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Removable isolated singularities for solutions of quasilinear parabolic equations

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Abstract. We have obtained the best possible conditions for removable singularity at the point for solutions of quasilinear parabolic equations of divergent form. Cases of interior singular point $(x_0, t_0) \in Q_T \subset \mathbb{R}^{n+1}$ we have established a removability result for a solution $u(x,t) \in V^{2,p}_{loc}(Q_T) \setminus (x_0, t_0)$ under a condition $u(x,t) = o([|x-x_0| + |t-t_0|^{\frac{1}{p+n(p-2)}}]^{-n})$. As a particular case we have a precise removability condition for p-Laplacian evolution equation. The proof is based on a new approach connected with point-wise estimates of solutions in puncturated domains.

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1. Introduction

The paper is devoted to the study of conditions for removable isolated singularities for solutions of quasilinear parabolic equations of divergent form

$$\frac{\partial u}{\partial t} - \sum_{j=1}^{n} \frac{\partial}{\partial x_j} a_j\Big(x, t, u, \frac{\partial u}{\partial x}\Big) + a_0\Big(x, t, u, \frac{\partial u}{\partial x}\Big) = 0 \tag{1.1}$$

in $Q_T \setminus (x_0, t_0)$ where $Q_T = \Omega \times [0, T]$, Ω is bounded open set in $\mathbb{R}^n$, $x_0 \in \Omega$, $t_0 \in [0, T]$. It means that we study conditions for the behavior of $u(x,t)$ near singular point (x_0, t_0) which guarantees that the extension $\tilde{u}(x,t)$ of $u(x,t)$ to Q_T satisfies the equation (1.1) in Q_T.

We will distinguish two cases: $t_0 = 0$ or $t_0 > 0$. In the first case we assume additionally an initial condition

$$u(x, 0) = 0 \quad \text{for} \quad x \in \Omega \setminus \{x_0\}. \tag{1.2}$$

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We consider the equation (1.1) with nonlinear growth of coefficients $a_j(x,t,u,\xi)$,
$j = 0, 1, \ldots, n$ with respect to u, ξ. In particular a parabolicity condition is formulated in the form

$$\sum_{j=1}^{n} a_j(x,t,u,\xi)\xi_j \geq \nu_1|\xi|^p - g_1(x,t)|u|^p - f_1(x,t) \tag{1.3}$$

with $p > 2$, positive constant ν_1 and some integrability conditions for functions $g_1(x,t)$, $f_1(x,t)$. It means that p-Laplace evolution equation is involved in our consideration.

We formulate the removability result for the problem (1.1), (1.2) in the form of the behavior of the function

$$M(r) = \sup\{|u(x,t)| : (x,t) \in \mathcal{D}(R_0, x_0) \setminus \mathcal{D}(r, x_0)\} \tag{1.4}$$

where

$$\mathcal{D}(r, x_0) = \Big\{(x,t) \in \mathbb{R}^n \times [0,\infty) : \Big(\frac{|x-x_0|}{r}\Big)^p + \frac{t}{r^{p+n(p-2)}} \leq 1\Big\}$$

and R_0 is some fixed number.

We prove in the Theorem 2.2 that the singularity of the solution $u(x,t)$ of the problem (1.1), (1.2) is removable if

$$\lim_{r\to 0} M(r)r^n = 0 \tag{1.5}$$

and the function r^n is best possible in (1.5) for removable singularity condition.

This type result is well-known for heat equation. It is known for nonnegative solutions for linear parabolic equations with measurable coefficients that follows from the paper [1] of D. G. Aronson. For nonnegative solutions of general quasilinear parabolic equations of the type (1.1) with $p = 2$ in (1.2) an analogous result follows from the paper of D. G. Aronson and J. Serrin [2] where it is proved an inequality

$$M(r)r^n \geq K \tag{1.6}$$

with positive constant K for non-removable singularity.

