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Expansions of solutions to the equation P_1^2 by algorithms of power geometry

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Abstract. Algorithms of Power Geometry allow to find all power expansions of solutions to ordinary differential equations of a rather general type. Among these, there are Painlevé equations and their generalizations. In the article we demonstrate how to find by these algorithms all power expansions of solutions to the equation P_1^2 at the points $z = 0$ and $z = \infty$. Two levels of the exponential additions to the expansions of solutions near $z = \infty$ are computed. We also describe an algorithm of computation of a basis of a minimal lattice containing a given set.

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1. Introduction

In [1] a hierarchy of the first Painlevé equation was suggested. It can be described by the relation

$$L^n[w] = z, \tag{1.1}$$

where L^n is the Lenard's operator determined by the relation [2]

$$\frac{d}{dz}L^{n+1} = L^n_{zzz} - 4wL^n_z - 2w_zL^n, \qquad L^0[w] = -\frac{1}{2}. \tag{1.2}$$

Assuming that $n = 0$ in (1.2), we have $L^1[w] = w$. In the case $n = 2$, using (1.1), we obtain the first Painlevé equation [3]

$$w_{zz} - 3w^2 = z. \tag{1.3}$$

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If $n = 3$ in (1.1), we obtain the fourth-order equation [1, 2]

$$w_{zzzz} - 10w\, w_{zz} - 5w_z^2 + 10w^3 = z. \tag{1.4}$$

Using $n = 4$, we obtain the sixth-order equation from (1.1)

$$w_{zzzzzz} - 14ww_{zzzz} - 28\, w_z\, w_{zzz} - 21\, w_{zz}^2 + 70\, ww_z^2 - 35\, w^4 = z. \tag{1.5}$$

In general, equation (1.1) has order $2(n-1)$.

Equation (1.4) is used in describing the waves on water [4, 5] and in the Hénon–Heiles model that characterizes the behavior of a star in the mean field of galaxy [6–8].

In papers [9–23], it was shown that equation (1.4) has properties that are typical for the Painlevé equations $P_1 \div P_6$. Equation (1.4) belongs to the class of exactly solvable equations, since it has the Lax pair and many other typical properties of the exactly solvable equations. It does not have the first integral in the polynomial form. It has the Painlevé property [41]. Equation (1.4) seems to determine new transcendental functions just as equations $P_1 \div P_6$, although the rigorous proof of the irreducibility of equation (1.4) remains an open problem. Thereupon the study of all the asymptotic forms and asymptotic expansions of solutions to equation (1.4) is the important stage of the analysis of this equation, as they can indirectly confirm the irreducibility of equation (1.4).

Recently developed methods of Power Geometry [24, 25] allow to obtain algorithmically asymptotic expansions of solutions for a wide class of ordinary differential equations (see also [26–29]). Further development of these methods see in [30, 31]. Applications of these methods for obtaining asymptotic expansions of solutions to the Painlevé equations see in [32–40].

We remark that algorithms in the articles [25, 30, 31] allow to find all power expansions as well as power-logarithmic, complicated and exotic ones. In particular, this made it possible to find all local and asymptotic expansions of solutions to the sixth Painlevé equation [40].

The power expansion of a function is an expansion over simpliest functions (power functions). It allows to clarify local properties of the expanded function. The Taylor and Laurent series are not for nothing used in Analysis. Certainly, expansions over other functions such as rational or elliptic ones are possible, but such expansions are not so convenient for local analysis of the expanded function and they are used not so frequently.

The main goal of this article is to demonstrate the application of these methods to the simplest equation, which is different from the Painlevé equations, i.e. to the equation (1.4), that is often denoted as P_1^2.

Let us find all the power expansions for the solution to equation (1.4) in the form of

$$w(z) = c_r\, z^r + \sum c_s\, z^s, \quad s \in \mathbf{K}; \tag{1.6}$$

as $z \to 0$, then $\omega \overset{\text{def}}{=} -1$, $s > r$; and as $z \to \infty$, then $\omega \overset{\text{def}}{=} 1$, $s < r$.

We use notions and notation from [25] or [26–29], but here we have no place to repeat them.

2. The general properties of equation (1.4)

Let us write the fourth-order equation (1.4) in the form

$$f(z,w) \overset{def}{=} w_{zzzz} - 10ww_{zz} - 5w_z^2 + 10w^3 - z = 0. \tag{2.1}$$

Monomials of equation (2.1) have vectorial power exponents

$$M_1 = (-4,1), \quad M_2 = (-2,2), \quad M_3 = (-2,2),$$
$$M_4 = (0,3), \quad M_5 = (1,0).$$

The support of the equation consists of four points

$$Q_1 = M_1, \quad Q_2 = M_4, \quad Q_3 = M_5 \quad \text{and} \quad Q_4 = M_2 = M_3. \tag{2.1$'$}$$

Their convex hull Γ is the triangle (Fig. 1).

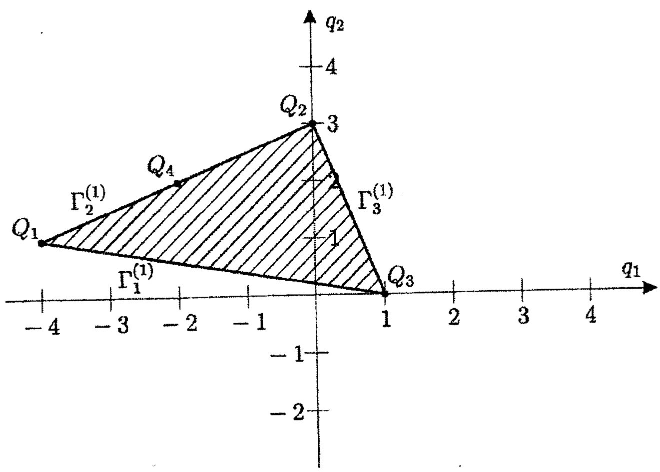

Figure 1

This triangle has vertexes Q_j $(j = 1, 2, 3)$ and edges $\Gamma_1^{(1)} = [Q_3, Q_1]$, $\Gamma_2^{(1)} = [Q_1, Q_2]$, $\Gamma_3^{(1)} = [Q_2, Q_3]$.

Outward normal vectors N_j $(j = 1, 2, 3)$ of edges $\Gamma_j^{(1)}$ $(j = 1, 2, 3)$ are

$$N_1 = (-1, -5), \quad N_2 = (-1, 2), \quad N_3 = (3, 1). \tag{2.2}$$

The normal cones $\mathbf{U}_j^{(1)}$ of edges $\Gamma_j^{(1)}$ are rays

$$\mathbf{U}_j^{(1)} = \mu N_j, \quad \mu > 0, \quad j = 1, 2, 3. \tag{2.3}$$

They and the normal cones $\mathbf{U}_j^{(0)}$ of vertices $\Gamma_j^{(0)} = Q_j$ $(j = 1, 2, 3)$ are represented in Fig. 2.

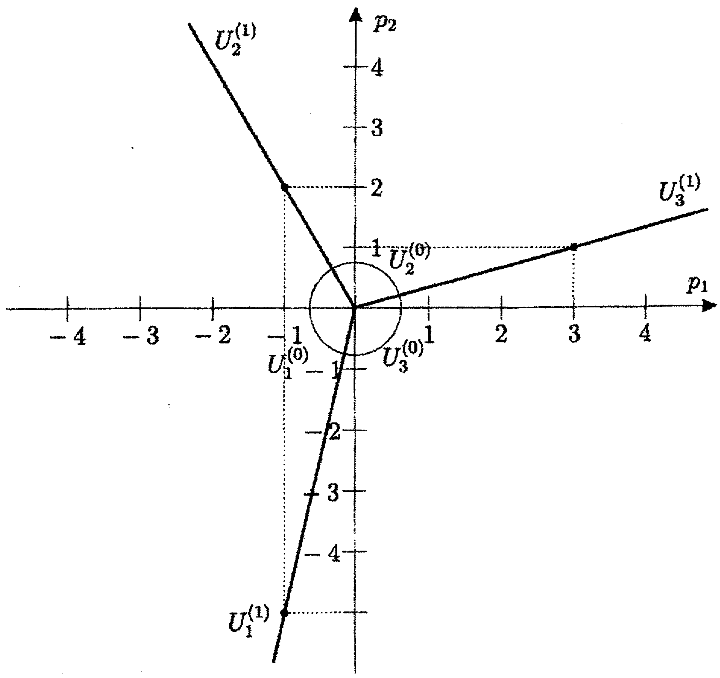

Figure 2

If the support $(2.1')$ of equation (2.1) is shifted by vector $-Q_3$, then it is situated at the lattice $\mathbf{Z}$, formed by vectors

