Український математичний вісник
Том 12 **(2015)**, № 1, 67 – 84

Large deviations for random evolutions with independent increments in a scheme of Lévy approximation

Igor V. Samoilenko

(*Presented by V. S. Koroliuk*)

Abstract. Asymptotic analysis of the problem of large deviations for random evolutions with independent increments in the circuit of the Lévy approximation is carried out. Large deviations for random evolutions in the circuit of the Lévy approximation are determined by the exponential generator for a jump process with independent increments.

2010 MSC. 60J55, 60B10, 60F17, 60K10.

Key words and phrases. Lévy approximation, nonlinear exponential generator, Markov process, process with independent increments, piecewise deterministic Markov process, singular perturbation..

1. Introduction

We conduct the asymptotic analysis of the problem of large deviations for random evolutions with independent increments and switching in the circuit of the Lévy approximation. To apply the Lévy approximation scheme, we use the nonlinear generator approach instead of the classical cumulant method useful in the case of an average scheme. The solution of the large deviations problem is determined by the exponential generator for a jump process with independent increments.

The theory of large deviations arose in work [3] and deals with the asymptotic estimations of probabilities of rare events. The main problem in the large deviations theory is to construct the rate functional estimating the probabilities of rare events. The method used in the majority of classical works is based on a change of the measure and the application of a variational formula to the cumulant of the process under study. Different aspects and applications of this problem were studied by many

Received 27.11.2013

ISSN 1810 – 3200.

mathematicians. Since we discuss Markov processes with independent increments, it is natural to refer to the fundamental works [4, 11, 12, 28].

Another approach arose in works [2, 13] and was applied to the large deviations problem in [10]. It is based on the asymptotic analysis of the nonlinear Hamilton–Jacobi equation corresponding to the process under study. Then the solution of the limit nonlinear Hamilton–Jacobi equation is given by a variational formula that defines the rate functional of the pre-limit process. The main problem here is to prove the uniqueness of the solution of the limit nonlinear equation.

The technical problems connected with the application of this last method to different classes of Markov problems are solved in book [9]. The basic idea is the following:

Let $\mathbf{L}$ be the generator of a Markov process $x(t), t \geq 0$, defined on a standard state space $(E, \mathcal{E})$ (i.e., E is a Polish space and $\mathcal{E}$ its Borel σ-algebra). It has a dense domain $\mathcal{D}(\mathbf{L}) \subseteq \mathcal{B}_E$ that contains continuous functions with continuous derivatives. Here $\mathcal{B}_E$ is a Banach space of real-valued finite test-functions $\varphi(x) \in E$, endowed by the norm: $\|\varphi\| := \sup_{x\in E} |\varphi(x)|$.

Unlike the classical martingale characterization of Markov processes (see [6])

$$\mu_t = \varphi(x(t)) - \varphi(x(0)) - \int_0^t \mathbf{L}\varphi(x(s))\, ds, \tag{1.1}$$

the large deviations theory is based on the exponential martingale characterization (see [8] and [9, Ch. 1]):

$$\widetilde{\mu}_t = \exp\{\varphi(x(t)) - \varphi(x(0)) - \int_0^t \mathbf{H}\varphi(x(s))\, ds\} \tag{1.2}$$

is a martingale.

Here the exponential nonlinear operator

$$\mathbf{H}\varphi(x) := e^{-\varphi(x)}\mathbf{L}e^{\varphi(x)}, \quad e^{\varphi(x)} \in \mathcal{D}(\mathbf{L})$$

is the Hamiltonian associated with the Markov process.

If $\varphi(x)$ is bounded away from zero, then the equivalence of martingales (1.1) and (1.2) follows from the observations:

Proposition 1.1 (see [6, p. 66]). *Let $x(t)$ and $y(t)$ be real-valued, right-continuous, $\{\mathcal{F}_t\}$-adapted processes. Suppose that, for each t, $\inf_{s\leq t} x(s) > 0$. Then*

$$\mu(t) = x(t) - \int_0^t y(s)\, ds$$

is an $\{\mathcal{F}_t\}$*-local martingale if and only if*

$$\widetilde{\mu}(t) = x(t)exp\left\{-\int\limits_0^t \frac{y(s)}{x(s)}\,ds\right\} \textit{ is an}\{\mathcal{F}_t\}\textit{-local martingale.}$$

We may assume that the domain $\mathcal{D}(\mathbf{L})$ contains constants, and if $\varphi(x) \in \mathcal{D}(\mathbf{L})$, then there exists a constant c such that $\varphi(x) + c \in \mathcal{D}(\mathbf{L})$ is positive and bounded away from zero.

The solution of the large deviations problem for a scaled Markov process $x_\varepsilon(t),\ t \geq 0,\ \varepsilon \to 0+$ consists in the verification of the large deviations principle. The large deviations principle is satisfied if there exists a lower semicontinuous function $I : E \to [0, \infty)$ such that, for each open set A,

$$\liminf_{\varepsilon \to 0} \varepsilon \ln \mathbf{P}\{x_\varepsilon(t) \in A\} \geq -\inf_{x \in A} I(x),$$

and, for each closed set B,

$$\limsup_{\varepsilon \to 0} \varepsilon \ln \mathbf{P}\{x_\varepsilon(t) \in B\} \leq -\inf_{x \in B} I(x).$$

I is called the rate function for the large deviations principle.

The problem of large deviations is solved in four stages ([9, Ch. 2]):

1) Verification of the convergence of the exponential (nonlinear) generator $\mathbf{H}^\varepsilon$ to the limit exponential (nonlinear) generator $\mathbf{H}$;

2) Verification of the exponential tightness of the pre-limit Markov processes. Convergence of the semigroups corresponding to $\mathbf{H}^\varepsilon$ and exponential tightness of the pre-limit Markov processes give the large deviations principle in $\mathbf{D}_E[0, \infty)$;

3) Verification of the comparison principle for the limit exponential generator, showing that the semigroups corresponding to $\mathbf{H}^\varepsilon$ really converge to the unique semigroup corresponding to $\mathbf{H}$;

4) Construction of a variational representation for the limit exponential generator that gives the rate function.

