(z)$ of finite mean oscillation at the point z_0 in the variable z. Then the Beltrami equation (1.1) *has a ring solution.*

The following theorem is a consequence of Theorem 5.1 and Corollary 2.1.

Theorem 5.2. *Let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L^1_{loc}$. Suppose that at every $z_0 \in D$*

$$\overline{\lim_{\varepsilon\to 0}} \;\; \fint_{D(z_0,\varepsilon)} \frac{\left|1 - \frac{\overline{z-z_0}}{z-z_0}\mu(z)\right|^2}{1-|\mu(z)|^2}\, dx\, dy < \infty. \tag{5.8}$$

Then the Beltrami equation (1.1) *has a ring solution f_μ.*

The following theorem is an important particular case of Theorem 5.1.

Theorem 5.3. *Let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. such that*

$$K_\mu(z) = \frac{1+|\mu(z)|}{1-|\mu(z)|} \leq Q(z) \;\in\; FMO. \tag{5.9}$$

Then the Beltrami equation (1.1) *has a ring solution.*

Since every ring solution is an ACL homeomorphic solution and since every BMO function is in FMO, the theorem generalizes and strengthens earlier results in [50, 51] about the existence of ACL homeomorphic solutions of the Beltrami equation when the conditions involve majorants of bounded mean oscillation.

Corollary 5.2. *If*

$$\overline{\lim_{\varepsilon\to 0}} \;\; \fint_{D(z_0,\varepsilon)} \frac{1+|\mu(z)|}{1-|\mu(z)|}\, dx\, dy < \infty \tag{5.10}$$

at every $z_0 \in D$, then (1.1) *has a ring solution.*

Applying Lemma 5.1 with $\psi(t) = 1/t$, we have also the following statement which is formulated in terms of the logarithmic mean, see (4.46), of $K_\mu^T(z, z_0)$ over the annuli $A(\varepsilon) = \{z \in \mathbb{C} : \varepsilon < |z - z_0| < \varepsilon_0\}$ for a fixed $\varepsilon_0 = \delta(z_0) \le \operatorname{dist}(z_0, \partial D)$.

Theorem 5.4. *Let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L^1_{loc}$. If at every point $z_0 \in D$ the logarithmic mean of K_μ^T over $A(\varepsilon)$ does not converge to ∞ as $\varepsilon \to 0$, i.e.,*

$$\liminf_{\varepsilon \to 0} M_{\log}^{K_\mu^T}(\varepsilon) < \infty, \tag{5.11}$$

then the Beltrami equation (1.1) *has a ring solution.*

Corollary 5.3. *Let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L^1_{loc}$. Denote by $q_{z_0}^T(t)$ the mean of $K_\mu^T(z, z_0)$ over the circle $C = \{z \in \mathbb{C} : |z - z_0| = t\}$. If*

$$\int\limits_0^{\delta(z_0)} q_{z_0}^T(t) \frac{dt}{t} < \infty \tag{5.12}$$

at every point $z_0 \in D$ for some $\delta(z_0) > 0$, then (1.1) *has a ring solution.*

Lehto considers in [34] degenerate Beltrami equations in the special case where the *singular set* S_μ

$$S_\mu = \{z \in \mathbb{C} : \lim_{\varepsilon \to 0} \ \|K_\mu\|_{L^\infty(D(z,\varepsilon))} \ = \ \infty\} \tag{5.13}$$

of the complex coefficient μ in (1.1) is of measure zero, and shows that, if for every $z_0 \in \mathbb{C}$ and every r_1 and $r_2 \in (0, \infty)$ the integral

$$\int\limits_{r_1}^{r_2} \frac{dr}{r(1 + q_{z_0}^T(r))} \ , \qquad r_2 > r_1 \tag{5.14}$$

is positive and tends to ∞ as either $r_1 \to 0$ or $r_2 \to \infty$ where

$$q_{z_0}^T(r) = \frac{1}{2\pi} \int\limits_0^{2\pi} \frac{|1 - e^{-2i\vartheta}\mu(z_0 + re^{i\vartheta})|^2}{1 - |\mu(z_0 + re^{i\vartheta})|^2} \, d\vartheta, \tag{5.15}$$

then there exists a homeomorphism $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ which is ACL in $\mathbb{C} \setminus S_\mu$ and satisfies (1.1) a.e. Note that the integrand in (5.15) is the tangential dilatation $K_\mu^T(z, z_0)$, see (1.3).

