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On integral conditions in the mapping theory

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Abstract. It is established interconnections between various integral conditions that play an important role in the theory of space mappings and in the theory of degenerate Beltrami equations in the plane.

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1. Introduction

In the theory of space mappings and in the Beltrami equation theory in the plane, the integral conditions of the following type

$$\int_0^1 \frac{dr}{rq^{\lambda}(r)} = \infty, \qquad \lambda \in (0,1], \tag{1.1}$$

are often met with the function Q is given say in the unit ball $\mathbb{B}^n = \{x \in \mathbb{R}^n : |x| < 1\}$ and $q(r)$ is the average of the function $Q(x)$ over the sphere $|x| = r$, see e.g. [2, 4, 6, 7, 14–17, 20–23, 27, 28].

On the other hand, in the theory of mappings called quasiconformal in the mean, conditions of the type

$$\int_{\mathbb{B}^n} \Phi(Q(x))\, dx < \infty \tag{1.2}$$

are standard for various characteristics Q of these mappings, see e.g. [1, 3, 8, 10–13, 17–19, 26]. Here dx corresponds to the Lebsgue measure in $\mathbb{R}^n$, $n \geq 2$.

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In this connection, we establish interconnections between integral conditions on the function Φ and between (1.1) and (1.2). More precisely, we give a series of equivalent conditions for Φ under which (1.2) implies (1.1). It makes possible to apply many known results formulated under the condition (1.1) to the theory of the mean quasiconformal mappings in space as well as to the theory of the degenerate Beltrami equations in the plane.

2. On some equivalent integral conditions

In this section we establish equivalence of integral conditions, see also Section 3 for one more related condition.

Further we use the following notion of the inverse function for monotone functions. For every non-decreasing function $\Phi : [0,\infty] \to [0,\infty]$, the *inverse function* $\Phi^{-1} : [0,\infty] \to [0,\infty]$ can be well defined by setting

$$\Phi^{-1}(\tau) = \inf_{\Phi(t)\geq\tau} t. \tag{2.1}$$

As usual, here inf is equal to ∞ if the set of $t \in [0,\infty]$ such that $\Phi(t) \geq \tau$ is empty. Note that the function Φ^{-1} is non-decreasing, too.

Remark 2.1. Immediately by the definition it is evident that

$$\Phi^{-1}(\Phi(t)) \leq t \quad \forall\, t \in [0,\infty] \tag{2.2}$$

with the equality in (2.2) except intervals of constancy of the function $\Phi(t)$.

Similarly, for every non-increasing function $\varphi : [0,\infty] \to [0,\infty]$, set

$$\varphi^{-1}(\tau) = \inf_{\varphi(t)\leq\tau} t. \tag{2.3}$$

Again, here inf is equal to ∞ if the set of $t \in [0,\infty]$ such that $\varphi(t) \leq \tau$ is empty. Note that the function φ^{-1} is also non-increasing.

Lemma 2.1. *Let $\psi : [0,\infty] \to [0,\infty]$ be a sense-reversing homeomorphism and $\varphi : [0,\infty] \to [0,\infty]$ a monotone function. Then*

$$[\psi\circ\varphi]^{-1}(\tau) = \varphi^{-1}\circ\psi^{-1}(\tau) \quad \forall\, \tau \in [0,\infty] \tag{2.4}$$

and

$$[\varphi\circ\psi]^{-1}(\tau) \leq \psi^{-1}\circ\varphi^{-1}(\tau) \quad \forall\, \tau \in [0,\infty] \tag{2.5}$$

with the equality in (2.5) *except a countable collection of* $\tau \in [0,\infty]$.

Remark 2.2. If ψ is a sense-preserving homeomorphism, then (2.4) and (2.1) are obvious for every monotone function φ. Similar notations and statements also hold for other segments $[a,b]$, where a and $b \in [-\infty,+\infty]$, instead of the segment $[0,\infty]$.