Stages 2)–4) were realized in [9] under rather general conditions for the exponential generators corresponding to the processes with independent increments. Namely, the verification of exponential tightness for the solutions of the martingale problem (1.2) was discussed in Ch. 4 on pages 67–71; the verification of the comparison principle with partial differential equations (PDEs) analysis for the limit exponential generator of the view $\mathbf{H}^0\varphi(u) = H^0(\varphi'(u))$ (see formula (4.4) below) was made in Ch. 9 on pages 172–179; finally, the construction of a variational representation for the limit exponential generator of view (4.4) may be found in Ch. 10

on page 202. The methods used there are similar for the Lévy approximation and small diffusion schemes discussed below. Thus, our aim is only to realize stage 1) of this program in the case of the Lévy approximation scheme, as soon as it demands a quite different scaling of the process.

Some of the stages are also presented in book [12], where the large deviations problem was studied, using the cumulant of the process with independent increments.

Remark 1.1. The cumulant and the exponential generator are obviously connected. Really, the generator of a Markov process may be presented in the form (see, e.g., [26])

$$\mathbf{L}\varphi(x) = \int\limits_{\mathbb{R}} e^{\lambda x} a(\lambda)\overline{\varphi}(\lambda) d\lambda,$$

where $a(\lambda)$ is the cumulant of the process, $\overline{\varphi}(\lambda) = \int_{\mathbb{R}} e^{-\lambda x}\varphi(x)dx$.

The inverse transformation gives

$$\int\limits_{\mathbb{R}} e^{-\lambda x}\mathbf{L}\varphi(x)dx = a(\lambda)\overline{\varphi}(\lambda).$$

Let us rewrite

$$\int\limits_{\mathbb{R}} e^{-\lambda x}\mathbf{L}\varphi(x)dx = \int\limits_{\mathbb{R}} e^{-\lambda x} a(\lambda)\varphi(x)dx.$$

Taking

$$e^{-\lambda x}\varphi(x) =: \widetilde{\varphi}(x),$$

we obtain

$$\int\limits_{\mathbb{R}} e^{-\lambda x}\mathbf{L}e^{\lambda x}\widetilde{\varphi}(x)dx = \int\limits_{\mathbb{R}} a(\lambda)\widetilde{\varphi}(x)dx.$$

Thus,

$$e^{-\lambda x}\mathbf{L}e^{\lambda x} = a(\lambda),$$

or, using the exponential generator,

$$\mathbf{H}\varphi_0(x) = a(\lambda), \text{ where } \varphi_0(x) = \lambda x.$$

The Lévy approximation scheme was proposed by V. S. Koroliuk and N. Limnios (see [15, Ch. 9] for examples and possible applications) for the asymptotic analysis of random evolutions. The basic idea of the Lévy approximation scheme is that the jump values of a stochastic system are split into two parts: small jumps taking values with probabilities close to one and big jumps taking values with probabilities tending to zero together with the series parameter $\delta \to 0$. So, in the Lévy approximation principle, the probabilities (or the intensities) of the jumps are normalized by the series parameter $\delta > 0$. This characteristic of the Lévy approximation scheme is defined by Lévy approximation conditions (see Section 3).

In this model, the big jumps are rare events, so we suppose that the Lévy approximation scheme is a natural media to apply the theory of large deviations. The main problem in the case of the Lévy approximation scheme is to choose correct scalings both for time and intensity of jumps (see Section 2).

We should also note that the majority of models discussed in [9] *a priori* contain a diffusion term as a part of the pre-limit process (see, e.g., the model by M. Freidlin and A. Wentzel ([12, Ch. 3,4])).

A model, where the small diffusion term is not defined *ad hoc*, but appears only in the limit generator, was developed by A. Mogulski [21] (see also Ch. 10, pp. 202–204 in [9]). This effect is reached due to the appropriate scaling of the process with independent increments. The exhaustive analysis of this model with different types of scalings may also be found in [17]. We may see that, under Lévy approximation conditions, the pre-limit generator (4.2)–(4.3), that defines the scaled random evolution with independent increments and switching, does not contain a diffusion part *a priori*, but the limit generator (4.4) has the small diffusion term (see Section 4).

Random evolutions with switching were also studied in [9, Ch. 11] by the classical methods of averaging and homogenization. This approach arose in works [19, 23] and involves perturbed PDEs operators and perturbed test functions. Recent books [24,27] include the large bibliography on this problem. The nonlinear case may also be found in work [7]. This approach is important for the infinite-dimensional state space models. But in this case, a lot of additional problems appear: correct description of the functional space for the solutions, a domain of infinitesimal operators, etc.

We use the generators of Markov processes with a locally compact vector state space (see [15] for more details). Similar methods for the average scheme were used in [22] in the case of the large deviations problem for stochastic additive functionals with switching. This simplifies the analysis because the generators are defined for all bounded measurable functions. We lose generality, but may present obvious algorithms for the verification of convergence conditions and the calculation of the limit generators. This approach is important for finite-dimensional models such as random evolutions in $\mathbb{R}^d$, queuing theory, etc.

The paper consists of five parts. In Section 2, we define the process of random evolutions with independent increments and its scaling by two small series parameters. The conditions of the Lévy approximation are introduced in Section 3. In Section 4, we prove the main result. Finally, the proofs of three auxiliary lemmas are presented in Section 5.

2. Basic definitions

Let $x(t), t \geq 0$ be a Markov switching process on the standard state space $(E, \mathcal{E})$ (here again, E is a Polish space, and $\mathcal{E}$ is its Borel σ-algebra) defined by the generator

$$Q\varphi(x) = q(x) \int_{\mathbf{E}} [\varphi(y) - \varphi(x)] P(x, dy), \quad x \in E,\ \varphi(u) \in \mathcal{B}_E, \tag{2.1}$$

where $q(x), x \in E$, is the intensity of jumps function of $x(t), t \geq 0$; $P(x, dy)$ is the transition kernel of the embedded Markov chain $x_n,\ n \geq 0$ defined by $x_n = x(\tau_n),\ n \geq 0$, with $0 = \tau_0 \leq \tau_1 \leq \cdots \leq \tau_n \leq \cdots$ is the jump times of $x(t),\ t \geq 0$.