Preciseness of our condition (1.5) confirms well known Barenblatt's singular solution [3]

$$u(x,t) = t^{-\frac{n}{p+n(p-2)}} \max{}^{\frac{p-1}{p-2}}\Big\{K_1 - K_2\Big(\frac{|x|}{t^{\frac{1}{p+n(p-2)}}}\Big)^{\frac{p}{p-1}}, 0\Big\} \tag{1.7}$$

of the p-Laplacian evolution equation

$$\frac{\partial u}{\partial t} - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0.$$

In (1.7) K_1 and K_2 are some positive numbers. It is clear that for the solution (1.7) an inequality $0 < C_1 \leq M(r)r^n \leq C_2 < +\infty$ is satisfied.

Remark that properties of solutions of the equation (1.1) for $p > 2$ essentially different from corresponding properties for $p = 2$. It is simple to see by the analysis of the behavior of the solution $u(x,t)$ given by the formula (1.7).

Many authors studied problems of singularities of solutions of special form parabolic equations with Laplace of p-Laplace operators in principle part. Review of these results can be found in monograph of L. Veron [12].

The study is based on precise point-wise estimates of solutions in puncturated domains. This method was developed by I. V Skrypnik (see, for example, [11]) and it was applied in [8] for the proof of precise removability condition for elliptic equations.

Our approach gives a possibility to prove a non-existence of singular solution for quasilinear parabolic equations with absorption term. For model p-Laplacian parabolic equations the results are in papers [5, 6]. Precise results for general quasilinear elliptic equations are published in [10]. We are planning to publish these results and results on solutions with singularities on smooth manifolds in forthcoming papers.

The paper is organized as follows. In Section 2 assumptions and main results are formulated. The boundedness of a singular solution satisfying an inequality

$$M(r) \leq K r^{\gamma - n} \quad \text{for} \quad 0 < r \leq R_0 \tag{1.8}$$

with positive constants K, γ is proved in Section 3. This result is analogous to known Serrin's result [9] for elliptic case. Auxiliary integral estimates for solutions with isolated singularity are established in Section 4. We prove the fundamental point-wise estimate of singular solution in Section 5. We establish in this section that the condition (1.5) implies the inequality (1.8). The theorem on the removability of the singularity is proved in Section 6.

2. Formulation of assumptions and main results

We assume that functions $a_j(x,t,u,\xi),\ j = 0,1,\ldots,n$ in the equation (1.1) satisfy the following conditions:

$a_1)$ $a_j(x,t,u,\xi),\ j = 0,1,\ldots,n$ are defined for $(x,t,u,\xi) \in \Omega \times (0,T) \times \mathbb{R}^1 \times \mathbb{R}^n)$ and they are measurable functions of x,t for all $(u,\xi) \in$

$\mathbb{R}^1 \times \mathbb{R}^n$ and continuous functions of u, ξ for almost all points $x, t \in \Omega \times (0, T)$;

a_2) there exist numbers $p \in [2, n), \nu_1, \nu_2 > 0$ such that for all values of x, t, u, ξ inequalities (1.3) and

$$\begin{gathered} |a_j(x,t,u,\xi)| \le \nu_2 |\xi|^{p-1} + g_2(x,t)|u|^{p-1} + f_2(x,t), \quad j = 1, \dots, n \\ |a_0(x,t,u,\xi)| \le \nu_3(x,t)|\xi|^{p-1} + g_3(x,t)|u|^{p-1} + f_3(x,t) \end{gathered} \tag{2.1}$$

hold with nonnegative functions $\nu_3(x,t), g_1(x,t), f_i(x,t)$ and such that

$$\begin{gathered} H(x,t) \in L^{r_0}(0,T; L^{q_0}(\Omega)), \\ \frac{p + n(p-2)}{r_0} + \frac{n}{q_0} = p(1-\delta); \\ r_0, q_0 \ge 1, \end{gathered}$$

$$\begin{aligned} H(x,t) = 1 + \nu_3^p(x,t) + f_1(x,t) + [f_2(x,t)]^{\frac{p}{p-1}} + f_3(x,t) \\ + g_1(x,t) + [g_2(x,t)]^{\frac{p}{p-1}} + g_3(x,t). \end{aligned}$$