Proof of Lemma 2.1. We first prove (2.4). If φ is non-increasing, then

$$[\psi\circ\varphi]^{-1}(\tau) = \inf_{\psi(\varphi(t))\geq\tau} t = \inf_{\varphi(t)\leq\psi^{-1}(\tau)} t = \varphi^{-1}\circ\psi^{-1}(\tau).$$

Similarly, if φ is non-decreasing, then

$$[\psi\circ\varphi]^{-1}(\tau) = \inf_{\psi(\varphi(t))\leq\tau} t = \inf_{\varphi(t)\geq\psi^{-1}(\tau)} t = \varphi^{-1}\circ\psi^{-1}(\tau).$$

Now, let us prove (2.5) and (2.1). If φ is non-increasing, then applying the substitution $\eta=\psi(t)$ we have

$$\begin{aligned}[\varphi\circ\psi]^{-1}(\tau) &= \inf_{\varphi(\psi(t))\geq\tau} t \\ &= \inf_{\varphi(\eta)\geq\tau}\psi^{-1}(\eta) = \psi^{-1}\Big(\sup_{\varphi(\eta)\geq\tau}\eta\Big) \\ &\leq \psi^{-1}\Big(\inf_{\varphi(\eta)\leq\tau}\eta\Big) = \psi^{-1}\circ\varphi^{-1}(\tau),\end{aligned}$$

i.e., (2.5) holds for all $\tau\in[0,\infty]$. It is evident that here the strict inequality is possible only for a countable collection of $\tau\in[0,\infty]$ because an interval of constancy of φ corresponds to every such τ. Hence (2.1) holds for all $\tau\in[0,\infty]$ if and only if φ is decreasing.

Similarly, if φ is non-decreasing, then

$$\begin{aligned}[\varphi\circ\psi]^{-1}(\tau) &= \inf_{\varphi(\psi(t))\leq\tau} t \\ &= \inf_{\varphi(\eta)\leq\tau}\psi^{-1}(\eta) = \psi^{-1}\Big(\sup_{\varphi(\eta)\leq\tau}\eta\Big) \\ &\leq \psi^{-1}\Big(\inf_{\varphi(\eta)\geq\tau}\eta\Big) = \psi^{-1}\circ\varphi^{-1}(\tau),\end{aligned}$$

i.e., (2.5) holds for all $\tau\in[0,\infty]$ and again the strict inequality is possible only for a countable collection of $\tau\in[0,\infty]$. In the case, (2.1) holds for all $\tau\in[0,\infty]$ if and only if φ is increasing. □

Corollary 2.1. *In particular, if* $\varphi:[0,\infty]\to[0,\infty]$ *is a monotone function and* $\psi=j$ *where* $j(t)=1/t$, *then* $j^{-1}=j$ *and*

$$[j \circ \varphi]^{-1}(\tau) = \varphi^{-1} \circ j(\tau) \quad \forall\, \tau \in [0, \infty] \tag{2.6}$$

i.e.,

$$\varphi^{-1}(\tau) = \Phi^{-1}(1/\tau) \quad \forall\, \tau \in [0, \infty] \tag{2.7}$$

where $\Phi = 1/\varphi$,

$$[\varphi \circ j]^{-1}(\tau) \leq j \circ \varphi^{-1}(\tau) \quad \forall\, \tau \in [0, \infty] \tag{2.8}$$

i.e., the inverse function of $\varphi(1/t)$ *is dominated by* $1/\varphi^{-1}$, *and except a countable collection of* $\tau \in [0, \infty]$

$$[\varphi \circ j]^{-1}(\tau) = j \circ \varphi^{-1}(\tau). \tag{2.9}$$

$1/\varphi^{-1}$ *is the inverse function of* $\varphi(1/t)$ *if and only if the function* φ *is strictly monotone.*

Further, in (2.11) and (2.12), we complete the definition of integrals by ∞ if $\Phi(t) = \infty$, correspondingly, $H(t) = \infty$, for all $t \geq T \in [0, \infty)$. The integral in (2.12) is understood as the Lebesgue–Stieltjes integral and the integrals (2.11) and (2.13)–(2.16) as the ordinary Lebesgue integrals.

Theorem 2.1. *Let* $\Phi : [0, \infty] \to [0, \infty]$ *be a non-decreasing function and let*

$$H(t) = \log \Phi(t). \tag{2.10}$$

Then the equality

$$\int\limits_{\Delta}^{\infty} H'(t)\, \frac{dt}{t} = \infty \tag{2.11}$$

implies the equality

$$\int\limits_{\Delta}^{\infty} \frac{dH(t)}{t} = \infty \tag{2.12}$$

and (2.12) *is equivalent to*

$$\int\limits_{\Delta}^{\infty} H(t)\, \frac{dt}{t^2} = \infty \tag{2.13}$$

for some $\Delta > 0$, *and* (2.13) *is equivalent to every of the equalities:*

$$\int\limits_{0}^{\delta} H\left(\frac{1}{t}\right) dt = \infty \tag{2.14}$$

for some $\delta > 0$,

$$\int\limits_{\Delta_*}^{\infty} \frac{d\eta}{H^{-1}(\eta)} = \infty \tag{2.15}$$

for some $\Delta_* > H(+0)$,

$$\int\limits_{\delta_*}^{\infty} \frac{d\tau}{\tau\Phi^{-1}(\tau)} = \infty \tag{2.16}$$

for some $\delta_* > \Phi(+0)$.