$$Q_1 - Q_3 = (-5, 1), \qquad Q_4 - Q_3 = (-3, 2). \tag{2.4}$$

We choose the basis of the lattice as

$$B_1 = (-5, 1), \quad B_2 = (-3, 2). \tag{2.5}$$

To each face $\Gamma_j^{(d)}$ there corresponds the truncated equation $\hat{f}_j^{(d)}(z,w) = 0$ where the sum $\hat{f}_j^{(d)}(z,w)$ contains only such monomials of the sum $f(z,w)$ in (2.1) whose vectorial power exponents M_k belong to the face $\Gamma_j^{(d)}$. If expansion (1.6) is a solution to equation (2.1), then its first term $w = c_r w^r$ is a solution of the truncated equation $\hat{f}_j^{(d)}(z,w) = 0$, if vector $\omega(1,r) \in \mathbf{U}_j^{(d)}$ [25]. Thus, to each face $\Gamma_j^{(d)}$ there corresponds a set of families of expansions (1.6) of solutions to equation (2.1).

Let us study solutions corresponding to the faces $\Gamma_j^{(d)}$, $d = 0, 1;\ j = 1, 2, 3$ taking into account the truncated equations corresponding to vertices $\Gamma_j^{(0)}$ $(j = 1, 2, 3)$

$$\hat{f}_1^{(0)} \stackrel{def}{=} w_{zzzz} = 0, \tag{2.6}$$

$$\hat{f}_2^{(0)} \stackrel{def}{=} 10\, w^3 = 0, \tag{2.7}$$

$$\hat{f}_3^{(0)} \stackrel{def}{=} -z = 0. \tag{2.8}$$

and truncated equations corresponding to edges $\Gamma_j^{(1)}$ $(j = 1, 2, 3)$

$$\hat{f}_1^{(1)} \stackrel{def}{=} w_{zzzz} - z = 0, \tag{2.9}$$

$$\hat{f}_2^{(1)} \stackrel{def}{=} w_{zzzz} - 10\, w\, w_{zz} - 5w_z^2 + 10w^3 = 0, \tag{2.10}$$

$$\hat{f}_3^{(1)} \stackrel{def}{=} 10w^3 - z = 0. \tag{2.11}$$

Note that the truncated equations (2.7) and (2.8) are algebraic ones. According to [25] they do not have non-trivial solutions.

3. Solutions, corresponding to the vertex Q_1

The vertex $Q_1 = (-4, 1)$ corresponds to the truncated equation (2.6). Let us find the truncated solutions

$$w = c_r z^r, \quad c_r \neq 0 \tag{3.1}$$

with $\omega(1,r) \in \mathbf{U}_1^{(0)}$.

Since $p_1 < 0$ in the cone $\mathbf{U}_1^{(0)}$, then $\omega = -1,\ z \to 0$, and the expansions are by ascending powers of z. The dimension of the face $d = 0$, therefore

$$g(z,w) = w^4\, w^{-1}\, w_{zzzz}. \tag{3.2}$$

We obtain the characteristic polynomial

$$\chi(r) \stackrel{def}{=} g(z, z^r) = r(r-1)(r-2)(r-3). \tag{3.3}$$

Its roots are

$$r_1 = 0, \quad r_2 = 1, \quad r_3 = 2, \quad r_4 = 3 \tag{3.4}$$

Let us explore all these roots. According to Fig. 2, all vectors $\omega(1, r_i)$ belong to the normal cone $\mathbf{U}_1^{(0)}$ $(i = 1, 2, 3, 4)$.

For the root $r_1 = 0$, we obtain the family $\mathcal{F}_1^{(1)}1$ of truncated solutions $y = c_0$, where $c_0 \neq 0$ is arbitrary constant. The first variation of equation (2.6)

$$\frac{\delta \hat{f}_1^{(0)}}{\delta w} = \frac{d^4}{dz^4} \tag{3.5}$$

gives the operator

$$\mathcal{L}(z) = \frac{d^4}{dz^4} \neq 0. \tag{3.6}$$

Its characteristic polynomial is

$$\nu(k) = z^{4-k} \mathcal{L}(z)\, z^k = k(k-1)(k-2)(k-3). \tag{3.7}$$

Equation

$$\nu(k) = 0 \tag{3.8}$$

has four roots

$$k_1 = 0, \quad k_2 = 1, \quad k_3 = 2, \quad k_4 = 3. \tag{3.9}$$

Since $\omega = -1$ and $r = 0$, then the cone of the problem is

$$\mathcal{K} = \{k > 0\}. \tag{3.10}$$

It contains the critical numbers $k_2 = 1$, $k_3 = 2$, and $k_4 = 3$. Expansions (1.6) for the solutions corresponding to the truncated solution (3.1) take the form

$$w = c_0 + c_1 z + c_2 z^2 + c_3 z^3 + \sum_{k=4}^{\infty} c_k z^k, \tag{3.11}$$

where all the coefficients are constants, $c_0 \neq 0$, c_1, c_2, c_3 are arbitrary ones, and c_k $(k \geq 4)$ are uniquely determined. Denote this family as $\mathcal{G}_1^{(0)}1$. Expansion (3.11) with eight terms is

$$\begin{aligned} w(z) = c_0 + c_1\, z + c_2\, z^2 + c_3\, z^3 + \Big(\frac{5}{6}\, c_0 c_2 - \frac{5}{12}\, {c_0}^3 + \frac{5}{24}\, {c_1}^2\Big) z^4 \\ + \Big(\frac{1}{120}\, \alpha - \frac{1}{4}\, c_1 {c_0}^2 + \frac{1}{2}\, c_0 c_3 + \frac{1}{3}\, c_1 c_2\Big) z^5 \\ + \Big(\frac{7}{36}\, c_2 {c_0}^2 - \frac{5}{36}\, {c_0}^4 - \frac{1}{72}\, c_0 {c_1}^2 + \frac{1}{9}\, {c_2}^2 + \frac{1}{4}\, c_1 c_3\Big) z^6 \end{aligned}$$

$$+\left(\frac{1}{36}\,{c_1}^3+\frac{1}{504}\,c_0+\frac{1}{12}\,c_3{c_0}^2+\frac{1}{6}\,c_0c_1c_2-\frac{5}{36}\,c_1{c_0}^3+\frac{1}{6}\,c_2c_3\right)z^7+\cdots \tag{3.12}$$

Similarly, for each root $r_{j+1} = j$, $j = 1, 2, 3$, the cone of the problem is $\mathcal{K} = \{k > j\}$, the truncated solution $\mathcal{F}_1^{(0)}(j+1)$: $w = c_j z^j$ and the expansion is (3.11), (3.12) with $c_i = 0$, $i < j$, $c_j \neq 0$, and $c_j, \ldots, c_3$ are arbitrary constants. Denote this family as $\mathcal{G}_1^{(0)}(j+1)$. For $j = 2$, the expansion (3.11) with eight terms is

$$\begin{aligned} w(z) = c_2z^2 + c_3z^3 + \frac{1}{120}\,z^5 + \frac{1}{9}\,{c_2}^2z^6 + \frac{1}{6}\,c_2c_3z^7 + \frac{1}{16}\,{c_3}^2z^8 \\ + \frac{1}{1134}\,c_2\,z^9 + \left(\frac{5}{648}\,{c_2}^3 + \frac{41}{60480}\,c_3\right)z^{10} + \cdots \end{aligned} \tag{3.13}$$

The expansions of solutions converge for sufficiently small $|z|$. The existence and analyticity of expansions (3.11), (3.12), (3.13) follows from the Cauchy theorem.