The corresponding counting process of jumps is

$$\nu(t) := \max\{k \geq 0 : \tau_k \leq t\}.$$

The Markov processes with independent increments $\eta(t; x), t \geq 0, x \in E$, are modulated by the process $x(t)$. Thus, the transition probabilities are generated by the Markov semigroup

$$\Gamma_t(x)\varphi(u) := \mathbf{E}[\varphi(\eta(t; x)) | \eta(0; x) = u], \quad u \in \mathbb{R}\ x \in E.$$

The family (indexed by x) of the corresponding generators is the following:

$$\Gamma(x)\varphi(u) = \int_{\mathbb{R}} [\varphi(u + v) - \varphi(u)] \Gamma(dv; x), \quad \varphi(u) \in \mathcal{B}_{\mathbb{R}},\ x \in E,$$

where the intensity kernel $\Gamma(dv; x)$ satisfies the boundedness property: $\Gamma(\mathbb{R}; x) \in \mathbb{R}_+$.

The random evolutions with independent increments (see Ch. 1 in book [15]) are defined by:

$$\xi(t) = \xi_0 + \int_0^t \eta(ds; x(s)), \quad t \geq 0. \tag{2.2}$$

Remark 2.1. Definition (2.2) may be rewritten in the following form:

$$\xi(t) = \xi_0 + \sum_{k=1}^{\nu(t)} [\eta(\tau_{k+1}, x_k) - \eta(\tau_k, x_k)] + \eta(t - \tau_{\nu(t)}, x(t)). \tag{2.3}$$

The increment of the random evolution on an interval between jumps of the switching Markov process $\xi(\tau_k + t) - \xi(\tau_k)$ is defined by the process with independent increments $\eta(t - \tau_k, x(t))$.

We see that the random evolution is independent of the switching process, but consists of the parts of trajectories of the Markov processes with independent increments $\eta(t; x)$, that are indexed by the switching process.

The processes of the type (2.3) may be applied, for instance, in the queuing theory (see [1]). For such problems, the Lévy approximation scheme models systems with the rare appearance of large information batches.

The random evolution (2.2) is characterized by the generator of a two-component Markov process $\xi(t), x(t),\ t \geq 0$ (see [15, Ch. 2])

$$\mathbf{L}\varphi(u, x) = Q\varphi(u, \cdot)(x) + \Gamma(x)\varphi(\cdot, x)(u).$$

The basic assumption about the switching Markov process is the following:

C1: The Markov process $x(t),\ t \geq 0$, is uniformly ergodic with the stationary distribution $\pi(A),\ A \in \mathcal{E}$.

Remark 2.2. Uniform ergodicity means the following:

Let Π be the projector onto the null-subspace of the reducible-invertible operator Q defined in (2.1):

$$\Pi\varphi(x) = \int_E \pi(dx)\varphi(x).$$

A Markov process $x(t),\ t \geq 0$, is called uniformly ergodic if, for the semigroup P_t generated by this process, the following limit exists:

$$\lim_{t \to \infty} P_t = \Pi \neq 0$$

in the uniform operator topology (see [5, 16] for details).

The following relation is true

$$Q\Pi = \Pi Q = 0.$$

The potential operator ([15, Ch. 1])

$$R_0 := \Pi - (Q + \Pi)^{-1} = (\Pi - Q)^{-1} - \Pi$$

has the following property:

$$QR_0 = R_0 Q = \Pi - I. \tag{2.4}$$

For a uniformly ergodic Markov process with the semigroup $P_t,\ t \geq 0$, the potential operator R_0 is a bounded operator and may be also defined by

$$R_0 := \int_0^\infty (P_t - \Pi) dt.$$

Remark 2.3. It follows from relation (2.4) that, under the solvability condition

$$\Pi\psi = 0,$$

the Poisson equation

$$Q\varphi = \psi$$

has the unique solution

$$\varphi = -R_0\psi,$$

when $\Pi\varphi = 0$.

The exponential operator in the series scheme with a small series parameter $\varepsilon \to 0 (\varepsilon > 0)$ has the form (see, e.g., [18]):

$$\mathbf{H}^\varepsilon \varphi(x) := e^{-\varphi(x)/\varepsilon} \varepsilon \mathbf{L}^\varepsilon e^{\varphi(x)/\varepsilon},$$

where the operators $\mathbf{L}^\varepsilon, \varepsilon > 0$, define some Markov processes $\zeta^\varepsilon(t),\ t \geq 0,\ \varepsilon > 0$ in the series scheme. The test-functions $\varphi(x)$ are real-valued and finite. In our case, the Markov processes $\zeta^\varepsilon(t),\ t \geq 0,\ \varepsilon > 0$ are two-component scaled Markov processes $\xi^\delta_\varepsilon(t), x(t),\ t \geq 0,\ \varepsilon, \delta > 0$.

The correct scaling of the random evolution (2.2) by small series parameters in the *Lévy approximation scheme* is not a trivial problem itself and is different from the *small diffusion approximation scheme* (see [18]). We use two small parameters: ε normalizing time and the range of jumps, and δ normalizing the intensity of big and small jumps. The order of scaling is conditioned by the proofs of Lemmas 4.1, 4.2. Thus, the scaling is the following:

$$\xi_\varepsilon^\delta(t) = \xi_\varepsilon^\delta(0) + \int_0^t \eta_\varepsilon^\delta(ds; x(s/\varepsilon^3)),\ t \geq 0,$$

$$\eta_\varepsilon^\delta(t; x) = \varepsilon\eta^\delta(t/\varepsilon^3; x).$$

To prove the main theorem, we use the solution of a singular perturbation problem with two small series parameters.

Remark 2.4. In work [18], V. S. Koroliuk considered a singular perturbation problem with one small series parameter to study the large deviations for the random evolutions with independent increments in the asymptotically small diffusion scheme.

The method of two small parameters was firstly proposed in [25] for a scheme of Poisson approximation.