Moreover, (2.11) *is equivalent to* (2.12) *and hence* (2.11)–(2.16) *are equivalent each to other if* Φ *is in addition absolutely continuous. In particular, all the conditions* (2.11)–(2.16) *are equivalent if* Φ *is convex and non-decreasing.*

Proof. The equality (2.11) implies (2.12) because except the mentioned special case

$$\int\limits_{\Delta}^{T} d\Psi(t) \geq \int\limits_{\Delta}^{T} \Psi'(t)\, dt \quad \forall\, T \in (\Delta,\ \infty)$$

where

$$\Psi(t) := \int\limits_{\Delta}^{t} \frac{d\,H(\tau)}{\tau}, \quad \Psi'(t) = \frac{H'(t)}{t},$$

see e.g. Theorem $IV.7.4$ in [25, p. 119], and hence

$$\int\limits_{\Delta}^{T} \frac{d\,H(t)}{t} \geq \int\limits_{\Delta}^{T} H'(t)\ \frac{dt}{t} \quad \forall\, T \in (\Delta,\ \infty)$$

The equality (2.12) is equivalent to (2.13) by integration by parts, see e.g. Theorem III.14.1 in [25, p. 102]. Indeed, again except the mentioned special case, through integration by parts we have

$$\int_{\Delta}^{T} \frac{d\,H(t)}{t} - \int_{\Delta}^{T} H(t)\ \frac{dt}{t^2} = \frac{H(T+0)}{T} - \frac{H(\Delta-0)}{\Delta} \quad \forall T \in (\Delta,\ \infty)$$

and, if

$$\liminf_{t\to\infty}\ \frac{H(t)}{t}\ < \infty,$$

then the equivalence of (2.12) and (2.13) is obvious. If

$$\lim_{t\to\infty}\ \frac{H(t)}{t}\ = \infty,$$

then (2.13) obviously holds, $\frac{H(t)}{t} \geq 1$ for $t > t_0$ and

$$\int_{t_0}^{T} \frac{d\,H(t)}{t} = \int_{t_0}^{T} \frac{H(t)}{t}\ \frac{d\,H(t)}{H(t)} \geq \log\ \frac{H(T)}{H(t_0)}$$

$$= \log\ \frac{H(T)}{T} + \log\ \frac{T}{H(t_0)} \to \infty$$

as $T \to \infty$, i.e. (2.12) holds, too.

Now, (2.13) is equivalent to (2.14) by the change of variables $t \to 1/t$.

Next, (2.14) is equivalent to (2.15) because by the geometric sense of integrals as areas under graphs of the corresponding integrands

$$\int_{0}^{\delta} \Psi(t)\ dt = \int_{\Psi(\delta)}^{\infty} \Psi^{-1}(\eta)\,d\eta + \delta\cdot\Psi(\delta)$$

where $\Psi(t) = H(1/t)$, and because by Corollary 2.1 the inverse function for $H\,(1/t)$ coincides with $1/H^{-1}$ at all points except a countable collection.

Further, set $\psi(\xi) = \log\xi$. Then $H = \psi\circ\Phi$ and by Lemma 2.1 and Remark 2.2 $H^{-1} = \Phi^{-1}\circ\psi^{-1}$, i.e., $H^{-1}(\eta) = \Phi^{-1}(e^{\eta})$, and by the substitutions $\tau = e^{\eta}$, $\eta = \log\ \tau$ we have the equivalence of (2.15) and (2.16).

Finally, (2.11) and (2.12) are equivalent if Φ is absolutely continuous, see e.g. Theorem IV.7.4 in [25, p. 119]. □

3. Connection with one more condition

In this section we establish useful connection of the conditions of the Zorich–Lehto–Miklyukov–Suvorov type (3.23) further with one of the integral conditions from the last section.