4. Solutions corresponding to the edge $\Gamma_1^{(1)}$

The edge $\Gamma_1^{(1)}$ corresponds to the truncated equation

$$\hat{f}_1^{(1)}(z, y) \stackrel{def}{=} y_{zzzz} - z = 0. \tag{4.1}$$

The normal cone is

$$\mathbf{U}_1^{(1)} = \{-\mu(1, 5),\ \mu > 0\}. \tag{4.2}$$

Therefore $\omega = -1$, i.e. $z \to 0$, and $r = 5$. We are looking for power solutions to (4.1) in the form

$$w = c_5 z^5.$$

For c_5 we have equation $120c_5 - 1 = 0$, i.e.

$$c_5 = \frac{1}{120}. \tag{4.3}$$

The only power solution is

$$\mathcal{F}_2^{(1)}1: \quad w = \frac{z^5}{120}. \tag{4.4}$$

Compute the critical numbers. The first variation of (2.9) is

$$\frac{\delta \hat{f}_1^{(1)}}{\delta w} = \frac{d^4}{dz^4}. \tag{4.5}$$

We obtain the eigenvalues (3.9)

$$k_1 = 0, \quad k_2 = 1, \quad k_3 = 2, \quad k_4 = 3. \tag{4.6}$$

The cone of the problem

$$\mathcal{K} = \{k > 5\}$$

does not contain them. Solution (4.4) corresponds to two vectorial power exponents $\tilde{Q}_1 = (0, 1)$, $\tilde{Q}_2 = (5, 0)$. Their difference $B = \tilde{Q}_1 - \tilde{Q}_2 = (-5,\ 1)$ equals the vector $Q_1 - Q_3$. So the solution (4.4) corresponds to the lattice $\mathbf{Z}$ which consists of the points $Q = (q_1,\ q_2) = k(-3,\ 2) + m(-5,\ 1) = (-3k - 5l,\ 2k + l)$, where k and l are integers. These points lie on the line $q_2 = -1$, if $l = -1 - 2k$. In this case $q_1 = 5 + 7k$. Since the cone of the problem here is $\mathcal{K} = \{k > 5\}$, then the support $\mathbf{K}$ of the solutions expansion takes the form

$$\mathbf{K} = \{5 + 7n,\ n \in \mathbb{N}\}. \tag{4.7}$$

Then the expansion of the solution can be written as

$$w(z) = z^5 \left(\frac{1}{120} + \sum_{m=1}^{\infty} c_{5+7m}\, z^{7m} \right). \tag{4.8}$$

Expansion (4.8) with three terms takes the form

$$w(z) = \frac{z^5}{120} \left(1 + \frac{13\, z^7}{57024} + \frac{2851\, z^{14}}{79569008640} + \cdots \right) \tag{4.9}$$

and coincides with expansion (3.13) with $c_2 = c_3 = 0$. It does not have exponential additional terms. Equation (4.1) does not give non-power asymptotics as well.

5. Solutions corresponding to the edge $\Gamma_2^{(1)}$

The edge $\Gamma_2^{(1)}$ corresponds to the truncated equation

$$\hat{f}_2^{(1)}(z, w) \stackrel{def}{=} w_{zzzz} - 10\, w\, w_{zz} - 5\, w^2 + 10\, w^3 = 0. \tag{5.1}$$

The normal cone is

$$\mathbf{U}_2^{(1)} = \{-\mu(1, -2),\ \mu > 0\}. \tag{5.2}$$

Therefore $\omega = -1$, i.e. $z \to 0$, and $r = -2$. Hence the solution to equation (5.1) must be found in the form

$$w = c_{-2} z^{-2}. \tag{5.3}$$

For c_{-2} we have the determining equation

$$c_{-2}^2 - 8\,c_{-2} + 12 = 0. \tag{5.4}$$

Consequently we obtain

$$c_{-2}^{(1)} = 2, \qquad c_{-2}^{(2)} = 6. \tag{5.5}$$

The truncated solutions are

$$\mathcal{F}_2^{(1)}1: \;\; w = 2z^{-2}, \tag{5.6}$$

$$\mathcal{F}_2^{(1)}2: \;\; w = 6z^{-2}. \tag{5.7}$$

Let us compute the corresponding critical numbers. The first variation is

$$\frac{\delta \hat{f}_2^{(1)}}{\delta w} = \frac{d^4}{dz^4} - 10w_{zz} - 10w\frac{d^2}{dz^2} - 10\,w_z\frac{d}{dz} + 30\,w^2. \tag{5.8}$$

Substituting the solution (5.6), it produces operator

$$\mathcal{L}^{(1)}(z) = \frac{d^4}{dz^4} - \frac{20}{z^2}\frac{d^2}{dz^2} + \frac{40}{z^3}\frac{d}{dz}, \tag{5.9}$$

which corresponds to the characteristic polynomial

$$\nu(k) = k^4 - 6k^3 - 9k^2 + 54k. \tag{5.10}$$

Equation

$$\nu(k) = 0 \tag{5.11}$$

has the roots

$$k_1 = -3, \quad k_2 = 0, \quad k_3 = 3, \quad k_4 = 6. \tag{5.12}$$

For the solution (5.7), the variation (5.8) gives the operator

$$\mathcal{L}^{(2)}(z) = \frac{d^4}{dz^4} + \frac{60}{z^2}\frac{d^2}{dz^2} - \frac{120}{z^3}\frac{d}{dz} + \frac{720}{z^4}, \tag{5.13}$$

which corresponds to the characteristic polynomial

$$\nu(k) = k^4 - 6k^3 - 49k^2 + 174k + 720 \tag{5.14}$$

with roots

$$k_1 = -5, \quad k_2 = -3, \quad k_3 = 6, \quad k_4 = 8. \tag{5.15}$$

The cone of the problem here is

$$\mathcal{K} = \{k > -2\}. \tag{5.16}$$

Therefore the truncated solution (5.6) has three critical numbers $k_2 = 0$, $k_3 = 3$, $k_4 = 6$; and there are two critical numbers $k_3 = 6$, $k_4 = 8$ for the truncated solution (5.7) in the cone of the problem.

Let us consider expansions of solutions beginning with (5.6). Similarly to Section 4, here the shifted support of the truncated solution (5.6) belongs to the lattice $\mathbf{Z}$. Hence, similarly to (4.7) we have

$$\mathbf{K} = \{-2 + 7n, \quad n \in \mathbb{N}\}. \tag{5.17}$$

The sets $\mathbf{K}(0)$, $\mathbf{K}(0,3)$, and $\mathbf{K}(0,3,6)$ are

$$\begin{aligned}\mathbf{K}(0) = \{-2 + 7n + 2m,\ n, m \in \mathbb{N} \cup \{0\},\ n + m > 0\}& \\ = \{0, 4, 6, 5, 7, 8, \dots\},&\end{aligned} \tag{5.18}$$

$$\begin{aligned}\mathbf{K}(0,3) = \{-2 + 7n + 2m + 5k,\ n, m, k \in \mathbb{N} \cup \{0\},\ m + n + k > 0\}& \\ = \{0, 2, 3, 4, 5, 6, 7, 8, \dots\},&\end{aligned} \tag{5.19}$$

$$\begin{aligned}\mathbf{K}(0,3,6) = \{-2 + 7n + 2m + 5k + 8l,\ n, m, k, l \in \mathbb{N} \cup \{0\},& \\ m + n + k + l > 0\} = \{0, 2, 3, 4, 5, 6, 7, 8, \dots\}.&\end{aligned} \tag{5.20}$$

Here the expansion of the solution to the equation (2.1) can be written as

$$w(z) = \frac{2}{z^2} + \sum_{n,m,k,l} c_{5+7n+2m+2k+8l}\, z^{5+7n+2m+2k+8l}. \tag{5.21}$$

Denote this family as $\mathcal{G}_2^{(1)}1$. The critical number 0 does not belong to the set $\mathbf{K}$, so the compatibility condition for c_0 holds automatically, and c_0 is an arbitrary constant. The critical number 3 also does not belong to set $\mathbf{K}(0)$, therefore the compatibility condition for c_3 holds as well, and c_3 is an arbitrary constant. But the critical number 6 belong to sets $\mathbf{K}(0)$ and $\mathbf{K}(0,3)$, so it is necessary to verify that the compatibility condition for c_6 holds and that c_6 is an arbitrary constant. The calculation shows that here the condition holds and c_6 is an arbitrary constant as well. The three-parameter power expansion of solutions corresponding to the truncated solution (5.6) takes the form

$$\begin{aligned}w(z) = \frac{2}{z^2} + c_0 - \frac{3}{2}\,{c_0}^2 z^2 + c_3 z^3 - \frac{5}{2}\,{c_0}^3 z^4 \\ + \left(\frac{3}{4}\,c_0 c_3 - \frac{1}{80}\right) z^5 + c_6 z^6 - \frac{1}{280}\,c_0\, z^7 \\ + \left(\frac{153}{352}\,{c_0}^5 + \frac{9}{44}\,c_0 c_6 + \frac{9}{176}\,{c_3}^2\right) z^8 + \left(\frac{19}{12096}\,{c_0}^2 - \frac{5}{16}\,{c_0}^3 c_3\right) z^9\end{aligned}$$

$$+\Big(\frac{25}{104}\,{c_0}^6-\frac{29}{29120}\,c_3-\frac{3}{26}\,{c_0}^2c_6+\frac{3}{52}\,c_0{c_3}^2\Big)z^{10}+\cdots \quad (5.22)$$