3. Lévy approximation conditions

C2: *Lévy approximation.* The family of the processes with independent increments $\eta^\delta(t; x)$, $x \in E$, $t \geq 0$ satisfies the Lévy approximation conditions:

LA1 Approximation of the first two moments:

$$a_\delta(x) = \int_{\mathbb{R}} v\Gamma^\delta(dv; x) = \delta a_1(x) + \delta^2[a(x) + \theta_a^\delta(x)],$$

and

$$c_\delta(x) = \int_{\mathbb{R}} v^2\Gamma^\delta(dv; x) = \delta^2[c(x) + \theta_c^\delta(x)],$$

where

$$\sup_{x\in E} |a_1(x)| \leq a_1 < +\infty, \sup_{x\in E} |a(x)| \leq a < +\infty,$$

$$\sup_{x\in E} |c(x)| \leq c < +\infty.$$

LA2 Asymptotic representation of the intensity kernel

$$\Gamma_g^\delta(x) = \int_{\mathbb{R}} g(v)\Gamma^\delta(dv;x) = \delta^2[\Gamma_g(x) + \theta_g^\delta(x)]$$

for all $g \in C_3(\mathbb{R})$ — measure-determining class of functions which are real-valued, bounded, and such that $\varphi(u)/u^2 \to 0, |u| \to 0$ (see [14]), $\Gamma_g(x)$ is a finite kernel

$$|\Gamma_g(x)| \leq \Gamma_g \quad \text{(a constant depending on } g\text{)}.$$

The kernel $\Gamma^0(dv;x)$ is defined on the measure-determining class of functions $C_3(\mathbb{R})$ by the relation

$$\Gamma_g(x) = \int_{\mathbb{R}} g(v)\Gamma^0(dv;x), \quad g \in C_3(\mathbb{R}).$$

The negligible terms $\theta_a^\delta, \theta_c^\delta, \theta_g^\delta$ satisfy the condition

$$\sup_{x\in E} |\theta_\cdot^\delta(x)| \to 0, \quad \delta \to 0.$$

LA3 The balance condition:

$$\int_E \pi(dx)a_1(x) = 0.$$

C3: *Uniform square integrability*:

$$\lim_{c\to\infty} \sup_{x\in E} \int_{|v|>c} v^2\Gamma^0(dv;x) = 0.$$

C4: *Exponential finiteness*:

$$\int_{\mathbb{R}} e^{p|v|}\Gamma^\delta(dv;x) < \infty, \quad \forall\, p \in \mathbb{R}.$$

Example 3.1. A simple example of the process with similar characteristics is the following. For the process α, let us have:

$$\mathbf{P}\{\alpha = b\} = \delta^2 p,$$

$$\mathbf{P}\{\alpha = \delta a_1 + \delta^2 b_1\} = 1 - \delta^2 p.$$

Then condition **LA1** is true:

$$\mathbf{E}\alpha = \delta a_1 + \delta^2(bp + b_1) + o(\delta^2),$$

$$\mathbf{E}\alpha^2 = \delta^2(b^2 p + a_1^2) + o(\delta^2).$$

4. Main result

The scaled random evolution with independent increments

$$\xi_\varepsilon^\delta(t) = \xi_\varepsilon^\delta(0) + \int_0^t \eta_\varepsilon^\delta(ds; x(s/\varepsilon^3)), \quad t \geq 0, \tag{4.1}$$

$$\eta_\varepsilon^\delta(t; x) := \varepsilon \eta^\delta(t/\varepsilon^3; x)$$

is defined by the generator of the two-component Markov process $\xi_\varepsilon^\delta(t)$, $x(t)$, $t \geq 0$

$$\mathbf{L}_\varepsilon^\delta \varphi(u, x) = \varepsilon^{-3} Q\varphi(u, \cdot)(x) + \Gamma_\varepsilon^\delta(x)\varphi(\cdot, x)(u), \tag{4.2}$$

where

$$\Gamma_\varepsilon^\delta(x)\varphi(u) = \varepsilon^{-3} \int_{\mathbb{R}} [\varphi(u + \varepsilon v) - \varphi(u)]\Gamma^\delta(dv; x), \quad x \in E. \tag{4.3}$$

By $C^3(\mathbb{R})$, we denote the space of continuous bounded functions with continuous bounded derivatives up to the third degree.

Theorem 4.1. *Let conditions **C1–C4** hold for the family of scaled processes with independent increments $\eta_\varepsilon^\delta(t; x)$.*

Then the exponential generator associated with the scaled random evolution (4.1)

$$\mathbf{H}^{\varepsilon,\delta}\varphi_\varepsilon^\delta(u) := e^{-\varphi_\varepsilon^\delta/\varepsilon} \varepsilon \mathbf{L}_\varepsilon^\delta e^{\varphi_\varepsilon^\delta/\varepsilon}$$

converges to the limit exponential generator, when $\varepsilon, \delta \to 0+,\ \varepsilon^{-1}\delta \to 1$.

The limit exponential generator has the view ($\varphi(u) \in C^3(\mathbb{R})$):

$$\mathbf{H}^0\varphi(u) = (\widetilde{a} - \widetilde{a}_0)\varphi'(u) + \frac{1}{2}\sigma^2(\varphi'(u))^2 + \int_{\mathbb{R}} [e^{v\varphi'(u)} - 1]\widetilde{\Gamma}^0(dv), \tag{4.4}$$

$$\widetilde{a} = \Pi a(x) = \int_E \pi(dx)a(x), \widetilde{a}_0 = \Pi a_0(x) = \int_E \pi(dx)a_0(x),$$

$$a_0(x) = \int_{\mathbb{R}} v\Gamma^0(dv; x), \widetilde{c} = \Pi c(x) = \int_E \pi(dx)c(x),$$

$$\widetilde{c}_0 = \Pi c_0(x) = \int_E \pi(dx)c_0(x), c_0(x) = \int_{\mathbb{R}} v^2\Gamma^0(dv; x),$$

$$\sigma^2 = (\widetilde{c} - \widetilde{c}_0) + 2 \int_E \pi(dx) a_1(x) R_0 a_1(x),$$

$$\widetilde{\Gamma}^0(v) = \Pi \Gamma^0(v; x) = \int_E \pi(dx) \Gamma^0(v; x).$$

Remark 4.1. Large deviations for random evolutions in the Lévy approximation scheme are determined by the exponential generator for a jumping process with independent increments. The large deviations problem for such type of processes was studied in book [12, Ch. 3,4].