Recall that a function $\Phi : [0,\infty] \to [0,\infty]$ is called *convex* if

$$\Phi(\lambda t_1 + (1-\lambda)t_2) \le \lambda\ \Phi(t_1)\ +\ (1-\lambda)\ \Phi(t_2)$$

for all t_1 and $t_2 \in [0,\infty]$ and $\lambda \in [0,1]$.

In what follows, $\mathbb{B}^n$ denotes the unit ball in the space $\mathbb{R}^n$, $n \ge 2$,

$$\mathbb{B}^n = \{\ x \in \mathbb{R}^n :\ |x|\ <\ 1\ \}. \tag{3.1}$$

Lemma 3.1. *Let $Q : \mathbb{B}^n \to [0,\infty]$ be a measurable function and let $\Phi : [0,\infty] \to [0,\infty]$ be a non-decreasing convex function. Then*

$$\int\limits_0^1 \frac{dr}{rq(r)} \ge \frac{1}{n} \int\limits_{\lambda_n M}^{\infty} \frac{d\tau}{\tau \Phi^{-1}(\tau)} \tag{3.2}$$

where $q(r)$ is the average of the function $Q(x)$ over the sphere $|x| = r$,

$$M = \int\limits_{\mathbb{B}^n} \Phi(Q(x))\, dx, \tag{3.3}$$

$\lambda_n = e/\Omega_n$ and Ω_n is the volume of the unit ball in $\mathbb{R}^n$.

Remark 3.1. In other words,

$$\int\limits_0^1 \frac{dr}{rq(r)} \ge \frac{1}{n} \int\limits_{eM_n}^{\infty} \frac{d\tau}{\tau \Phi^{-1}(\tau)} \tag{3.4}$$

where

$$M_n :\ =\ \frac{M}{\Omega_n}\ = ⨍\limits_{\mathbb{B}^n} \Phi(Q(x))\, dx \tag{3.5}$$

is the mean value of the function $\Phi \circ Q$ over the unit ball. Recall also that by the Jacobi formula

$$\Omega_n = \frac{\omega_{n-1}}{n} = \frac{2}{n} \cdot \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$$

where ω_{n-1} is the area of the unit sphere in $\mathbb{R}^n$, Γ is the well-known gamma function of Euler, $\Gamma(t+1) = t\Gamma(t)$. For $n = 2$ we have that $\Omega_n = \pi$, $\omega_{n-1} = 2\pi$, and, thus, $\lambda_2 = e/\Omega_2 < 1$. Consequently, we have in the case $n = 2$ that

$$\int_0^1 \frac{dr}{rq(r)} \geq \frac{1}{2} \int_M^\infty \frac{d\tau}{\tau\Phi^{-1}(\tau)}. \tag{3.6}$$

In the general case we have that

$$\Omega_{2m} = \frac{\pi^m}{m!}, \quad \Omega_{2m+1} = \frac{2(2\pi)^m}{(2m+1)!!},$$

i.e., $\Omega_n \to 0$ and, correspondingly, $\lambda_n \to \infty$ as $n \to \infty$.

Proof. Note that the result is obvious if $M = \infty$. Hence we assume further that $M < \infty$. Consequently, we may also assume that $\Phi(t) < \infty$ for all $t \in [0, \infty)$ because in the contrary case $Q \in L^\infty(\mathbb{B}^n)$ and then the left hand side in (3.2) is equal to ∞. Moreover, we may assume that $\Phi(t)$ is not constant (because in the contrary case $\Phi^{-1}(\tau) \equiv \infty$ for all $\tau > \tau_0$ and hence the right hand side in (3.2) is equal to 0), $\Phi(t)$ is (strictly) increasing, convex and continuous in a segment $[t_*, \infty]$ for some $t_* \in [0, \infty)$ and

$$\Phi(t) \equiv \tau_0 = \Phi(0) \quad \forall\, t \ \in [0, t_*]. \tag{3.7}$$

Next, setting

$$H(t) := \log\ \Phi(t), \tag{3.8}$$

we see by Proposition 2.1 and Remark 2.2 that

$$H^{-1}(\eta) = \Phi^{-1}(e^\eta), \quad \Phi^{-1}(\tau) = H^{-1}(\log\ \tau). \tag{3.9}$$

Thus, we obtain that

$$q(r) = H^{-1}\Big(\log \frac{h(r)}{r^n}\Big) = H^{-1}\Big(n \log \frac{1}{r} + \log\ h(r)\Big) \quad \forall\, r \in R_* \tag{3.10}$$

where $h(r) : = r^n\Phi(q(r))$ and $R_* = \{r \in (0,1) : \ q(r) \ > \ t_*\}$. Then also

$$q(e^{-s}) = H^{-1}\left(ns + \log\ h(e^{-s})\right) \quad \forall\, s \ \in S_* \tag{3.11}$$

where $S_* = \{s \in (0, \infty) : \ q(e^{-s}) \ > \ t_*\}$.