Let us consider a solution $u(x,t)$ of the equation (1.1) that has isolated singularity at the point $(0,0)$ and satisfies the initial condition (1.2). By a solution of the problem (1.1), (1.2) we mean a function $u(x,t)$ satisfying a including

$$u(x,t)\zeta(x,t) \in V^{2,p}(Q_T) = C(0,T; L^2(\Omega)) \cap L^p(0,T; W^{1,p}(\Omega))$$

and an integral identity

$$\begin{aligned} I(u,\varphi) \equiv \int\limits_\Omega u(x,\tau)\varphi(x,t)\,dx \\ + \int\limits_0^r \int\limits_\Omega \Big\{ -u\frac{\partial \varphi}{\partial t} + \sum_{i=1}^n a_i\Big(x,t,u,\frac{\partial u}{\partial x}\Big)\frac{\partial \varphi}{\partial x_i} + a_0\Big(x,t,u,\frac{\partial u}{\partial x}\Big)\varphi \Big\} dx\,dt = 0, \end{aligned} \tag{2.2}$$

with $\varphi(x,t) = \psi(x,t)\zeta(x,t)$ where $\psi \in V^{2,p}(Q_T)$ is an arbitrary function such that $\frac{\partial \psi}{\partial t} \in L^2(Q_T)$. Here τ is a number satisfying an inequality $0 < \tau \le T$, ζ is a arbitrary function such that $\zeta(x,t) \in C^\infty(\overline{Q}_T)$ and $\zeta(x,t)$ is equal to zero near $(0,0) \cup \{\partial\Omega \times (0,T)\}$.

We will say that the singularity at the point $(0,0)$ of the solution $u(x,t)$ of the problem (1.1), (1.2) is removable if the integral identity (2.2) is satisfied for all functions $\varphi = \psi\bar{\zeta}$ and all $\tau \in (0,T)$ where ψ is the same function as above and $\bar{\zeta}$ is an arbitrary function such that $\bar{\zeta} \in C^\infty(\overline{Q}_T)$ and $\bar{\zeta}$ us equal to zero near $\partial\Omega \times (0,T)$.

Introduce the Steklov averaging of $w \in L^1(Q_T)$

$$
\begin{aligned}
&[w(x,t)]_h = \frac{1}{h}\int\limits_t^{t+h} w(x,s)\,ds \quad t \in (0, T-h],\\
&[w(x,t)]_h = 0 \quad \text{for } t > T-h.
\end{aligned}
\tag{2.3}
$$

Standard argument (see, for example, [7]) implies that the identity (2.2) can be equivalently formulated as

$$
\int\limits_{t_1}^{t_2}\int\limits_{\Omega}\Big\{\frac{\partial[u]_h}{\partial t}\varphi + \sum_{i=1}^n \Big[a_i\Big(x,t,u,\frac{\partial u}{\partial x}\Big)\Big]_h \frac{\partial\varphi}{\partial x_i}
+ \Big[a_0\Big(x,t,u,\frac{\partial u}{\partial x}\Big)\Big]_h\varphi\Big\}dx\,dt = 0 \tag{2.4}
$$

with $\varphi = \psi\zeta$ where ζ is the same function as in (2.1), h, t_1, t_2 are numbers satisfying an inequality $0 < h < t_1 < t_2 < T-h$, $\psi \in L^p(0,T;W^{1,p}(\Omega))$ is an arbitrary function.

Let R_0 be some number satisfying a condition

$$
\mathcal{D}(R_0) \subset \Omega \times [0,T) \tag{2.5}
$$

where $\mathcal{D}(r) = \mathcal{D}(r,0)$ and $\mathcal{D}(r,0)$ is the same as in (1.4) and let $M(r)$ be defined by the equality (1.4) with $x_0 = 0$. It follows from [4] that $u(x,t)$ is Hölder function on $\mathcal{D}(R_0)\setminus\mathcal{D}(r)$ and therefore $M(r)$ is a finite number. We omit R_0 in the notation of $M(r)$ since main results formulated below do not depend on the choice of R_0 and the number R_0 will be fixed later.

We will understand numbers $\nu_1, \nu_2, n, p, q_0, r_0, \delta, R_0$, norm of the functions $H(x,t)$ in respective spaces as known parameters.

The first result on the behavior of the solution near isolated singularity is the following theorem.

