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Application of the Thomas precession to the deformations of a rotating disk

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Abstract. Within the framework of Special Relativity, the recent paper [1] describes the shrinking determined by the rotation of a planar disk, and an apparent paradox emerging from the anisotropic outcoming contraction. In this paper, using the Thomas precession, we obtain the same result in a different way.

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Introduction

This year, 2009, marks the 100th anniversary of the rotating disk paradox. Namely, the problem known as "rotating disk paradox" originates from 1909 paper [2] where an ideally rigid cylinder rotating about its axis of symmetry is considered. It is discussed that the radius R as seen in the laboratory frame is always perpendicular to its motion and should be equal to its value R_0 when stationary. But the circumference $2\pi R$ should appear Lorentz-contracted to a smaller value than at rest, by the Lorentz factor γ. So, a contradiction arises: $R = R_0$ and $R < R_0$. Ehrenfest concludes that motions of extended bodies cannot be Born rigid.

From 1910 many mathematicians and physicists worked on the resolution of this problem, with different approaches. Plank [3] suggests that elastic contraction as a result of the acquired spin velocity of the disk should not be confused with the measurement differences between a stationary and rotating observer. Einstein, evolving from the discussions of

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simultaneity [4] to General Relativity [5] introduced the Equivalence Principle via rotating disk, apparently noticing the underlying non-Euclidean geometry.

Much later, Langevin introduces a frame field corresponding to the family of rotating observers [6]. The concept of moving frames applied to the rotating disk paradox was improved by Weyssenhoff and Rosen. On the other side, in [7] is given a pure kinematical explanation the problem.

The history and an overview of the paradox is given in [8] up to that date, but the relativistically rotating reference frame still continues to be a research topic of interest. It is widely accepted that this topic should be described by tools of differential geometry and therefore the most accepted paradox solution is described by the curved Landau–Lifschitz metric.

In this paper we show that the coefficient of shrinking from [1] can be derived in a different way, using the Thomas precession as presented in [9].

1. Rotating disk and the shrinking coefficient

Consider the disk $C = \{(x, y) \mid x^2 + y^2 = R^2\}$ of radius R in the xOy Euclidean plane. We assume that the disk rotates around the origin in trigonometric sense, with constant angular velocity ω. In this section, we remind of the recent results [1]. In [1] is determined the maximal radius ρ of the moving disk, observed from the inertial system in which the center of the disk rests. The velocity of the periphery points of C is $\omega\rho$, and its periphery of length $2\pi\rho$ is observed under contraction with the coefficient $\sqrt{1 - (\frac{\omega\rho}{c})^2}$. Hence we obtain the equality

$$2\pi R = \frac{2\pi\rho}{\sqrt{1 - (\omega\rho/c)^2}},$$

which implies the dependence $R = \rho\,(1 - (\omega\rho/c)^2)^{-1/2}$,

$$\rho = \frac{R}{\sqrt{1 + (\omega R/c)^2}}. \tag{1.1}$$

We notice that $\rho < \frac{c}{\omega} = \rho_{max}$ (see Fig. 1), hence the boundary radius c/ω can never be achieved. Note also that the dependence (1.1) is true not only for the points of the periphery of the disk, but also for the points which are interior to the disk.

The basic idea of this paper is to deduce the previous equality (1.1) (up to c^{-2}), in a different way, as a consequence of the Thomas precession.

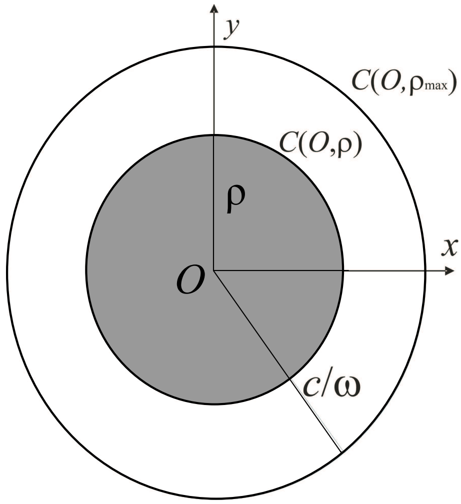

Figure 1: The disk during rotation

This will be made in Section 2. Further we shall present the consequences of (1.1), omitting the technical details.