Remark 4.2. The limit exponential generator in the Euclidean space $\mathbb{R}^d$, $d > 1$ can be represented in the following form:

$$\mathbf{H}^0 \varphi(u) = \sum_{k=1}^{d} (\widetilde{a}_k - \widetilde{a}_k^0) \varphi_k' + \frac{1}{2} \sum_{k,r=1}^{d} \sigma_{kr} \varphi_k' \varphi_r' + \int_{\mathbb{R}^d} [e^{v \varphi'(u)} - 1] \widetilde{\Gamma}^0(dv),$$

$$\varphi_k' := \partial \varphi(u) / \partial u_k, \quad 1 \le k \le d.$$

Here $\sigma^2 = [\sigma_{kr}; 1 \le k, r \le d]$ is the variance matrix.

The last exponential generator may be extended on the space of absolutely continuous functions (see [9])

$$C_b^1(R^d) = \Big\{ \varphi : \ \exists \lim_{|u| \to \infty} \varphi(u) = \varphi(\infty), \ \lim_{|u| \to \infty} \varphi'(u) = 0 \Big\}.$$

Proof. The limit transition for the exponential nonlinear generator of a random evolution is performed on the perturbed test-functions

$$\varphi_\varepsilon^\delta(u, x) = \varphi(u) + \varepsilon \ln[1 + \delta \varphi_1(u, x) + \delta^2 \varphi_2(u, x)],$$

where $\varphi(u) \in C^3(\mathbb{R})$. Thus, relation (4.2) yields

$$\begin{aligned} \mathbf{H}^{\varepsilon,\delta} \varphi_\varepsilon^\delta &= e^{-\varphi_\varepsilon^\delta / \varepsilon} \varepsilon \mathbf{L}_\varepsilon^\delta e^{\varphi_\varepsilon^\delta / \varepsilon} = e^{-\varphi_\varepsilon^\delta / \varepsilon} [\varepsilon^{-2} Q + \varepsilon \Gamma_\varepsilon^\delta(x)] e^{\varphi_\varepsilon^\delta / \varepsilon} \\ &= e^{-\varphi / \varepsilon} [1 + \delta \varphi_1 + \delta^2 \varphi_2]^{-1} [\varepsilon^{-2} Q + \varepsilon \Gamma_\varepsilon^\delta(x)] e^{\varphi / \varepsilon} [1 + \delta \varphi_1 + \delta^2 \varphi_2]. \end{aligned}$$

To see the asymptotic behavior of the last exponential generator, we use Lemmas 4.1 and 4.2 (see Section 5 for the proofs).

Lemma 4.1. *The exponential generator*

$$H_Q^\varepsilon \varphi_\varepsilon^\delta(u, x) = e^{-\varphi_\varepsilon^\delta / \varepsilon} \varepsilon^{-2} Q e^{\varphi_\varepsilon^\delta / \varepsilon} \tag{4.5}$$

has the following asymptotic representation:

$$H_Q^\varepsilon \varphi_\varepsilon^\delta = \varepsilon^{-1} Q \varphi_1 + Q \varphi_2 - \varphi_1 Q \varphi_1 + \theta_Q^{\varepsilon,\delta}(x), \tag{4.6}$$

where $\sup_{x \in E} |\theta_Q^{\varepsilon,\delta}(x)| \to 0, \varepsilon, \delta \to 0.$

Lemma 4.2. *Under conditions **C2–C4**, the exponential generator*

$$H_\Gamma^{\varepsilon,\delta}(x)\varphi_\varepsilon^\delta(u,x) = e^{-\varphi_\varepsilon^\delta/\varepsilon}\varepsilon\Gamma_\varepsilon^\delta(x)e^{\varphi_\varepsilon^\delta/\varepsilon} \tag{4.7}$$

has the asymptotic representation

$$H_\Gamma^{\varepsilon,\delta}(x)\varphi_\varepsilon^\delta = H_\Gamma(x)\varphi(u) + \varepsilon^{-1}a_1(x)\varphi'(u) + \theta_\Gamma^{\varepsilon,\delta}(x),$$

where

$$H_\Gamma(x)\varphi(u) = (a(x) - a_0(x))\varphi'(u) + \frac{1}{2}(c(x) - c_0(x))(\varphi'(u))^2 +$$

$$\int_{\mathbb{R}} [e^{v\varphi'(u)} - 1]\Gamma^0(dv; x), \tag{4.8}$$

and $\sup_{x\in E} |\theta_\Gamma^{\varepsilon,\delta}(x)| \to 0, \varepsilon, \delta \to 0.$

From (4.5) and (4.7), we see that

$$\mathbf{H}^{\varepsilon,\delta}\varphi_\varepsilon^\delta = H_Q^\varepsilon\varphi_\varepsilon^\delta(u,x) + H_\Gamma^{\varepsilon,\delta}(x)\varphi_\varepsilon^\delta(u,x).$$

Using Lemmas 4.1 and 4.2, we obtain the following asymptotic representation:

$$\mathbf{H}^{\varepsilon,\delta}\varphi_\varepsilon^\delta = \varepsilon^{-1}[Q\varphi_1 + a_1(x)\varphi'(u)] + Q\varphi_2 - \varphi_1 Q\varphi_1 + H_\Gamma(x)\varphi(u) + h^{\varepsilon,\delta}(x),$$

where $h^{\varepsilon,\delta}(x) = \theta_Q^{\varepsilon,\delta}(x) + \theta_\Gamma^{\varepsilon,\delta}(x)$.

The equations that give the solution of the singular perturbation problem for the reducibly invertible operator Q (see [15, Ch. 1]) take the form

$$Q\varphi_1 + a_1(x)\varphi'(u) = 0,$$
$$Q\varphi_2 - \varphi_1 Q\varphi_1 + H_\Gamma(x)\varphi(u) = \mathbf{H}^0\varphi(u).$$

Due to the balance condition **LA3**, the first equation yields

$$\varphi_1(u,x) = R_0 a_1(x)\varphi'(u), \quad Q\varphi_1(u,x) = -a_1(x)\varphi'(u).$$

After the substitution to the second equation, we have

$$Q\varphi_2 + a_1(x)R_0 a_1(x)(\varphi'(u))^2 + H_\Gamma(x)\varphi(u) = \mathbf{H}^0\varphi(u)$$

From the solvability condition, we have

$$\mathbf{H}^0\varphi(u) = \Pi H_\Gamma(x)\Pi\varphi(u) + \Pi a_1(x)R_0 a_1(x)\mathbf{1}(\varphi'(u))^2,$$

where **1** is the unit vector.