Theorem 2.1. *Let conditions* a_1), a_2) *be satisfied and suppose that* $u(x,t)$ *is the solution of the equation* (1.1) *in* Q_T *satisfying the condition* (1.2) *and the inequality*

$$
M(r) \le \frac{K_1}{r^{n-\gamma_1}} \quad \textit{for} \quad 0 < r \le R_0 \tag{2.6}
$$

with some positive constants K_1, γ_1*. Then there exists a constant* M_1 *depending only on known parameters and* K_1, γ_1 *such that an inequality*

$$|u(x,t)| \le M_1 \quad \textit{for} \quad (x,t) \in \mathcal{D}\Big(\frac{R_0}{2}\Big) \tag{2.7}$$

holds.

We will see that the inequality (2.7) implies the removability of the singularity at $(0,0)$ immediately.

This theorem is analogous to the well-known Serrin's result on removable singularity at the point for quasilinear elliptic equations [9].

Using Theorem 2.1 and precise analysis of the behavior of $u(x,t)$ near $(0,0)$ we establish the best condition on removable singularity.

We will formulate the following result in the form

$$\lim_{r\to 0} M(r)R(r) = 0 \tag{2.8}$$

where R such positive function that $R(r) \to 0$ as $r \to 0$. We will say that the function R is the best possible for the removability of the singularity at the point $(0,0)$ if the assumption (2.8) implies the removability of the singularity at $(0,0)$ an it is not possible to find another function $\overline{R}$ such that the assumption

$$\lim_{r\to 0} M(r)\overline{R}(r) = 0$$

guarantees the removability of the singularity at $(0,0)$ and $\lim_{r\to 0} \frac{\overline{R}(r)}{R(r)} = 0$.

Theorem 2.2. *Let conditions* $a_1)$, $a_2)$ *be satisfied and suppose that* $u(x,t)$ *is the solution of the equation* (1.1) *in* Q_T *satisfying the condition* (1.2) *and the equality*

$$\lim_{r\to 0} M(r)r^n = 0. \tag{2.9}$$

Then the singularity of u *at the point* $(0,0)$ *is removable and the function* r^n *is the best possible for the removability of the singularity at the point* $(0,0)$*.*

This theorem is analogous to the precise result on removable singularity at the point for quasilinear elliptic equations that was published by authors in [8].

Our results on the removable singularity at the point $(0,t_0)$, $t_0 > 0$, are analogous to Theorems 2.1, 2.2. By a solution of the equation (1.1) with the singularity at the point $(0,t_0)$ we mean a function $u(x,t)$ satisfying an including $u\xi \in V^{2,p}(Q_T)$ and an integral identity $I(u,\varphi) = 0$

with $\varphi = \psi\xi$ where $I(u,\varphi)$ is defined by (2.2), $\psi \in L^p(0,T;W^{1,p}(\Omega))$ is an arbitrary function such that $\frac{\partial\psi}{\partial t} \in L^2(Q_T)$, ξ is an arbitrary function such that $\xi \in C^\infty(\overline{Q}_T)$ and ζ is equal to zero near $(0,t_0)\cup\{\partial\Omega\times(0,T)\}\cup\{\Omega\times\{0\}\}$. Removable singularity at $(0,t_0)$ is understood analogously as for the point $(0,0)$.

Define for $r > 0$

$$\mathcal{D}_*(r); \Big\{(x,t) \in \mathbb{R}^n \times \mathbb{R}^1 : \Big(\frac{|x|}{r}\Big)^p + \frac{|t-t_0|}{r^{p+n(p-2)}} \le 1\Big\}$$

and let R_0^* be some number satisfying a condition $\mathcal{D}_*(R_0^*) \subset Q_T$. Denote for $0 < r < R_0^*$

$$M_*(r) = \sup\{|u(x,t)| : (x,t) \in \mathcal{D}_*(R_0^*) \setminus \mathcal{D}_*(r)\}. \tag{2.10}$$

Theorem 2.3. *Let conditions $a_1)$, $a_2)$ be satisfied and suppose that $u(x,t)$ is the solution of the equation (1.1) in Q_T with the singularity at $(0,t_0)$, $t_0 > 0$. Assume that the inequality*

$$M_*(r) \le \frac{K_2}{r^{n-\gamma_2}} \quad \text{for} \quad 0 < r \le R_0^* \tag{2.11}$$

is satisfied with some positive constants K_2, γ_2. Then there exists a constant M_2 depending only on known parameters and R_0^*, K_2, γ_2 such that the estimate

$$|u(x,t)| \le M_2 \quad \text{for} \quad (x,t) \in \mathcal{D}_*\Big(\frac{R_0^*}{2}\Big) \tag{2.12}$$

holds.