From Special relativity (SR), it is well known that we observe a contraction of lengths only in the direction of motion, but not orthogonal to the motion. Since the radial direction is orthogonal to the direction of motion, we have apparently a paradox: we still observe a contraction in the radial direction. To explain this paradox we choose two infinitesimally close points A and B in the radial direction (before rotation) and examine their places A' and B' at a chosen moment during the rotation. If O, A' and B' are collinear, then we have a paradox in SR (the rotation is transversal to OB, hence it is supposed to affect differently A and B). So these points should be non-collinear (Fig. 2).

Assume that the coordinates of A' in the xOy plane are (x, y) and the coordinates of B' are $(x + \Delta x, y + \Delta y)$. Let C' be a point on OB', such that the angle $A'C'B'$ is right. Since A' and B' are infinitesimally close, without loss of generality we assume $OA' \cong OC'$. Let us denote $\rho = d(O, A') = \sqrt{x^2 + y^2}$. Using elementary geometrical calculation, it is easy to find that

$$|A'C'| = |x\Delta y - y\Delta x|/\rho, \quad |B'C'| = |x\Delta x + y\Delta y|/\rho. \tag{1.2}$$

Since the distance $|A'C'|$ is observed under contraction with factor $\sqrt{1 - (\omega\rho/c)^2}$, and there is no contraction in the direction $B'C'$, we obtain the following equality

$$|AB|^2 = |B'C'|^2 + \frac{|A'C'|^2}{1 - (\omega\rho/c)^2}. \tag{1.3}$$

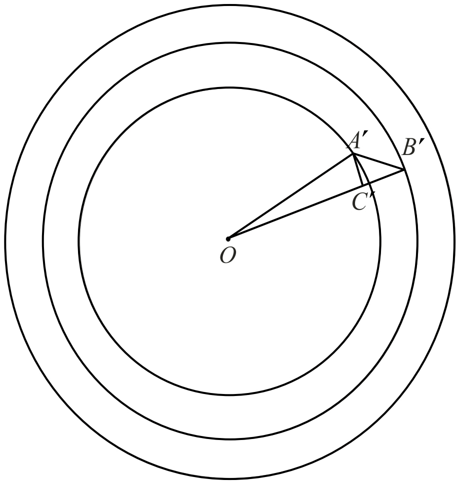

Figure 2: The disk during rotation

On the other side,

$$\frac{y - xy'}{x + yy'} = \frac{\pm\sqrt{3\frac{\omega^2\rho^2}{c^2} - 3\frac{\omega^4\rho^4}{c^4} + \frac{\omega^6\rho^6}{c^6}}}{1 - (\omega\rho/c)^2}, \tag{1.4}$$

where $y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$, i.e. y' is the usual derivative of $y = y(x)$. In polar coordinates (ρ, ϕ), we have

$$\frac{d\phi}{d\rho} = \frac{y'x - y}{\rho(x + yy')},$$

where $\phi = \operatorname{atan} \frac{y}{x}$. Now,

$$\rho\frac{d\phi}{d\rho} = \frac{\pm\sqrt{3\frac{\omega^2\rho^2}{c^2} - 3\frac{\omega^4\rho^4}{c^4} + \frac{\omega^6\rho^6}{c^6}}}{1 - (\omega\rho/c)^2}. \tag{1.5}$$

Hence for the function $\phi(\rho)$ we get

$$\phi(\rho) = \phi_0 - \int_0^\rho \frac{\sqrt{3\frac{\omega^2}{c^2} - 3\frac{\omega^4 z^2}{c^4} + \frac{\omega^6 z^4}{c^6}}}{1 - (\omega\rho/c)^2}\, dz, \tag{1.6}$$

where ϕ_0 is the initial value of ϕ, i.e. $\phi_0 = \phi(0)$.

Notice that if $\rho\omega/c \approx 0$, then we have the following approximation

$$\phi(\rho) \approx \phi_0 - \sqrt{3}\frac{\omega\varrho}{c}. \tag{1.7}$$

The previous results do not contradict the SR fact that there is no shrinking in the direction orthogonal to the moving direction, because the rotating systems are not inertial systems. Notice that in this case:

- the effect of shrinking in radial direction appears because the points in the radial direction have different velocities, and
- the same coefficient of shrinking is observed from both the inertial system and the rotating system, and it essentially differs from the Fitzgerald contraction.

Note that shrinking in the radial direction depends on the distance to the center, and it can be calculated as follows:

From (1.1) we get

$$d\rho = \frac{dR}{\left(\sqrt{1+(\frac{\omega R}{c})^2}\right)^3} = dR\left(\sqrt{1-\left(\frac{\omega\rho}{c}\right)^2}\right)^3,$$

and the required coefficient k of local shrinking on distance R (before rotation), or distance r (during rotating) is equal to

$$k = \frac{1}{\left(\sqrt{1+(\omega R/c)^2}\right)^3} = \left(\sqrt{1-\left(\frac{\omega\rho}{c}\right)^2}\right)^3.$$

Note that this coefficient of shrinking as well as the change $d\phi$ given with the differential equation (1.5) can be applied generally for nonconstant ω and more generally, for particles which arbitrarily move including rotation.