Now, using (4.8), we finally obtain (4.4).

The negligible term $h^{\varepsilon,\delta}(x)$ may be found explicitly, using the solution of the Poisson equation (see Remark 2.3 in [15] for details)

$$\varphi_2(u,x) = R_0\widetilde{H}(x)\varphi(u) - R_0a_1(x)R_0a_1(x)\mathbf{1}(\varphi'(u))^2,$$

$$\widetilde{H}(x) := \mathbf{H}^0 - H_\Gamma(x).$$

The theorem is proved. □

5. Proofs of the auxiliary lemmas

Proof of Lemma 4.1. We have

$$\begin{aligned} H^\varepsilon_Q\varphi^\delta_\varepsilon &= e^{-\varphi/\varepsilon}[1+\delta\varphi_1+\delta^2\varphi_2]^{-1}\varepsilon^{-2}Qe^{\varphi/\varepsilon}[1+\delta\varphi_1+\delta^2\varphi_2] \\ &= \left[1-\delta\varphi_1+\delta^2\frac{\varphi_1^2+\delta\varphi_1\varphi_2-\varphi_2}{1+\delta\varphi_1+\delta^2\varphi_2}\right][\delta\varepsilon^{-2}Q\varphi_1+\delta^2\varepsilon^{-2}Q\varphi_2] \\ &= \delta\varepsilon^{-2}Q\varphi_1+\delta^2\varepsilon^{-2}Q\varphi_2-\delta^2\varepsilon^{-2}\varphi_1Q\varphi_1+\theta^{\varepsilon,\delta}_Q(x), \end{aligned}$$

where

$$\theta^{\varepsilon,\delta}_Q(x) = \delta^3\varepsilon^{-2}\frac{\varphi_1^2+\delta\varphi_1\varphi_2-\varphi_2}{1+\delta\varphi_1+\delta^2\varphi_2}[Q\varphi_1+\delta Q\varphi_2]-\delta^3\varepsilon^{-2}\varphi_1Q\varphi_2.$$

By the limit condition $\varepsilon^{-1}\delta\to1,\ \varepsilon,\delta\to0$, we finally obtain (4.6).

Lemma 4.1 is proved. □

Proof of Lemma 4.2. We have

$$\begin{aligned} H^{\varepsilon,\delta}_\Gamma(x)\varphi^\delta_\varepsilon &= e^{-\varphi/\varepsilon}[1+\delta\varphi_1+\delta^2\varphi_2]^{-1}\varepsilon\Gamma^\delta_\varepsilon(x)e^{\varphi/\varepsilon}[1+\delta\varphi_1+\delta^2\varphi_2] \\ = e^{-\varphi/\varepsilon}&\left[1-\delta\varphi_1+\delta^2\frac{\varphi_1^2+\delta\varphi_1\varphi_2-\varphi_2}{1+\delta\varphi_1+\delta^2\varphi_2}\right]\Big[\varepsilon\Gamma^\delta_\varepsilon(x)e^{\varphi/\varepsilon}+\varepsilon\delta\Gamma^\delta_\varepsilon(x)e^{\varphi/\varepsilon}\varphi_1 \\ &\qquad +\varepsilon\delta^2\Gamma^\delta_\varepsilon(x)e^{\varphi/\varepsilon}\varphi_2\Big]. \end{aligned}$$

Thus,

$$H^{\varepsilon,\delta}_\Gamma(x)\varphi^\delta_\varepsilon = H^{\varepsilon,\delta}_\Gamma(x)\varphi(u)+e^{-\varphi/\varepsilon}\varepsilon\delta\{\Gamma^\delta_\varepsilon(x)e^{\varphi/\varepsilon}\varphi_1-\varphi_1\Gamma^\delta_\varepsilon(x)e^{\varphi/\varepsilon}\}+\widetilde{\theta}^{\varepsilon,\delta}_\Gamma(x), \tag{5.1}$$

where

$$\begin{aligned} \widetilde{\theta}^{\varepsilon,\delta}_\Gamma(x) &= \varepsilon\delta^2[e^{-\varphi/\varepsilon}\Gamma^\delta_\varepsilon(x)e^{\varphi/\varepsilon}\varphi_2-e^{-\varphi/\varepsilon}\varphi_1\Gamma^\delta_\varepsilon(x)e^{\varphi/\varepsilon}\varphi_1] \\ &+\varepsilon\delta^2\frac{\varphi_1^2+\delta\varphi_1\varphi_2-\varphi_2}{1+\delta\varphi_1+\delta^2\varphi_2}[e^{-\varphi/\varepsilon}\Gamma^\delta_\varepsilon(x)e^{\varphi/\varepsilon}+e^{-\varphi/\varepsilon}\delta\Gamma^\delta_\varepsilon(x)e^{\varphi/\varepsilon}\varphi_1 \end{aligned}$$

$$+ e^{-\varphi/\varepsilon}\delta^2\Gamma^\delta_\varepsilon(x)e^{\varphi/\varepsilon}\varphi_2] - \varepsilon\delta^3 e^{-\varphi/\varepsilon}\varphi_1\Gamma^\delta_\varepsilon(x)e^{\varphi/\varepsilon}\varphi_2.$$

Now let us rewrite $H^{\varepsilon,\delta}_\Gamma(x)\varphi(u)$ in (5.1), using generator (4.3). We have

$$H^{\varepsilon,\delta}_\Gamma(x)\varphi(u) = \varepsilon^{-2}\int_{\mathbb{R}}[e^{\Delta_\varepsilon\varphi(u)} - 1]\Gamma^\delta(dv;x),$$

where

$$\Delta_\varepsilon\varphi(u) := \varepsilon^{-1}[\varphi(u+\varepsilon v) - \varphi(u)].$$

We may rewrite it in a following way:

$$H^{\varepsilon,\delta}_\Gamma(x)\varphi(u) = \varepsilon^{-2}\int_{\mathbb{R}}\Big[e^{\Delta_\varepsilon\varphi(u)} - 1 - \Delta_\varepsilon\varphi(u) - \frac{1}{2}(\Delta_\varepsilon\varphi(u))^2\Big]\Gamma^\delta(dv;x)$$
$$+ \varepsilon^{-2}\int_{\mathbb{R}}\Big[\Delta_\varepsilon\varphi(u) + \frac{1}{2}(\Delta_\varepsilon\varphi(u))^2\Big]\Gamma^\delta(dv;x).$$

It is easy to see that the function $\psi^\varepsilon_u(v) = e^{\Delta_\varepsilon\varphi(u)} - 1 - \Delta_\varepsilon\varphi(u) - \frac{1}{2}(\Delta_\varepsilon\varphi(u))^2$ belongs to the class $C_3(\mathbb{R})$. Really,

$$\psi^\varepsilon_u(v)/v^2 \to 0, v \to 0.$$

In addition, this function is continuous and bounded for every ε under the condition that $\varphi(u)$ is bounded. Moreover, the function $\psi^\varepsilon_u(v)$ is bounded uniformly by u under conditions **C3, C4** and if $\varphi'(u)$ is bounded.