The support of the power expansion corresponding to the truncated solution (5.7), is determined by the sets

$$\begin{aligned}&\mathbf{K}(6)=\{-2+7n+8m,\ n,m\in\mathbb{N}\cup\{0\},\ m+n>0\}\\&=\{5,6,12,14,20,21,22,27,28,29,30,34,35,36,37,38,41,\dots\},\end{aligned} \quad (5.23)$$

$$\begin{aligned}\mathbf{K}(6,8)=\{-2+7n+8m+10k,\ n,m,k\in\mathbb{N}\cup\{0\},\ m+n+k>0\}\\=\{5,6,8,12,13,14,15,16,18,19,20,21,\dots\}.\end{aligned} \quad (5.24)$$

The expansion of solution to the equation can be written as

$$w(z)=\frac{6}{z^2}+\sum_{n,m,k}c_{5+7n+8m+10k}\,z^{5+7n+8m+10k}. \quad (5.25)$$

Denote this family as $\mathcal{G}_2^{(1)}2$. The critical numbers 6 and 8 do not belong to the set $\mathbf{K}$, and the number 8 does not belong to the set $\mathbf{K}(6)$. For numbers 6 and 8 the compatibility conditions holds automatically, therefore the coefficients c_6 and c_8 are arbitrary constants. The two-parameter expansion of solution corresponding to the truncated solution (5.7) is

$$\begin{aligned}w(z)=\frac{6}{z^2}+\frac{1}{240}\,z^5+c_6z^6+c_8z^8+\frac{29}{70502400}\,z^{12}+\\+\frac{11}{60480}\,c_6z^{13}+\frac{25}{1292}\,{c_6}^2z^{14}+\frac{1}{6804}\,c_8z^{15}+\cdots\end{aligned} \quad (5.26)$$

According to [25, §§7,5], the expansions of solutions (5.25) and (5.26) do not have exponential additions and equation (5.1) does not give non-power asymptotics.

In Sections 3–5, we obtained all power expansions of solutions to the equation (1.4) in the neighborhood of the origin $z=0$. All these expansions were Taylor or Laurent series. According to [25], they all are convergent in some (punctured) neighborhood of the origin. Similarly, we can obtain all power expansions of solutions at an arbitrary point $z=z_0\neq 0$. For this we need to introduce a new independent variable $\tilde{z}=z-z_0$. Then the equation (1.4) would take the form

$$w^{(4)}-10ww''-5{w'}^2+10w^3-\tilde{z}-z_0=0. \quad (5.27)$$

It differs from the equation (2.1) only by the term $-z_0$, which corresponds to the new point $Q_5=(0,0)$ of the support. This point also is a vertex of a new polygon; other its vertices are Q_1, Q_2, Q_3. Using the described technique, we can obtain expansions of solutions to the equation (5.27) as $\tilde{z} \to 0$. These expansions are analogous to the already found ones, and for them, the consistency conditions for the critical power exponents are satisfied identically with respect to z_0. The solutions to the equation (1.4) do not have movable singularities, i.e. the equation (1.4) possesses the Painlevé property [41].

6. Solutions corresponding to the edge $\Gamma_3^{(1)}$

The edge $\Gamma_3^{(1)}$ corresponds to the truncated equation

$$\hat{f}_3^{(1)}(z,w) \stackrel{def}{=} 10\,w^3 - z = 0. \tag{6.1}$$

It has three power solutions

$$\mathcal{F}_3^{(1)}1: \ w = \varphi^{(1)}(z) = c_{1/3}^{(1)}\, z^{1/3}, \quad c_{1/3}^{(1)} = \left(\frac{1}{10}\right)^{1/3}; \tag{6.2}$$

$$\mathcal{F}_3^{(1)}2: \ w = \varphi^{(2)}(z) = c_{1/3}^{(2)}\, z^{1/3}, \quad c_{1/3}^{(2)} = \left(\frac{1}{2} + i\sqrt{3}\right)\left(\frac{1}{10}\right)^{1/3}; \tag{6.3}$$

$$\mathcal{F}_3^{(1)}3: \ w = \varphi^{(3)}(z) = c_{1/3}^{(3)}\, z^{1/3}, \quad c_{1/3}^{(3)} = \left(\frac{1}{2} - i\sqrt{3}\right)\left(\frac{1}{10}\right)^{1/3}. \tag{6.4}$$

The shifted support of the truncated solutions (6.2)–(6.4) gives the vector

$$B = \left(\frac{1}{3}, -1\right), \tag{6.5}$$

which equals the third part of the vector $Q_2 - Q_3$.

Let us find a basis of the lattice generated by vectors B_1, B_2 from (2.5) and vector B. According to the algorithm of Appendix, we compute determinants $D_1 = |B_1 B_2|$, $D_2 = |B_1 B|$, $D_3 = |B_2 B|$:

$$D_1 = \begin{vmatrix} -5 & 1 \\ -3 & 2 \end{vmatrix} = -7, \quad D_2 = \begin{vmatrix} -5 & 1 \\ 1/3 & -1 \end{vmatrix} = \frac{14}{3},$$

$$D_3 = \begin{vmatrix} -3 & 2 \\ 1/3 & -1 \end{vmatrix} = \frac{7}{3}.$$

Since $|D_3|$ is the minimum of absolute values and other determinants are its multiplies, then vectors B_2 and B form a basis of the lattice corresponding to solutions (6.2)–(6.4). Points of the lattice have the form

$$Q = (q_1, q_2) = l(-3,2) + m(1/3,-1) = (-3l + m/3, 2l - m),$$

where l, m are integer numbers. On the line $q_2 = -1$ we have $2l - m = -1$. Hence, $m = 2l + 1$ and $q_1 = (2l + 1 - 9l)/3$. So the support of the solution is

$$\mathbf{K} = \left\{k = \frac{1-7n}{3},\ n \in \mathbb{N}\right\} = \{-2, -11/3, \dots\}, \tag{6.6}$$

and the expansions of solutions take the form

$$\mathcal{G}_3^{(1)} l: \ w = \varphi^{(l)}(z) = c_{1/3}^{(l)} z^{1/3} + \sum_{n=1}^{\infty} c_{(1-7n)/3}^{(l)} z^{(1-7n)/3}. \tag{6.7}$$

Here $c_{1/3}^{(l)}$ are given in (6.2)–(6.4); coefficients $c_{(1-7n)/3}^{(l)}$ are computed sequentially. The expansion of the solution with five terms is

$$\varphi^{(l)}(z) = c_{1/3}\, z^{1/3} - \frac{1}{18}\, z^{-2} - \frac{7}{108}\, \frac{1}{c_{1/3}} z^{-13/3} - \frac{4199}{17496}\, {c_{1/3}}^{-2}\, z^{-20/3} - \frac{28006583}{23514624}\, \frac{1}{{c_{1/3}}^3} z^{-9} + \cdots \tag{6.8}$$

The obtained expansions seem to diverge [25, §7].