We present now an extension of Lehto's existence theorem which enables to derive many other existence theorems as it was shown in [54]. In this extension we prove the existence of a ring solution in a domain $D \subset \mathbb{C}$ which by the definition is ACL in D and not only in $D \setminus S_\mu$. Note that, in the following theorem, the situation where $S_\mu = D$ is possible. Note also that the condition (5.16) in the following theorem is weaker than the condition in Lehto's existence theorem.

Theorem 5.5. *Let D be a domain in $\mathbb{C}$ and let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L^1_{loc}$. Suppose that at every point $z_0 \in D$*

$$\int\limits_0^{\delta(z_0)} \frac{dr}{r q_{z_0}^T(r)} = \infty \tag{5.16}$$

where $\delta(z_0) < dist\,(z_0, \partial D)$ and $q_{z_0}^T(r)$ is the mean of $K_\mu^T(z, z_0)$ over $|z - z_0| = r$. Then the Beltrami equation (1.1) *has a ring solution.*

Proof. Theorem 5.5 follows from Lemma 5.1 by special choosing the functional parameter

$$\psi_{z_0,\varepsilon}(t) \equiv \psi_{z_0}(t) := \begin{cases} 1/[t q_{z_0}^T(t)], & t \in (0, \varepsilon_0), \\ 0, & \text{otherwise} \end{cases} \tag{5.17}$$

where $\varepsilon_0 = \delta(z_0)$. □

Corollary 5.4. *If $K_\mu \in L^1_{loc}$ and at every point $z_0 \in D$*

$$q_{z_0}^T(r) = O\Big(\log \frac{1}{r}\Big) \qquad as\ r \to 0, \tag{5.18}$$

then (1.1) has a ring solution.

Since $K_\mu^T(z, z_0) \le K_\mu(z)$ we obtain as a consequence of Theorem 5.5 the following result which is due to Miklyukov and Suvorov [41] for the case $K_\mu \in L^p_{loc}$, $p > 1$.

Corollary 5.5. *If $K_\mu \in L^p_{loc}$ for $p \ge 1$ and* (5.16) *holds for $K_\mu(z)$ instead of $K_\mu^T(z, z_0)$ for every point $z_0 \in D$, then* (1.1) *has a $W^{1,s}_{loc}$ homeomorphic solution with $s = 2p/(p+1)$.*

Further corollaries from the generalization of the Lehto existence theorem can be found in the paper [54] and other discussions in the survey [58].

6. Representation, factorization and uniqueness theorems

In Section 5 we have established a series of theorems on the existence of ring solutions f_μ for the Beltrami equation (1.1) for a variety of different conditions on the complex coefficient μ. We now show that, in each of these cases, f_μ generates all $W^{1,2}_{loc}$ solutions by composition with analytic functions.

Lemma 6.1. *Let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L^1_{loc}$. Suppose that for every $z_0 \in D$ there exist $\varepsilon_0 = \delta(z_0) \le \operatorname{dist}(z_0, \partial D)$ and a one parameter family of measurable functions $\psi_{z_0,\varepsilon} : (0,\infty) \to (0,\infty)$, $\varepsilon \in (0, \varepsilon_0)$, such that*

$$0 < I_{z_0}(\varepsilon) := \int\limits_{\varepsilon}^{\varepsilon_0} \psi_{z_0,\varepsilon}(t)\, dt < \infty, \qquad \varepsilon \in (0,\varepsilon_0), \tag{6.1}$$

and such that

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} K^T_\mu(z,z_0) \cdot \psi^2_{z_0,\varepsilon}(|z-z_0|)\, dx\, dy = o(I^2_{z_0}(\varepsilon)) \tag{6.2}$$

as $\varepsilon \to 0$ and let f_μ be a ring solution of (1.1)*. Then every $W^{1,2}_{loc}$ solution g of* (1.1) *has the representation*

$$g = h \circ f_\mu \tag{6.3}$$

for some holomorphic function h in $f_\mu(D)$.