Thus, by condition **C2**, we have

$$H^{\varepsilon,\delta}_\Gamma(x)\varphi(u) = \varepsilon^{-2}\delta^2\int_{\mathbb{R}}\Big[e^{\Delta_\varepsilon\varphi(u)} - 1 - \Delta_\varepsilon\varphi(u) - \frac{1}{2}(\Delta_\varepsilon\varphi(u))^2\Big]\Gamma^0(dv;x)$$
$$+ \varepsilon^{-2}\int_{\mathbb{R}}\Big[\Delta_\varepsilon\varphi(u) - v\varphi'(u) - \varepsilon\frac{v^2}{2}\varphi''(u)\Big]\Gamma^\delta(dv;x)$$
$$+ \varepsilon^{-2}\delta a_1(x)\varphi'(u) + \varepsilon^{-2}\delta^2 a(x)\varphi'(u) + \varepsilon^{-1}\delta^2 c(x)\varphi''(u)$$
$$+ \varepsilon^{-2}\int_{\mathbb{R}}\Big[\frac{1}{2}(\Delta_\varepsilon\varphi(u))^2 - \frac{v^2}{2}(\varphi'(u))^2\Big]\Gamma^\delta(dv;x) + \varepsilon^{-2}\delta^2\frac{1}{2}c(x)(\varphi'(u))^2.$$

The functions in the second and third integrals obviously belong to $C_3(\mathbb{R})$. Using the Taylor formula for the test-functions $\varphi(u) \in C^3(\mathbb{R})$ and condition **LA2**, we obtain

$$H^{\varepsilon,\delta}_\Gamma(x)\varphi(u) = \varepsilon^{-2}\delta^2\int_{\mathbb{R}}\Big[e^{v\varphi'(u)} - 1 - v\varphi'(u) - \frac{v^2}{2}(\varphi'(u))^2\Big]\Gamma^0(dv;x)$$

$$+\varepsilon^{-2}\delta^2\int_{\mathbb{R}}\Big(e^{v\varphi'(u)}\varepsilon\frac{v^2}{2}\varphi''(\widetilde{u})-\varepsilon\frac{v^2}{2}\varphi''(\widetilde{u})-\varepsilon^2\frac{v^4}{8}(\varphi''(\widetilde{u}))^2\Big)\Gamma^0(dv;x)$$

$$+\varepsilon^{-2}\delta^2\int_{\mathbb{R}}\varepsilon^2\frac{v^3}{3!}\varphi'''(\widetilde{u})\Gamma^0(dv;x)+\varepsilon^{-2}\delta a_1(x)\varphi'(u)+\varepsilon^{-2}\delta^2 a(x)\varphi'(u)$$

$$+\varepsilon^{-1}\delta^2 c(x)\varphi''(u)+\varepsilon^{-2}\delta^2\int_{\mathbb{R}}\varepsilon^2\frac{v^4}{4}(\varphi''(\widetilde{u}))^2\Gamma^0(dv;x)$$

$$+\varepsilon^{-2}\delta^2\frac{1}{2}c(x)(\varphi'(u))^2.$$

By the limit condition $\varepsilon^{-1}\delta\to 1$, we finally have

$$H^{\varepsilon,\delta}_\Gamma(x)\varphi(u)=H_\Gamma(x)\varphi(u)+\varepsilon^{-1}a_1(x)\varphi'(u)+\theta^{\varepsilon,\delta}(x), \tag{5.2}$$

where

$$H_\Gamma(x)\varphi(u)=(a(x)-a_0(x))\varphi'(u)+\frac{1}{2}(c(x)-c_0(x))(\varphi'(u))^2$$
$$+\int_{\mathbb{R}}[e^{v\varphi'(u)}-1]\Gamma^0(dv;x),$$

and $\sup_{x\in E}|\theta^{\varepsilon,\delta}(x)|\to 0,\ \varepsilon,\delta\to 0$.

To finish the proof, we have to show that the term in the braces in (5.1) is equal to 0. Thus, we use the following lemma.

Lemma 5.1.

$$\Gamma^\delta_\varepsilon(x)e^{\varphi(u)/\varepsilon}\varphi_1(u,x)=\varphi_1(u,x)\Gamma^\delta_\varepsilon(x)e^{\varphi(u)/\varepsilon}+(\varepsilon\delta)^{-1}\widehat{\theta}^{\varepsilon,\delta}_\Gamma(x),$$

where the negligible term

$$\sup_{x\in E}|\widehat{\theta}^{\varepsilon,\delta}_\Gamma(x)|\to 0,\quad \varepsilon,\delta\to 0.$$

Proof. Really, by (4.3), we have

$$\Gamma^\delta_\varepsilon(x)e^{\varphi(u)/\varepsilon}\varphi_1(u,x)=\varepsilon^{-3}\int_{\mathbb{R}}[e^{\varphi(u+\varepsilon v)/\varepsilon}\varphi_1(u+\varepsilon v,x)$$
$$-e^{\varphi(u)/\varepsilon}\varphi_1(u,x)]\Gamma^\delta(dv;x)=\varphi_1(u,x)\Gamma^\delta_\varepsilon(x)e^{\varphi(u)/\varepsilon}$$
$$+(\varepsilon\delta)^{-1}\left[\varphi_1'(u,x)\varepsilon^{-1}\delta\int_{\mathbb{R}}e^{\varphi(u+\varepsilon v)/\varepsilon}v\Gamma^\delta(dv;x)\right].$$

Let us estimate the last integral. As soon as the function $\varphi(u)$ is bounded, we have, for fixed ε,

$$\int_{\mathbb{R}} e^{\varphi(u+\varepsilon v)/\varepsilon} v\Gamma^{\delta}(dv;x) < e^{C} \int_{\mathbb{R}} v\Gamma^{\delta}(dv;x)$$
$$= \delta e^{C}[a_1(x) + \delta a(x) + \delta\theta_a^{\delta}(x)].$$

Thus, we see that the last term is negligible, as $\varepsilon, \delta \to 0$.