7. Exponential additions of the first level

Let us find the exponential additions [25, §7] to solutions (6.7). We look for the solutions in the form

$$w = \varphi^{(l)}(z) + u^{(l)}, \quad l = 1, 2, 3.$$

The truncated equation for the addition $u^{(l)}$ is

$$\mathcal{M}_l^{(1)}(z) u^{(l)} = 0, \tag{7.1}$$

where $\mathcal{M}_l^{(1)}(z)$ is the first variation of $f(z, w)$ from (2.1) at the solution $w = \varphi^{(l)}(z)$. As we have

$$\frac{\delta f}{\delta w} = \frac{d^4}{dz^4} - 10 w_{zz} - 10 w \frac{d^2}{dz^2} - 10 w_z \frac{d}{dz} + 30 w^2, \tag{7.2}$$

then

$$\mathcal{M}_l^{(1)}(z) = \frac{d^4}{dz^4} - 10 \varphi_{zz}^{(l)} - 10 \varphi^{(l)} \frac{d^2}{dz^2} - 10 \varphi_z^{(l)} \frac{d}{dz} + 30 {\varphi^{(l)}}^2. \tag{7.3}$$

Equation (7.1) takes the form

$$\frac{d^4u^{(l)}}{dz^4} - 10\varphi_{zz}^{(l)}u^{(l)} - 10\varphi(l)\frac{d^2u^{(l)}}{dz^2} - 10\varphi_z^{(l)}\frac{du^{(l)}}{dz} + 30{\varphi^{(l)}}^2u^{(l)} = 0,$$
$$l = 1,2,3. \quad (7.4)$$

Denote

$$\zeta^{(l)} = \frac{d\ln u^{(l)}}{dz}, \tag{7.5}$$

then from (7.5) we have

$$\frac{du^{(l)}}{dz} = \zeta^{(l)}u^{(l)}, \qquad \frac{d^2u^{(l)}}{dz^2} = \zeta_z^{(l)}u^{(l)} + {\zeta^{(l)}}^2u^{(l)},$$

$$\frac{d^3u^{(l)}}{dz^3} = \zeta_{zz}^{(l)}u^{(l)} + 3\zeta^{(l)}\zeta_z^{(l)}u^{(l)} + {\zeta^{(l)}}^3u^{(l)},$$

$$\frac{d^4u^{(l)}}{dz^4} = \zeta_{zzz}^{(l)}u^{(l)} + 4\zeta^{(l)}\zeta_{zz}^{(l)}u^{(l)} + 3{\zeta_z^{(l)}}^2u^{(l)} + 6{\zeta^{(l)}}^2\zeta_z^{(l)}u^{(l)} + {\zeta^{(l)}}^4u^{(l)}.$$

Substituting the derivatives

$$\frac{du^{(l)}}{dz}, \qquad \frac{d^2u^{(l)}}{dz^2}, \qquad \frac{d^4u^{(l)}}{dz^4}$$

into equation (7.4) we obtain the equation in the form

$$u^{(l)}[\zeta_{zzz}^{(l)} + 4\zeta^{(l)}\zeta_{zz}^{(l)} + 3{\zeta_z^{(l)}}^2 + 6{\zeta^{(l)}}^2\zeta_z^{(l)} + {\zeta^{(l)}}^4 - 10\varphi_{zz}^{(l)}$$
$$- 10\varphi^{(l)}\zeta_z^{(l)} - 10\varphi^{(l)}{\zeta^{(l)}}^2 - 10\varphi_z^{(l)}\zeta^{(l)} + 30{\varphi^{(l)}}^2] = 0. \quad (7.6)$$

Let us find the power expansions for solutions to the equation (7.6). The support of the equation (7.6), divided by $u^{(l)}$, consists of points

$$\begin{gathered} Q_1 = (-3,1), \quad Q_2 = (-2,2), \quad Q_3 = (-1,3), \\ Q_4 = (0,4), \quad Q_5 = \Big(\frac{1}{3},2\Big), \quad Q_6 = \Big(\frac{2}{3},0\Big), Q_7 = \Big(-\frac{2}{3},1\Big), \\ Q_8 = \Big(-\frac{5}{3},0\Big), \quad Q_{8+n} = \Big(-\frac{5+7n}{3},0\Big), \\ Q_{9+m} = \Big(-\frac{2+7m}{3},1\Big), \\ Q_{10+k} = \Big(\frac{1-7k}{3},2\Big), \quad Q_{11+k} = \Big(\frac{12-14l}{3},0\Big), \quad m,n,k,l \in \mathbb{N}. \end{gathered} \tag{7.7}$$

The closure of the convex hull of the points of the support of the equation (7.6) is the band in Fig. 3.

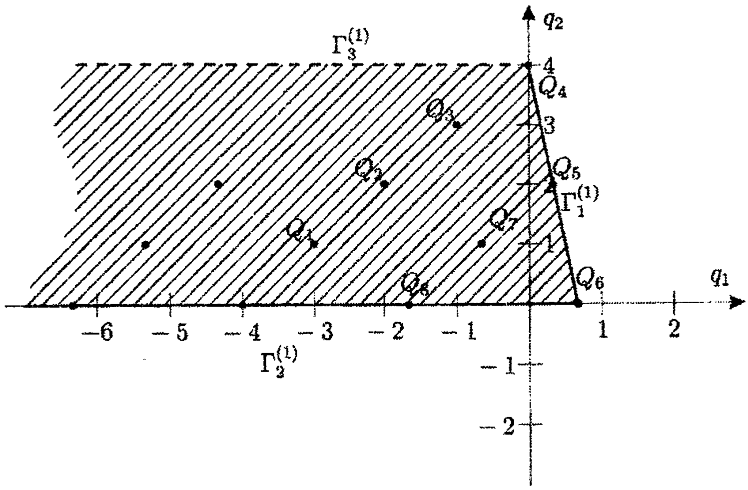

Figure 3

The boundary of the band contains the edges $\Gamma_j^{(1)}$ $(j = 1, 2, 3)$ with the normal vectors $N_1 = (6, 1)$, $N_2 = (0, -1)$, $N_3 = (0, 1)$. We must consider the edge $\Gamma_1^{(1)}$ only. This edge corresponds to the truncated equation

$$h_1^{(1)}(z, \zeta) \stackrel{def}{=} \zeta^4 - 10\hat{\varphi}^{(l)}\zeta^2 + 30\hat{\varphi}^{(l)2} = 0, \tag{7.8}$$

where $\hat{\varphi}^{(l)} = c_{1/3}^{(l)} z^{1/3}$ from (6.2)–(6.4). Wherefrom we have

$$\zeta^2 = \big(5 + (-1)^{m-1} i\sqrt{5}\big)\hat{\varphi}^{(l)}, \quad m = 1, 2. \tag{7.9}$$

We obtain twelve solutions to the equation (7.8)

$$\zeta^{(l,m,k)} = g_{1/6}^{(l,m,k)} z^{1/6}, \quad l = 1, 2, 3;\ m, k = 1, 2, \tag{7.10}$$

where

$$g_{1/6}^{(1,m,k)} = \left(\frac{1}{10}\right)^{1/3} (-1)^{k-1} \left(5 + (-1)^{m-1} i\sqrt{5}\right)^{1/2}, \quad m, k = 1, 2, \tag{7.11}$$

$$g_{1/6}^{(2,m,k)} = \left(\frac{1}{2} + i\sqrt{3}\right)\left(\frac{1}{10}\right)^{1/3} (-1)^{k-1} \left(5 + (-1)^{m-1} i\sqrt{5}\right)^{1/2}, \quad m, k = 1, 2, \tag{7.12}$$

$$g_{1/6}^{(3,m,k)} = \left(\frac{1}{2} - i\sqrt{3}\right)\left(\frac{1}{10}\right)^{1/3} (-1)^{k-1} \left(5 + (-1)^{m-1} i\sqrt{5}\right)^{1/2}, \quad m, k = 1, 2. \tag{7.13}$$

The truncated equation (7.8) is an algebraic one, so it gives no critical numbers. Let us compute the support of the expansion for solution to the equation (7.6). The shifted support of equation (7.6) is contained in the lattice generated by vectors $\tilde{B}_1 = (7/3, 0)$, $\tilde{B}_2 = (1, 1)$. The shifted support of solutions (7.10) gives the vector $\tilde{B}_3 = (-1/6, 1)$.