Proof. Let $\varphi = f_\mu^{-1}$ and $h = g \circ \varphi$. Since $g \in \mathrm{W}^{1,2}_{loc}$ and $\varphi \in \mathrm{W}^{1,2}_{loc}$ it follows that $h \in \mathrm{W}^{1,1}_{loc}(f(D))$, see [33], p. 151. Thus, by Weyl's lemma, see e.g. [1, p. 33] it suffices to show that $\partial h = 0$ a.e. in $f_\mu(D)$. Let E denote the set of points z in D where either f_μ or g do not satisfy (1.1) or $J_{f_\mu} = 0$. A direct computation, cf. [1, p. 9] shows that $\overline{\partial} h = 0$ in $f_\mu(D) \setminus f_\mu(E)$. Moreover, $\varphi \in \mathrm{W}^{1,2}_{loc}$ admits the change of variables, see e.g. [33, p. 121, 128–130 and 150]:

$$\iint\limits_{f_\mu(E)} |\partial\varphi|^2\, du\, dv = \iint\limits_{f_\mu(E)} J_\varphi(w) \frac{dudv}{1-|\mu(\varphi(w))|^2} = \iint\limits_{E} \frac{dxdy}{1-|\mu(z)|^2} = 0$$

which implies that $|\partial\varphi| = 0$ a.e. on $f_\mu(E)$, and since a.e. $|\overline{\partial}\varphi| \le |\partial\varphi|$ and

$$\overline{\partial} h = \overline{\partial}\varphi \cdot \partial g \circ \varphi + \overline{\partial\varphi} \cdot \overline{\partial} g \circ \varphi$$

it follows that $|\overline{\partial} h| = 0$ a.e. on $f_\mu(E)$, and thus $\overline{\partial} h = 0$ a.e. in $f_\mu(D)$ and, consequently, h is holomorphic in $f_\mu(D)$ and (6.3) holds. $\square$

Iwaniec and Sverak [29] showed that, if $K_\mu \in L^1_{loc}$, then every $\mathrm{W}^{1,2}_{loc}$ solution g of (1.1) has the representation $g = h \circ f$ for some holomorphic function h and some homeomorphism f. The conditions in Lemma 6.1 are more restrictive, however, the representation in the lemma is more specific and the proof is simpler.

Remark 6.1. Since all theorems on the existence of a ring solution f_μ in Section 5 are based on Lemma 5.1, where the conditions are as in Lemma 6.1, every $\mathrm{W}^{1,2}_{loc}$ solution g of the Beltrami equation (1.1) in each of these theorems has the representation (6.3).

It is not clear, even if μ satisfies the conditions of Lemma 6.1, whether an ACL homeomorphic solution of (1.1) is unique up to a composition with a conformal mapping, namely whether, for any two ACL homeomorphic solutions f_1 and f_2 of (1.1), $f_2 \circ f_1^{-1}$ is conformal. However, by (6.3) in Lemma 6.1 the answer is affirmative if f_1 and f_2 are in $W^{1,2}_{loc}$ and μ is as in Lemma 6.1, see Corollary 6.1 below. Another type of conditions for the uniqueness of a homeomorphic ACL solution can be obtained by imposing some conditions on the "size" of the singular set of μ. This will be done in Lemma 6.2 and Theorem 6.1 below.

Corollary 6.1. *Suppose that μ satisfies the conditions of one of the existence theorems in Section 5. If f_1 and f_2 are homeomorphic $W^{1,2}_{loc}$ solutions of* (1.1), *then $f_2 \circ f_1^{-1}$ is conformal.*

Iwaniec and Martin have constructed ACL solutions for the Beltrami equation which are not in $\mathrm{W}^{1,2}_{loc}$ and not open and discrete and, thus, are not generated by a homeomorphic solution in the sense of (6.3), see e.g. [25]. However, for discrete open solutions, it is easy to obtain by Stoilow's theorem the following proposition.