Lemma 5.1 is proved. □

Applying equality (5.2) and Lemma 5.1 to (5.1), we finally obtain

$$H_{\Gamma}^{\varepsilon,\delta}(x)\varphi_{\varepsilon}^{\delta} = H_{\Gamma}(x)\varphi(u) + \varepsilon^{-1}a_1(x)\varphi'(u) + \theta_{\Gamma}^{\varepsilon,\delta}(x),$$

where $\sup_{x\in E} |\theta_{\Gamma}^{\varepsilon,\delta}(x)| \to 0, \ \varepsilon, \delta \to 0$.

Lemma 4.2 is proved. □

References

[1] V. V. Anisimov, *Switching Processes in Queueing Models*, London: ISTE and Hoboken, New York: Wiley, 2008.

[2] J. D. Cole, *On a quasi-linear parabolic equation occuring in aerodynamics* // Quart. Appl. Math., **9** (1951), 225–236.

[3] H. Cramér, *Sur un nouveau théoreme-limite de la théorie des probabilités* // Acta. Sci. et Ind., **736** (1938), 5–23.

[4] M. D. Donsker, S. R. S. Varadhan, *Asymptotic evaluation of certain Markov process expectations for large time, I,II,III* // Comm. Pure Appl. Math., **27** (1975), 1–47, **28** (1977), 279–301, **29** (1976), 389–461.

[5] J. L. Doob, *Stochastic processes. Reprint of the 1953 original*, New York: Wiley, 1990.

[6] S. N. Ethier, T. G. Kurtz, *Markov Processes: Characterization and Convergence*, New York: Wiley, 1985.

[7] L. C. Evans, *The perturbed test function method for viscosity solutions of nonlinear PDE* // Proc. Roy. Soc. Edinburgh Sect. A., **111** (1989), Nos. 3–4, 359–375.

[8] J. Feng, *Martingale problems for large deviations of Markov processes* // Stochastic Processes and their Applications, **81** (1999), 165–216.

[9] J. Feng, T. G. Kurtz, *Large Deviation for Stochastic Processes*, Providence, RI: Amer. Math. Soc., 2006.

[10] W. H. Fleming and P. E. Souganidis, *A PDE approach to some large deviations problems. Nonlinear systems of partial differential equations in applied mathematics, Part 1*, (Santa Fe, N.M., 1984), 441–447, *Lectures in Appl. Math.*, **23**, Providence, RI: Amer. Math. Soc., 1986.

[11] M. I. Freidlin, *The averaging principle and theorems on large deviations* // *Russian Math. Surveys* **33** (1978), 117–176.

[12] M. J. Freidlin, A. D. Wentzel, *Random Perturbations of Dynamical Systems*, New York: Springer, 1998.

[13] E. Hopf, *The partial differential equation* $u_t + uu_x = \mu u_{xx}$ // Comm. Pure Appl. Math., **3** (1950), 201–230.

[14] J. Jacod, A. N. Shiryaev, *Limit Theorems for Stochastic Processes*, Berlin: Springer, 2003.

[15] V. S. Koroliuk, N. Limnios, *Stochastic Systems in Merging Phase Space*, Singapore: VSP, 2005.

[16] V. S. Koroliuk, A. F. Turbin, *Mathematical Foundations of the State Lumping of Large Systems*, Dordrecht: Kluwer, 1993.

[17] V. S. Koroliuk, *Random evolutions with locally independent increments on increasing time intervals* // J. of Math. Sci., **179** (2011), No. 2, 273–289.

[18] V. S. Koroliuk, *Large deviation problems for Markov random evolution with independent increments in the scheme of asymptotically small diffusion // Commun. in Statistics — Theory and Methods*, **40(19-20)** (2011), 3385–3395.

[19] T. G. Kurtz, *A limit theorem for perturbed operator semigroups with applications to random evolutions* // J. Functional Analysis, **12** (1973), 55–67.

[20] T. G. Kurtz, *Convergence of sequences of semigroups of nonlinear operators with an application to gas kinetics* // Trans. Amer. Math. Soc., **186** (1973), 259–272.

[21] A. A. Mogulskii, *Large deviations for processes with independent increments* // Ann. Probab., **21** (1993), 202–215.

[22] A. Oprisan, A. Korzeniowski, *Large deviations for ergodic processes in split spaces* // Dynamic Systems and Applications, **18** (2009), No. 3–4, 589–604.

[23] G. C. Papanicolaou, D. Stroock, S. R. S. Varadhan, *Martingale approach to some limit theorems* // Duke Univ. Math. Ser., **21** (1977), 202–215.

[24] G. A. Pavliotis and A. M. Stuart, *Multiscale Methods: Averaging and Homogenization*, New York: Springer Science+Business Media, 2008.

[25] I. V. Samoilenko, *Large deviation for random evolutions with independent increments in the scheme of Poisson approximation // Prob. Theor. and Math. Stat.*, **85** (2011), 107–114.

[26] A. V. Skorokhod, *Asymptotic Methods in the Theory of Stochastic Differential Equations*, Providence, RI: Amer. Math. Soc., 1989.

[27] A. V. Skorokhod, F. C. Hoppensteadt, and H. Salehi, *Random Perturbation Methods with Applications in Science and Engineering.* New York: Springer, 2002.

[28] A. D. Wentzel, *Rough limit theorems on large deviations for Markov stochastic processes, I, II, III* // Theory Probab. Appl., **21** (1976), 227–242, **24** (1979), 675–672, **27** (1982), 215–234.

CONTACT INFORMATION

Igor V. Samoilenko Taras Shevchenko National University of Kyiv,
64, Volodymyrs'ka Str.,
Kyiv 01601,
Ukraine
E-Mail: isamoil@i.ua