According to Appendix, we compute the determinants $|\tilde{B}_1\tilde{B}_2| = 7/3 = |\tilde{B}_1\tilde{B}_3|$, $|\tilde{B}_2\tilde{B}_3| = 7/6$. Hence, the vectors $\tilde{B}_2$, $\tilde{B}_3$ or $\tilde{B}_2$, $\tilde{B}_4 \stackrel{\text{def}}{=} \tilde{B}_2 - \tilde{B}_3 = (7/6, 0)$ form a basis of the lattice containing $\tilde{B}_1$, $\tilde{B}_2$, $\tilde{B}_3$. The points of this lattice can be written as

$$Q = (q_1, q_2) = k(1, 1) + m\Big(\frac{7}{6}, 0\Big) = \Big(k + \frac{7m}{6}, k\Big).$$

On the line $q_2 = -1$ we have $k = -1$, and so $q_1 = -1 + 7m/6$. Since the cone of the problem here is $\mathcal{K} = \{k < 1/6\}$, then the support $\mathbf{K}$ of expansions is

$$\mathbf{K} = \Big\{\frac{1-7n}{6},\ n \in \mathbb{N}\Big\}. \tag{7.14}$$

The expansion of the solution to the equation (7.6) takes the form

$$\zeta^{(l,m,k)} = g^{(l,m,k)}_{1/6} z^{1/6} + \sum_{n} g^{(l,m,k)}_{(1-7n)/6} z^{(1-7n)/6}, \qquad l = 1, 2, 3;\ m = 1, 2;\ k = 1, 2. \tag{7.15}$$

Coefficients $g^{(l,m,k)}_{1/6}$ are determined by expressions (7.11), (7.12), and (7.13). The expansion of the solution with four terms takes the form

$$\zeta^{(l,m,k)} = g_{1/6} z^{1/6} - \frac{1}{4} z^{-1} - \frac{7}{288} \frac{\left(30\, {g_{1/6}}^2 - 7\, 10^{2/3}\right)}{g_{1/6}\left(2\, {g_{1/6}}^2 - 10^{2/3}\right)} z^{-13/6} - \frac{49}{1728} \frac{30\, {g_{1/6}}^4 - 6\, 10^{2/3} {g_{1/6}}^2 + 35\sqrt[3]{10}}{{g_{1/6}}^2\left(5\sqrt[3]{10} - 2\, 10^{2/3}\, {g_{1/6}}^2 + 2\, {g_{1/6}}^4\right)} z^{-10/3} + \cdots \tag{7.16}$$

In view of (7.5) we can find the additions $u^{(l,m,k)}(z)$. We have

$$u^{(l,m,k)}(z) = C \exp \int \zeta^{(l,m,k)}(z)\, dz. \tag{7.17}$$

Wherefrom we obtain

$$u^{(l,m,k)}(z)$$
$$= C_1\, z^{-1/4} \exp\left[\frac{6}{7}\, g_{1/6}^{(l,m,k)}\, z^{7/6} + \sum_{n=2}^{\infty} \frac{6}{7(1-n)} g_{(1-7n)/6}^{(l,m,k)} z^{7(1-n)/6}\right],$$
$$l = 1,2,3;\ m = 1,2;\ k = 1,2. \quad (7.18)$$

Here C_1 and further C_2 and C_3 are arbitrary constants. The addition $u^{(l,m,k)}(z)$ near $z \to \infty$ is exponentially small in those sectors of the complex plane z where

$$\operatorname{Re}\left[g_{1/6}^{(l,m,k)}\, z^{7/6}\right] < 0. \quad (7.19)$$

Thus for each of three expansions $\mathcal{G}_3^{(1)} l$ we obtain four one-parameter families of additions $\mathcal{G}_3^{(1)} l \mathcal{G}_1^{(1)} mk$, where $l = 1,2,3,\ m = 1,2$ and $k = 1,2$.

8. Exponential additions of the second level

Let us find exponential additions of the second level $v^{(p)}$, i.e. the additions to $\zeta^{(l,m,k)}(z)$. The truncated equation for the addition $v^{(p)}$ is

$$\mathcal{M}_p^{(2)}(z) v^{(p)} = 0, \quad (8.1)$$

where the operator $\mathcal{M}_p^{(2)}$ is the first variation of the square brackets in (7.6). Equation (8.1) for $v = v^{(p)}$ after substitution $d \ln v / dz = \xi$ takes the form

$$\xi_{zz} + 3\,\xi\,\xi_z + \xi^3 + 4\zeta_{zz} + 4\xi_z\,\zeta + 4\xi^2\,\zeta + 6\,\xi\,\zeta_z + 12\zeta\zeta_z$$
$$+ 6\,\xi\,\zeta^2 + 4\,\zeta^3 - 10\,\varphi^{(l)}\xi - 20\,\zeta\,\varphi^{(l)} - 10\,\varphi_z^{(l)} = 0. \quad (8.2)$$

Monomials of the equation (8.2) have the following power exponents

$$M_1 = (-2,1), \quad M_2 = (-1,2), \quad M_3 = (0,3), \quad M_4 = \Big(-\frac{11}{6}, 0\Big),$$
$$M_5 = \Big(-\frac{5}{6}, 1\Big), \quad M_6 = \Big(\frac{1}{6}, 2\Big), \quad M_7 = \Big(-\frac{5}{6}, 1\Big),$$
$$M_8 = \Big(-\frac{2}{3}, 0\Big), \quad M_9 = \Big(\frac{1}{3}, 1\Big), \quad M_{10} = \Big(\frac{1}{2}, 0\Big),$$
$$M_{11} = \Big(\frac{1}{3}, 1\Big), \quad M_{12} = \Big(\frac{1}{2}, 0\Big), \quad M_{13} = \Big(-\frac{2}{3}, 0\Big), \ldots \quad (8.3)$$

The support of the equation (8.2) consists of points of the set (8.3). Its convex hull forms the band, which is similar to the band represented in Fig. 3. We must examine the edge $\Gamma_1^{(1)}$, passing through the points $Q_1 = (1/2, 0)$, $Q_2 = (1/3, 1)$, $Q_3 = (0, 3)$.

The truncated equation corresponding to this edge is

$$\xi^3 + 4\,\xi^2\,\zeta + 6\,\xi\,\zeta^2 + 4\,\zeta^3 - 20\,\zeta\,\varphi^{(l)} - 10\,\xi\,\varphi^{(l)} = 0. \tag{8.4}$$

The basis of the lattice corresponding to the support of the equation (8.2) is

$$B_1 = (1, 1), \quad B_2 = \Big(\frac{7}{6}, 0\Big).$$

The solutions to the equation (8.4) take the form

$$\xi^{(l,m,k,p)} = r_{1/6}^{(l,m,k,p)}\, z^{1/6}, \quad m, k = 1, 2;\ l = 1, 2, 3;\ p = 1, 2, 3, \tag{8.5}$$

where $r = r_{1/6}^{(l,m,k,p)}$, $p = 1, 2, 3$ are the roots of the equation

$$5\,r^3 + 4\,r^2\,g_{1/6}^{(l,m,k)} + \Big(6\,g_{1/6}^{(l,m,k)^2} - 10\,c_{1/3}^{(l)}\Big)\,r \\ + 4\,g_{1/6}^{(l,m,k)^3} - 20\,g_{1/6}^{(l,m,k)}\,c_{1/3}^{(l)} = 0. \tag{8.6}$$

Equation (8.6) has the roots

$$\begin{aligned} r_{1/6}^{(l,m,k,1)} &= -2\,g_{1/6}^{(l,m,k)}, \\ r_{1/6}^{(l,m,k,2)} &= -g_{1/6}^{(l,m,k)} + \big(10\,c_{1/3}^{(l)} - g_{1/6}^{(l,m,k)}\big)^{1/2}, \\ r_{1/6}^{(l,m,k,3)} &= -g_{1/6}^{(l,m,k)} - \big(10\,c_{1/3}^{(l)} - g_{1/6}^{(l,m,k)}\big)^{1/2}. \end{aligned} \tag{8.7}$$

The support $\mathbf{K}$ of expansions for the solution coincides with (7.14). The expansion of solution for $\xi^{(l,m,k,p)}$ takes the form

$$\xi^{(l,m,k,p)} = r_{1/6}^{(l,m,k,p)} z^{1/6} + \sum_{n=1}^{\infty} r_{(1-7n)/6}^{(l,m,k,p)}\, z^{(1-7n)/6}, \\ l = 1, 2, 3;\ m = 1, 2;\ k = 1, 2;\ p = 1, 2, 3. \tag{8.8}$$

The expansion of the solution with three terms is

$$\xi^{(l,m,k,p)} = r_{1/6} z^{1/6} + \frac{1}{6}\, z^{-1} + (-30\, g_{1/6}^4 - 9\, 10^{2/3} g_{1/6}^2 \\ + 150\, g_{1/6}^2 c_{1/3} - 35\, c_{1/3} 10^{2/3} - 30\, r_{1/6}^2 g_{1/6}^2 + 7\, r_{1/6}^2 10^{2/3} \\ - 60\, g_{1/6}^3 r_{1/6} + 6\, r_{1/6} g_{1/6} 10^{2/3})(10^{2/3} - 2\, g_{1/6}^2)^{-1} \\ \times (6\, g_{1/6}^2 - 10\, c_{1/3} + 3\, r_{1/6}^2 + 8\, g_{1/6} r_{1/6})^{-1} g_{1/6}^{-1} z^{-\frac{13}{6}}. \tag{8.9}$$

The exponential addition $v^{(l,m,k,p)}(z)$ to $\zeta^{(l,m,k)}(z)$ is

$$
\begin{gathered}
v^{(l,m,k,p)}(z) \\
= C_2\, z^{1/6} \exp\left[\frac{6}{7}\, r_{1/6}^{(l,m,k,p)}\, z^{7/6} + \sum_{n=2}^{\infty} \frac{6}{7(1-n)} r_{(1-7n)/6}^{(l,m,k,p)} z^{7(1-n)/6}\right], \\
l=1,2,3;\ m=1,2;\ k=1,2;\ p=1,2,3. \quad (8.10)
\end{gathered}
$$

Solutions $v^{(l,m,k,p)}(z)$ seem to diverge as well.