Now, by the Jensen inequality

$$\int_0^{\infty} h(e^{-s})\,ds = \int_0^1 h(r)\,\frac{dr}{r} = \int_0^1 \Phi(q(r))\; r^{n-1}dr$$
$$\leq \int_0^1 \left(\fint_{S(r)} \Phi(Q(x))\; d\mathcal{A} \right) r^{n-1}dr = \frac{M}{\omega_{n-1}} \qquad (3.12)$$

where we use the mean value of the function $\Phi \circ Q$ over the sphere $S(r) = \{x \in \mathbb{R}^n : |x| = r\}$ with respect to the area measure. Then

$$|T| = \int_T ds \leq \frac{\Omega_n}{\omega_{n-1}} = \frac{1}{n} \qquad (3.13)$$

where $T = \{\ s \in (0,\infty) :\ \ h(e^{-s}) > M_n\}$, $M_n = M/\Omega_n$. Let us show that

$$q(e^{-s}) \leq H^{-1}\,(ns\ +\ \log\ M_n) \quad \forall\, s \in (0,\infty) \setminus T_* \qquad (3.14)$$

where $T_* = T \cap S_*$. Note that $(0,\infty) \setminus T_* = [(0,\infty) \setminus S_*] \cup [(0,\infty) \setminus T] = [(0,\infty) \setminus S_*] \cup [S_* \setminus T]$. The inequality (3.14) holds for $s \in S_* \setminus T$ by (3.11) because H^{-1} is a non-decreasing function. Note also that by (3.7)

$$e^{ns} M_n = e^{ns} \fint_{\mathbb{B}^n} \Phi(Q(x))\,dx > \Phi(0) = \tau_0 \quad \forall\, s \in (0,\infty). \qquad (3.15)$$

Hence, since the function Φ^{-1} is non-decreasing and $\Phi^{-1}(\tau_0) = t_*$, we have by (3.9) that

$$t_* < \Phi^{-1}\,(M_n\; e^{ns}) = H^{-1}\,(ns\ +\ \log\ M_n) \quad \forall\, s \in (0,\infty). \qquad (3.16)$$

Consequently, (3.14) holds for $s \in (0,\infty) \setminus S_*$, too. Thus, (3.14) is true.

Since H^{-1} is non-decreasing, we have by (3.13) and (3.14) that

$$\int_0^1 \frac{dr}{rq(r)} = \int_0^{\infty} \frac{ds}{q(e^{-s})}$$
$$\geq \int_{(0,\infty)\setminus T_*} \frac{ds}{H^{-1}(ns+\Delta)} \geq \int_{|T_*|}^{\infty} \frac{ds}{H^{-1}(ns+\Delta)}$$
$$\geq \int_{\frac{1}{n}}^{\infty} \frac{ds}{H^{-1}(ns+\Delta)} = \frac{1}{n} \int_{1+\Delta}^{\infty} \frac{d\eta}{H^{-1}(\eta)} \qquad (3.17)$$

where $\Delta = \log M_n$. Note that $1 + \Delta = \log\ eM_n = \log\ \lambda_n M$. Thus,

$$\int\limits_0^1 \frac{dr}{rq(r)} \geq \frac{1}{n} \int\limits_{\log \lambda_n M}^{\infty} \frac{d\eta}{H^{-1}(\eta)} \tag{3.18}$$

and, after the replacement $\eta = \log\ \tau$, we obtain (3.2). □

Corollary 3.1. *Let $Q : \mathbb{B}^n \to [0, \infty]$ be a measurable function and let $\Phi : [0, \infty] \to [0, \infty]$ be a non-decreasing convex function. Then*

$$\int\limits_0^1 \frac{dr}{rq^{\lambda}(r)} \geq \frac{1}{n} \int\limits_{\lambda_n M_*}^{\infty} \frac{d\tau}{\tau \Phi^{-1}(\tau)} \quad \forall\, \lambda \in (0, 1) \tag{3.19}$$

where $q(r)$ is the average of the function $Q(x)$ over the sphere $|x| = r$,

$$M_* = \int\limits_{\mathbb{B}^n} \Phi(Q_*(x))\, dx, \tag{3.20}$$

Q_* is the lower cut-off function of Q, i.e., $Q_*(x) = 1$ if $Q(x) < 1$ and $Q_*(x) = Q(x)$ if $Q(x) \geq 1$.