Proposition 6.1. *Let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. such that*

$$K_\mu(z) = \frac{1 + |\mu(z)|}{1 - |\mu(z)|} \in \ L^1_{loc}. \tag{6.4}$$

Then every (*continuous*) *discrete and open ACL solution g of the Beltrami equation* (1.1) *has the representation $g = h \circ f$ where f is a homeomorphic $W^{1,1}_{loc}$ solution of* (1.1) *and h is a holomorphic function in $f(D)$.*

Remark 6.2. As a consequence of the proposition we obtain that, if $K_\mu \in L^1_{loc}$, then either the Beltrami equation (1.1) has a homeomorphic $W^{1,1}_{loc}$ solution or has no continuous, discrete and open ACL solution. Note that, for every $p \in [1, \infty)$, there are examples of measurable functions $\mu : \mathbb{C} \to \mathbb{C}$ such that $|\mu(z)| < 1$ a.e. and $K_\mu(z) \in L^p_{loc}$ and for which the Beltrami equation (1.1) has no homeomorphic ACL solution, see Proposition 6.3 in [50].

Let (X, d) be a metric space and let $H = \{h_x(r)\}_{x\in X}$ be a family of functions $h_x : (0, \rho_x) \to (0, \infty)$, $\rho_x > 0$, such that $h_x(r) \to 0$ as $r \to 0$. Let

$$L_H^\rho(X) = \inf\ \Sigma h_{x_k}(r_k) \tag{6.5}$$

where the infimum is taken over all finite collections of $x_k \in X$ and $r_k \in (0, \rho)$ such that the balls

$$B(x_k, r_k) = \{x \in X : d(x, x_k) < r_k\} \tag{6.6}$$

cover X. The limit

$$L_H(X) := \lim_{\rho\to 0}\ L_H^\rho(X) \tag{6.7}$$

exists. We call $L_H(X)$ by *H-length* of X. In the particular case where $h_x(r) = r$ for all $x \in X$ and $r > 0$, H-length is the usual (Hausdorff) length of X.

The *singular set* S_μ of $\mu : D \to \mathbb{C}$ is defined by

$$S_\mu = \{z \in D : \lim_{\varepsilon\to 0} \|K_\mu\|_{L^\infty(D(z,\varepsilon))} = \infty\}. \tag{6.8}$$

Obviously that the set S_μ is closed relatively to the domain D.

Lemma 6.2. *Let be as in Lemma 6.1 hold and let f_μ be a ring solution of (1.1). Suppose that the singular set S_μ is of H-length zero for $H = \{h_{z_0}(r)\}_{z_0\in S_\mu}$ with*

$$h_{z_0}(r)\ = \exp\Big(-\frac{2\pi}{\omega_{z_0}(r)}\Big), \qquad z_0 \in S_\mu,\ r \in (0, \delta(z_0))\ , \tag{6.9}$$

and

$$\omega_{z_0}(\varepsilon) = \frac{1}{I_{z_0}^2(\varepsilon)} \int\limits_{A(\varepsilon)} K_\mu^T(z, z_0) \cdot \psi_{z_0,\varepsilon}^2(|z - z_0|)\, dx\, dy. \tag{6.10}$$

Then every homeomorphic ACL solution f of (1.1) *has the representation $f = h \circ f_\mu$ for some conformal mapping h in $f_\mu(D)$.*

Proof. If $L_H(S_\mu) = 0$, then $S'_\mu = f_\mu(S_\mu)$ is of length zero Lemma 4.2. Consequently, S'_μ does not locally disconnect $f(D)$, see e.g. [61], and hence $G = D\setminus S_\mu$ is a domain. The homeomorphisms f and f_μ are locally quasiconformal in the domain G and hence $h = f \circ f_\mu^{-1}$ is conformal in the domain $f_\mu(D) \setminus S'_\mu$. Since S'_μ is of the length zero it is removable for h, i.e., h can be extended to a conformal mapping in $f_\mu(D)$ by the Painleve theorem, see e.g. [5]. □