Thus for each one-parameter family of additions $\mathcal{G}_3^{(1)} l \mathcal{G}_1^{(1)} mk$ of the first level we obtain 3 families of additions $\mathcal{G}_3^{(1)} l \mathcal{G}_1^{(1)} mk \mathcal{G}_1^{(1)} p$, where $p = 1,2,3$, of the second level.

9. Exponential additions of the third level

Similarly we obtain that the exponential addition $y^{(s,p,l,m,k)}(z)$ to the log $v^{(l,m,k,p)}(z)$ is

$$
\begin{gathered}
y^{(l,m,k,p,s)}(z) \\
= C_3\, z^{1/6} \exp\left[\frac{6}{7}\, q_{1/6}^{(l,m,k,p,s)}\, z^{7/6} + \sum_{n=2}^{\infty} \frac{6}{7(1-n)} q_{(1-7n)/6}^{(l,m,k,p,s)} z^{7(1-n)/6}\right] \\
l=1,2,3;\ m=1,2;\ k=1,2;\ p=1,2,3;\ s=1,2, \quad (9.1)
\end{gathered}
$$

where

$$
\begin{gathered}
q_{1/6}^{(l,m,k,p,s)} = -\frac{3}{2}\, r_{1/6}^{(l,m,k,p)} - 2\, g_{1/6}^{(l,m,k)} \\
+ (-1)^{s-1} \left(\frac{9}{4}\, r_{1/6}^{(l,m,k,p)\,2} - 2\, r_{1/6}^{(l,m,k,p)}\, g_{1/6}^{(l,m,k)} - 2\, g_{1/6}^{(l,m,k)\,2} + 10\, c_{1/3}^{(l)}\right)^{1/2}, \\
l=1,2,3;\ m,k=1,2;\ p=1,2,3;\ s=1,2. \quad (9.2)
\end{gathered}
$$

Thus we have found three levels of exponential additions to the expansions of solutions to the equation near the point $z=\infty$. Solution $w(z)$ with exponential additions as $z \to \infty$ has the expansion

$$
\begin{gathered}
w(z) = c_{1/3}^{(l)} z^{1/3} - \frac{1}{18z^2} + \sum_{n=2}^{\infty} c_{(1-7n)/3}^{(l)}\, z^{(1-7n)/3} \\
+ C_1\, z^{-1/4} \exp\left\{F_1(z) + C_2 \int z^{1/6} \exp\left[F_2(z)\right.\right. \\
\left.\left. + C_3 \int (z^{1/6} \exp F_3(z))\, dz\right] dz\right\}, \quad (9.3)
\end{gathered}
$$

where $c_{1/3}^{(l)}$ can be computed by formulas (6.2), (6.3), and (6.4); $F_1(z) = F_1^{(l,m,k)}(z)$, $F_2(z) = F_2^{(l,m,k,p)}(z)$, and $F_3(z) = F_3^{(l,m,k,p,s)}(z)$, $(l = 1,2,3;$ $m,k = 1,2;$ $p = 1,2,3;$ $s = 1,2)$ are

$$F_1^{(l,m,k)}(z) = \frac{6}{7}\, g_{1/6}^{(l,m,k)}\, z^{7/6} + \sum_{n=2}^{\infty} \frac{6}{7(1-n)} g_{(1-7n)/6}^{(l,m,k)} z^{7(1-n)/6}, \qquad (9.4)$$

$$F_2^{(p,l,m,k)}(z) = \frac{6}{7}\, r_{1/6}^{(l,m,k,p)}\, z^{7/6} + \sum_{n=2}^{\infty} \frac{6}{7(1-n)} r_{(1-7n)/6}^{(l,m,k,p)} z^{7(1-n)/6}, \qquad (9.5)$$

$$F_3^{(l,m,k,p,s)}(z) = \frac{6}{7}\, q_{1/6}^{(l,m,k,p,s)}\, z^{7/6} + \sum_{n=2}^{\infty} \frac{6}{7(1-n)} q_{(1-7n)/6}^{(l,m,k,p,s)}\, z^{7(1-n)/6}. \qquad (9.6)$$

Coefficients $g_{1/6}^{(l,m,k)}$, $r_{1/6}^{(l,m,k,p)}$, and $q_{1/6}^{(l,m,k,p,s)}$ are defined by formulas (7.11), (7.12), (7.13), (8.7), and (9.2). Other coefficients are computed sequentially.

Here the additions of the first level $F_1^{(l,m,k)}(z)$ exist only in those sectors of the complex plane, where the condition (7.19) holds, i.e.

$$\mathrm{Re}\left[g_{1/6}^{(l,m,k)} z^{7/6}\right] < 0. \qquad (9.7)$$

Additions of the second level $F_2^{(l,m,k,p)}(z)$ exist there, where the condition

$$\mathrm{Re}\left[r_{1/6}^{(l,m,k,p)} z^{7/6}\right] < 0 \qquad (9.8)$$

holds together with (9.7). Additions of the third level $F_3^{(l,m,k,p,s)}(z)$ exist there, where the condition

$$\mathrm{Re}\left[q_{1/6}^{(l,m,k,p,s)} z^{7/6}\right] < 0 \qquad (9.9)$$

holds together with (9.7) and (9.8).

The inequality

$$\mathrm{Re}[a\,x] < 0$$

means that

$$\frac{\pi}{4} - \arg a < \arg x < \frac{3\pi}{4} - \arg a.$$

Hence the inequalities (9.7) are always satisfied in the open half-plane of the complex plane $x = z^{7/6}$. We obtain all 12 cases. The inequalities (9.7) and (9.8) are satisfied simultaneously in some part of this plane if

$$\arg r_{1/6}^{(l,m,k,p)} \neq -\arg g_{1/6}^{(l,m,k)}.$$

According to (8.7), this inequality is satisfied for $p = 2, 3$, since

$$10 \neq (-1)^{k-1}(5 + (-1)^{m-1} i\sqrt{5})^{1/2}.$$

We obtain 24 cases of two inequalities (9.7) and (9.8) being satisfied simultaneously. The three inequalities (9.7)–(9.9) are never satisfied simultaneously, which is demonstrated by the computation of the complex numbers $g_{1/6}$, $r_{1/6}$, $q_{1/6}$. Hence additions of the third level (9.6) are absent.

According to [25, §7], the expansions (6.8) diverge, since the exponential additions are present. The exponential additions themselves describe the Stokes phenomenon.

Since the truncated equation (6.1) is algebraic, then it has not non-power solutions and does not give non-power asymptotics.

Note that the asymptotic study of P_1^2 has rather long history. In [42], asymptotics were obtained corresponding to the first terms in the expantions (6.7), (6.8). In [43], a one-parameter asymptotics were obtained corresponding to the first terms of the exponential additions of the first level (7.16), (7.18). In [44], the following equation was suggested

$$X = TU - \left[\frac{1}{6}U^3 + \frac{1}{24}(U'^2 + 2UU'') + \frac{1}{24}U^{(4)}\right], \tag{9.10}$$

which differs from the equation (1.4) by the additional term TU containing a new parameter T. For solutions to this equation, in [44], the first term of the asymptotics was written, which coincides with the first terms in expansions (6.7), (6.8) for $T = 0$. A more accurate asymptotics of solutions to the equation (9.10) was written in [45], but for $T = 0$, it coincides with already mentioned.