Indeed, let $q_*(r)$ be the average of the function $Q_*(x)$ over the sphere $|x| = r$. Then $q(r) \leq q_*(r)$ and, moreover, $q_*(r) \geq 1$ for all $r \in (0, 1)$. Thus, $q^{\lambda}(r) \leq q_*^{\lambda}(r) \leq q_*(r)$ for all $\lambda \in (0, 1)$ and hence by Lemma 3.1 applied to the function $Q_*(x)$ we obtain (3.19).

Theorem 3.1. *Let $Q : \mathbb{B}^n \to [0, \infty]$ be a measurable function such that*

$$\int\limits_{\mathbb{B}^n} \Phi(Q(x))\, dx < \infty \tag{3.21}$$

where $\Phi : [0, \infty] \to [0, \infty]$ is a non-decreasing convex function such that

$$\int\limits_{\delta_0}^{\infty} \frac{d\tau}{\tau \Phi^{-1}(\tau)} = \infty \tag{3.22}$$

for some $\delta_0 > \tau_0 = \Phi(0)$. Then

$$\int\limits_0^1 \frac{dr}{rq(r)} = \infty \tag{3.23}$$

where $q(r)$ is the average of the function $Q(x)$ over the sphere $|x| = r$.

Remark 3.2. Note that (3.22) implies that

$$\int\limits_{\delta}^{\infty} \frac{d\tau}{\tau\Phi^{-1}(\tau)} = \infty \tag{3.24}$$

for every $\delta \in [0, \infty)$ but (3.24) for some $\delta \in [0, \infty)$, generally speaking, does not imply (3.22). Indeed, for $\delta \in [0, \delta_0)$, (3.22) evidently implies (3.24) and, for $\delta \in (\delta_0, \infty)$, we have that

$$0 \le \int\limits_{\delta_0}^{\delta} \frac{d\tau}{\tau\Phi^{-1}(\tau)} \le \frac{1}{\Phi^{-1}(\delta_0)} \ \log \frac{\delta}{\delta_0} \ < \ \infty \tag{3.25}$$

because Φ^{-1} is non-decreasing and $\Phi^{-1}(\delta_0) > 0$. Moreover, by the definition of the inverse function $\Phi^{-1}(\tau) \equiv 0$ for all $\tau \in [0, \tau_0]$, $\tau_0 = \Phi(0)$, and hence (3.24) for $\delta \in [0, \tau_0)$, generally speaking, does not imply (3.22). If $\tau_0 > 0$, then

$$\int\limits_{\delta}^{\tau_0} \frac{d\tau}{\tau\Phi^{-1}(\tau)} = \infty \qquad \forall\, \delta \in [0, \tau_0) \tag{3.26}$$

However, (3.26) gives no information on the function $Q(x)$ itself and, consequently, (3.24) for $\delta < \Phi(0)$ cannot imply (3.23) at all.

By (3.24) the proof of Theorem 3.1 is reduced to Lemma 3.1.

Corollary 3.2. *If* $\Phi : [0, \infty] \to [0, \infty]$ *is a non-decreasing convex function and* Q *satisfies the condition* (3.21)*, then every of the conditions* (2.11)–(2.16) *implies* (3.23)*.*

Moreover, if in addition $\Phi(1) < \infty$ *or* $q(r) \ge 1$ *on a subset of* $(0, 1)$ *of a positive measure, then*

$$\int\limits_{0}^{1} \frac{dr}{rq^{\lambda}(r)} = \infty \quad \forall\, \lambda \in (0, 1) \tag{3.27}$$

and also

$$\int\limits_{0}^{1} \frac{dr}{r^{\alpha}q^{\beta}(r)} = \infty \quad \forall\, \alpha \ge 1,\ \beta \in (0, \alpha] \tag{3.28}$$