Theorem 6.1. *Let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L^1_{loc}$. Suppose that every point $z_0 \in D$ has a neighborhood U_{z_0} and a measurable function $Q_{z_0}(z) : U_{z_0} \to [0, \infty]$ such that*

$$K^T_\mu(z, z_0) \le Q_{z_0}(z) \qquad a.e.\ in\ U_{z_0} \tag{6.11}$$

and that, for some $\delta(z_0) > 0$,

$$\int\limits_0^{\delta(z_0)} \frac{dt}{tq_{z_0}(t)} = \infty \tag{6.12}$$

where $q_{z_0}(t)$ is the mean of $Q_{z_0}(z)$ over the circle $|z - z_0| = t$. Let f_μ be a ring solution of (1.1).

If the singular set S_μ has H-length zero for $H = \{h_{z_0}(r)\}_{z_0 \in S_\mu}$ where

$$h_{z_0}(r) = \exp\left(-\int\limits_r^{\delta(z_0)} \frac{dt}{tq_{z_0}(t)}\right), \qquad z_0 \in S_\mu,\ r \in (0, \delta(z_0)), \tag{6.13}$$

then every homeomorphic ACL solution f of (1.1) has the representation $f = h \circ f_\mu$ for some conformal mapping h in $f_\mu(D)$.

Proof. Theorem 6.1 follows from Lemma 6.2 with

$$\psi_{z_0,\varepsilon}(t) \equiv \psi_{z_0}(t) := \begin{cases} 1/[tq_{z_0}(t)], & t \in (0, \varepsilon_0), \\ 0, & \text{otherwise} \end{cases} \tag{6.14}$$

where $\varepsilon_0 = \delta(z_0)$ because

$$\int\limits_{\varepsilon < |z - z_0| < \varepsilon_0} Q(z) \cdot \psi^2_{z_0}(|z - z_0|)\, dx\, dy = 2\pi \int\limits_\varepsilon^{\varepsilon_0} \psi_{z_0}(t)\, dt. \tag{6.15}$$

□

Corollary 6.2. *Let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L^1_{loc}$. Suppose that every point $z_0 \in D$ has a neighborhood U_{z_0} where (6.11) holds with a function $Q_{z_0}(z)$ of finite mean oscillation at z_0 in the variable z. Suppose also that the singular set of S_μ is of H-length zero for $H = \{h_{z_0}(r)\}_{z_0 \in S_\mu}$,*

$$h_{z_0}(r) = \left(\log \frac{\delta(z_0)}{r}\right)^{-\beta(z_0)}, \qquad z_0 \in S_\mu,\ r \in (0, \delta(z_0)), \tag{6.16}$$

where $\delta(z_0) < \mathrm{dist}(z_0, \partial D)$ *and* $2\beta(z_0) = (q(z_0) + 6d(z_0))^{-1}$, $q(z_0)$ *is the mean value of* $Q_{z_0}(z)$ *over* $D(z_0, \delta(z_0)/2)$ *and* $d(z_0)$ *is the maximal dispersion of* $Q_{z_0}(z)$ *in* $D(z_0, \delta(z_0)/2)$. *Let* f_μ *be a ring solution of* (1.1).

Then every homeomorphic ACL solution f *of (1.1) has the representation* $f = h \circ f_\mu$ *for some a conformal mapping* h *in* $f_\mu(D)$.

Corollary 6.2 follows immediately from Lemmas 6.2 and 2.1.

Remark 6.3. In view of Remark 2.1, if the condition

$$Q^*(z_0) := \overline{\lim_{\varepsilon \to 0}} \fint_{D(z_0,\varepsilon)} Q_{z_0}(z)\, dx\, dy < \infty \qquad (6.17)$$

holds for all $z_0 \in D$, then one may take $\beta(z) = \gamma / Q^*(z)$ in (6.16) for any $\gamma < 1/26$.

On the base of Lemma 6.2, for every existence theorem in Section 5, one can formulate a corresponding uniqueness theorem in the spirit of Theorem 6.1.

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