Finally, in [46], the equation (9.10) was cited, but there was considered a more complicated equation of the fourth order containing two parameters. There the existence of asymptotics depending on four parameters was shown for solutions of this equation. It is possible that two new parameters in the asymptotics of this solution correspond to arbitrary constants in the exponential additions of the first and the second levels. But this is a subject of a separate study, since algorithms of Power Geometry are applicable to this equation as well and allow to obtain asymptotic expansions and exponential additions of different levels. In [42–46] there are only asymptotic forms of solutions, they do not contain asymptotic expansions of solutions, but they contain global results: the existence and uniqueness of a remarkable solution of equation (9.10) which has the asymptotic behaviour $\pm(6|x|)^{1/3}$ as x approaches $\pm\infty$.

10. Summary of the results and discussion

We obtained the following expansions for the solutions to the fourth-order equation (2.1).

Near point $z = 0$:

1. The four-parameter (with arbitrary constants c_0, c_1, c_2 and c_3) family $\mathcal{G}_1^{(0)}1$ of expansions (3.11), (3.12).
2. The three-parameter (with arbitrary constants c_1, c_2 and c_3) family $\mathcal{G}_1^{(0)}2$ of expansions (3.11), (3.12) with $c_0 = 0$.
3. Two-parameter (with arbitrary constants c_2 and c_3) family $\mathcal{G}_1^{(0)}3$ of expansions (3.11), (3.13) with $c_0 = c_1 = 0$.
4. One-parameter (with arbitrary constant c_3) family $\mathcal{G}_1^{(0)}4$ of expansions (3.11), (3.13) with $c_0 = c_1 = c_2 = 0$.
5. Family $\mathcal{G}_1^{(1)}1$ of one expansion (4.8).
6. Three-parameter (with arbitrary constants c_0, c_3 and c_6) family $\mathcal{G}_2^{(1)}1$ of expansions (5.21).
7. Two-parameter (with arbitrary constants c_6 and c_8) family $\mathcal{G}_2^{(1)}2$ of expansions (5.25).

All listed expansions converge for sufficiently small $|z|$, $z \neq 0$.

Near point $z = \infty$:

8. Three expansions $\mathcal{G}_3^{(1)}l$ ($l = 1, 2, 3$) described by formulas (6.7), (6.2)–(6.4). For each of these expansions we found four exponential additions $\mathcal{G}_3^{(1)}l\mathcal{G}_1^1mk$ ($m, k = 1, 2$) of the first level expressed by the formula (7.18). For them we also computed two exponential additions $\mathcal{G}_3^{(1)}l\mathcal{G}_1^{(1)}mk\mathcal{G}_1^{(1)}p$ ($p = 2, 3$) of the second level expressed by the formula (8.10).

Theorem. *All power expansions of solutions to the equation* (1.4) *at the points $z = 0$ and $z = \infty$ are exhausted by the enumerated families* 1–8.

The existence and analyticity of expansions described in items 1–5 follows from the Cauchy theorem. Families $\mathcal{G}_2^{(1)}1$ and $\mathcal{G}_2^{(1)}2$ were first found in the paper [13]. However the structure of supports of expansions $\mathcal{G}_2^{(1)}1$ and $\mathcal{G}_2^{(2)}2$ was not discussed earlier. The other families of expansions of solutions and their exponential additions of two levels are found for the first time.

Asymptotic expansions allow us to find asymptotic forms with any accuracy. New precise asymptotic forms of solutions to the equation (1.4) can be applied to study the waves on water [4, 5] and to study the behaviour of a star [6–8], but such studies are subjects for separate papers.

11. Appendix. The computation of the basis of a lattice

Let there be a set $\mathbf{S}$ of points $Q_1, \dots, Q_m$ in the plane $\mathbb{R}^2$ with the origin among them. Our aim is to compute the basis $B_1,\ B_2$ of the minimal lattice $\mathbf{Z}$ that contains all the points of the set $\mathbf{S}$. The minimality of the lattice $\mathbf{Z}$ means that there is no other lattice $\mathbf{Z}_1 \subset \mathbf{Z}$ and $\mathbf{Z}_1 \neq \mathbf{Z}$ which also contains the set $\mathbf{S}$. The computation is divided into three steps.

Step 1. Let $Q_m = 0$, and the others $Q_j \neq 0$. For all pairs of vectors $Q_j,\ Q_k,\ 1 \le j,\ k < m, j \neq k$ compose the determinants

$$\det(Q_j Q_k) \overset{def}{=} \Delta_{jk}. \tag{11.1}$$

Among pairs with $\Delta_{jk} \neq 0$ we find one with $|\Delta_{jk}| = \min |\Delta_{jk}| \neq 0$ for all $j, k = 1, \dots, m-1$. If there are several such pairs, then we take any one of them. Suppose for the sake of simplicity that it is the pair $Q_1,\ Q_2$. Other points $Q_3, \dots, Q_{m-1}$ are arbitrary ordered.

Step 2. Let us find the basis of the lattice generated by the vectors Q_1, Q_2, Q_3. Let $Q_3 = aQ_1 + bQ_2$, where a and b are rational. Denote the integral part of the real number a as $[a]$ and the fractional part as $\{a\}$, i.e. $\{a\} = a - [a]$. Denote $Q_3' = \{a\}Q_1 + \{b\}Q_2$. Suppose that $\min|\det(Q_3' Q_i)|$ for $i = 1, 2$ is attained at $i = 1$. Then we take Q_1 and Q_3' as the basis vectors and use them to express Q_2, i.e. we get $Q_2 = a_1 Q_1 + bQ_3'$. Replace the vector Q_2 by $Q_2' = \{a_1\}Q_1 + \{b_1\}Q_3'$. Among the three vectors Q_1, Q_2', Q_3' we find the pair with the minimal modulus of the determinant. Using this pair we expand the third vector, take its fractional part and so on. At some step l we obtain that the fractional part of the third vector equals zero. The latest pair of vectors $Q_2^{(l)},\ Q_3^{(l)}$ gives the basis of the minimal lattice containing the points $Q_1,\ Q_2,\ Q_3$.

Step 3. For the vectors $Q_2^{(l)},\ Q_3^{(l)},\ Q_4$ we repeat Step 2 and obtain vectors $\tilde{Q}_3,\ \tilde{Q}_4$ and so on. After looking through all $Q_j,\ j \le m-1$, we obtain the pair of vectors $Q_{m-2}^*,\ Q_{m-1}^*$ which is the basis of the minimal lattice containing the set $\mathbf{S}$.

Remark. An analogous algorithm allows to find the basis of the minimal lattice in $\mathbb{R}^n$ containing a given finite set $\mathbf{S}$. If $n = 1$ it is the Euclid algorithm.

Example. Let us consider the equation (2.1). Its support consists of four points (2.1$'$). Shift them by the vector $-Q_3 = -(1, 0)$. We obtain

$$Q_1' = (-5,\ 1), \quad Q_2' = (-1,\ 3), \quad Q_3' = 0, \quad Q_4' = (-3,\ 2).$$

For vectors Q_1', Q_2', Q_4' we compute the pairwise determinants

$$\Delta_{14} = \begin{vmatrix} -5 & 1 \\ -3 & 2 \end{vmatrix} = -7, \quad \Delta_{12} = \begin{vmatrix} -5 & 1 \\ -1 & 3 \end{vmatrix} = -14, \qquad \Delta_{42} = \begin{vmatrix} -3 & 2 \\ -1 & 3 \end{vmatrix} = -7. \tag{11.2}$$

Thus we can use the vectors Q_1', Q_4' or Q_2', Q_4' as the initial pair. Let us take Q_1', Q_4' for example. We are looking for the expansion $Q_2' = aQ_1' + bQ_4' = a(-5, 1) + b(-3, 2)$; i.e. we solve the linear system of equations

$$\begin{aligned} -5a - 3b &= -1, \\ a + 2b &= 3. \end{aligned} \tag{11.3}$$

We obtain $a = -1$, $b = 2$. Since $\{a\} = \{b\} = 0$, then the vectors $B_1 = Q_1'$ and $B_2 = Q_4'$ generate the basis of the lattice of shifted support of the equation (2.1).

The preliminary versions of that article are preprints [47, 48].

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