Proof. First of all, by Theorems 2.1 and 3.1 every of the conditions (2.11)–(2.16) implies (3.23). Now, if $q(r) \geq 1$ on a subset of $(0,1)$ of a positive measure, then also $Q(x) \geq 1$ on a subset of $\mathbb{B}^n$ of a positive measure and, consequently, in view of (3.21) we have that $\Phi(1) < \infty$. Then

$$\int\limits_{\mathbb{B}^n} \Phi(Q_*(x))\, dx < \infty \tag{3.29}$$

where Q_* is the lower cut off function of Q from Corollary 3.1. Thus, by Corollary 3.1 and Remark 3.2 we obtain that every of the conditions (2.11)–(2.16) implies (3.27). Finally, (3.28) follows from (3.27) by the Jensen inequality. □

Remark 3.3. Note that if we have instead of (3.21) the condition

$$\int\limits_{D} \Phi(Q(x))\, dx < \infty \tag{3.30}$$

for some measurable function $Q : D \to [0,\infty]$ given in a domain $D \subset \mathbb{R}^n$, $n \geq 2$, then also

$$\int\limits_{|x-x_0|<r_0} \Phi(Q(x))\, dx < \infty \tag{3.31}$$

for every $x_0 \in D$ and $r_0 < \operatorname{dist}(x_0, \partial D)$ and by Theorem 3.1 and Corollary 3.2, after the corresponding linear replacements of variables, we obtain that

$$\int\limits_{0}^{r_0} \frac{dr}{r q_{x_0}^{\lambda}(r)} = \infty \quad \forall\, \lambda \in (0,1] \tag{3.32}$$

where $q_{x_0}(r)$ is the average of the function $Q(x)$ over the sphere $|x-x_0| = r$.

If D is a domain in the extended space $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$ and $\infty \in D$, then in the neighborhood $|x| > R_0$ of ∞ we may use the condition

$$\int\limits_{|x|>R_0} \Phi(Q(x))\, \frac{dx}{|x|^{2n}} < \infty \tag{3.33}$$

that is equivalent to the condition

$$\int\limits_{|x|<r_0} \Phi(Q'(x))\, dx < \infty \tag{3.34}$$

where $r_0 = 1/R_0$ and $Q'(x) = Q(x/|x|^2)$, i.e. $Q'(x)$ is obtained from $Q(x)$ by the inversion of the independent variable $x \to x/|x|^2$, $\infty \to 0$, with respect to the unit sphere $|x| = 1$.

Thus, by Theorem 3.1 and Corollary 3.2 the condition (3.33) imply the equality

$$\int\limits_{R_0}^{\infty} \frac{dR}{R q_\infty^\lambda(R)} = \infty \quad \forall\, \lambda \in (0,1] \tag{3.35}$$

where $q_\infty(R)$ is the average of Q over the sphere $|x| = R$.

Finally, if D is an unbounded domain in $\mathbb{R}^n$ or a domain in $\overline{\mathbb{R}^n}$ (3.30) should be replaced by the following condition

$$\int\limits_{D} \Phi(Q(x))\; dS(x) < \infty \tag{3.36}$$

where $dS(x) = dx/(1+|x|^2)^n$ is of a cell of the spherical volume. Here the spherical distance

$$s(x,y) \;=\; \frac{|x-y|}{(1+|x|^2)^{\frac{1}{2}}(1+|y|^2)^{\frac{1}{2}}} \quad \text{if} \quad x \,\neq\, \infty \neq y, \tag{3.37}$$

$$s(x,\infty) = \frac{1}{(1+|x|^2)^{\frac{1}{2}}} \quad \text{if} \quad x \,\neq\, \infty.$$

It is easy to see that

$$dS(x) \geq (1+\rho^2)^{-n} dx$$

in every bounded part of D where $|x| < \rho$ and

$$dS(x) \geq 2^{-n} \frac{dx}{|x|^{2n}}$$

in a neighborhood of ∞ where $|x| \geq 1$. Hence the condition (3.36) implies (3.31) as well as (3.33). Thus, under at least one of the conditions (2.12)–(2.16) the condition (3.36) implies (3.32) and (3.35) if the function Φ is convex and non-decreasing.

Recently it was established that the conditions (2.12)–(2.16) are not only sufficient but also necessary for the degenerate Beltrami equations with integral constraints of the type (3.21) on their characteristics to have homeomorphic solutions of the class $W^{1,1}_{loc}$, see [5, 7, 9, 24].

Moreover, the above results will have significant corollaries to the local and boundary behavior of space mappings in various modern classes with integral constrains for dilatations, see e.g. [15].

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