2. Application of the Thomas precession in deducing the shrinking coefficient

Thomas precession is a kinematic effect closely related to the problem of finding Lorentz transformation as a result of two successive non-collinear Lorentz transformations. More precisely, Thomas precession is defined as an additional spatial rotation R_t contained in the Lorentz transformation which represents the resultant of two successive non-collinear Lorentz transformations. This is more obvious for non-collinear Lorentz boosts B_1 and B_2 giving Lorentz transformation L which is not Lorentz boost, but it contains additional rotation R_t. Thomas precession is explicitly given as a vector of angular velocity and is expressed as

$$\vec{\omega}_{Thomas} = -\frac{1}{2}\frac{\vec{v}\times\vec{a}}{c^2}$$

where $\vec{v}$ is the velocity and $\vec{a}$ is the acceleration of a point as seen in the inertial system.

Let us consider a point with small mass m on the periphery of the rotating disk. This point moves with velocity $\vec{v}$ tangent to the circle of the periphery, with centripetal acceleration $\vec{a} = -\vec{\rho}\omega^2$, so the observer from the inertial system observes also the Thomas precession

$$\vec{\omega}_{Thomas} = \frac{1}{2}\frac{\vec{v}\times\vec{\rho}}{c^2}\omega^2. \tag{2.1}$$

Notice that this angular velocity $\vec{\omega}_{Thomas}$ has the opposite direction of the initial angular velocity $\vec{\omega}$, and so the module of the total angular velocity depends on the distance ρ to the center of the body and it is given by

$$\omega_\rho = \omega - \frac{1}{2}\frac{\omega\rho^2}{c^2}\omega^2 = \omega\Big(1 - \frac{(\omega\rho)^2}{2c^2}\Big). \tag{2.2}$$

We give a physical interpretation of this fact. Assume that the disk is not a solid body, but it is soft where cohesive forces completely disappear. In this case it is possible for different particles to move with different angular velocities, which happens according to (2.2). Notice that in this case each particle still preserves its distance to the center of the body; for example the particles on the periphery of the body are located at distance R from the center of the body. Now, assume that the cohesive forces become gradually stronger, until the body becomes a solid one, which means that at the end all particles must rotate with the same angular velocity ω. But, since the energy of each particle must be preserved in that process, the initial velocity of the each particle of the periphery must be equal to the final velocity of the particle, i.e.,

$$R\omega_R = \rho\omega, \tag{2.3}$$

and according to (2.2)

$$R\omega\Big(1 - \frac{(\omega\rho)^2}{2c^2}\Big) = \rho\omega. \tag{2.4}$$

Hence, the equality (1.1) just follows, namely, if the equality (1.1) is represented as the Taylor series with respect to $\omega\rho/c$ and all terms $(\omega\rho/c)^n$ for $n > 2$ are rejected, then we get (2.4). Thus, the formula for the shrinking coefficient holds true.

References

[1] K. Trencevski, V. Balan, Shrinking of rotational configurations and associated inertial forces, *J. of Calcuta Math. Soc.* **1**(3) & 4 165-180 (2005).

[2] P. Ehrenfest, Similar type rotation of rigid bodies and the theory of relativity, *Phys. Zeitschrift* **10** 918-918 (1909).

[3] M. Planck, Similar form rotation and the Lorentz-contraction. *Phys. Zeitschrift* **11** 294-294 (1910).

[4] A. Einstein, About the Ehrenfest's paradox. *Phys. Zeitschrift* **12** 509-510 (1911).

[5] A. Einstein, *The Principle of Relativity*, Dover Publications, New York, (1952).

[6] P. Langevin, Remarques au sujet de la Note de Prunier, *Comptes Rendus Acad. Sci. Paris* **200** 48 (1935)

[7] V. Cantoni, What is wrong with Relativistic Kinematics?, *Il Nuovo Cimento* **57** B 220–223 (1968).

[8] O. Gron, Relativistic description of a rotating disc, *Am. J. of Phys.* **43** (10) 869-876 (1975).

[9] W. Misner, K.S. Thorne, I.A. Wheeler, *Gravitation*, Freeman Eds., San Francisco, 1973.

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