Theorem 2.4. *Let conditions $a_1)$, $a_2)$ be satisfied and suppose that $u(x,t)$ is the solution of the equation (1.1) in Q_T with the singularity at $(0,t_0)$. Assume that the condition*

$$\lim_{r\to 0} M_*(r) r^n = 0 \tag{2.13}$$

is satisfied. Then the singularity of u at the point $(0,t_0)$ is removable and the function r^n is the best possible for the removability of the singularity at the point $(0,t_0)$.

Remark 2.1. For a fixed solution $u(x,t)$ we have from the assumption (2.9) the following estimate

$$|u(x,t)| \le K_0\{|x| + t^{\frac{1}{p+(p-2)n}}\}^{-n} \tag{2.14}$$

for $(x,t) \in \mathcal{D}(R_0)$ with some constant K_0 dependent on $u(x,t)$.

Remark 2.2. Equations with more strong growth of coefficients $a_j(x,t,u,\xi)$ can be considered as a special case of introduced above class of equations. Let us suppose that conditions (1.3), (2.1) are replaced by the following inequalities

$$\begin{aligned} &\sum_{j=1}^{n} a_j(x,t,u,\xi)\xi_j \geq \nu_1|\xi|^p - \nu_2|u|^q - f_1(x,t), \\ &|a_j(x,t,u,\xi)| \leq \nu_2|\xi|^{p-1} + \nu_2|u|^{q\frac{p-1}{p}} + f_2(x,t), \\ &|a_0(x,t,u,\xi)| \leq \nu_3(x,t)|\xi|^{p-1} + \nu_2|u|^{q-1} + f_3(x,t) \end{aligned} \tag{2.15}$$

with $q < p + \frac{p}{n}$ and the same functions $\nu_3(x,t), f_i(x,t)$ as in (2.1).

We consider the removability of a isolated singularity for a fixed solution satisfying the assumption (2.9). Denoting

$$g_1(x,t) = g_3(x,t) = \nu_2|u(x,t)|^{q-p}, \qquad g_2(x,t) = \nu_2|u(x,t)|^{(\frac{q}{p}-1)(p-1)}$$

and using the inequality (2.14) we check that such functions $g_i(x,t)$ satisfy assumption a_2). Thus the assertions of formulated above theorems remain true for coefficients satisfying inequalities (2.15).

3. Proof of Theorem 2.1

We fix the notation $\eta_r(x,t)$ for a function $\omega((\frac{|x|}{r})^p + \frac{t}{r^{p+n(p-2)}}), r > 0$ where ω is a function of the space $C^\infty(\mathbb{R}^1)$ satisfying conditions

$$\begin{gathered} \omega(s) = 0 \quad \text{for } s \leq 1, \qquad \omega(s) = 1 \quad \text{for } s \geq 2^p, \\ 0 \leq \frac{d\omega(s)}{ds} \leq 1, \quad 0 \leq \omega(s) \leq 1. \end{gathered} \tag{3.1}$$

Define for $k, l \geq 0$, $u \in \mathbb{R}^1$

$$F_{kl}(u) = (1 + u_k^2)^l [1 + u^2]^{-\alpha} u \tag{3.2}$$

where $u_k^2 = \min\{u^2, k^2\}$ and a number α is defined by the equality

$$\alpha = \frac{1}{4} + \frac{1}{4} \max\left\{\frac{n - 2\gamma_1}{n - \gamma_1}, \frac{2n - \gamma_1 p}{2n - \gamma_1}\right\} \tag{3.3}$$

with the number γ_1 from the condition (2.6).

We substitute in the integral identity (2.4) a test function

$$\varphi_1(x,t) = F_{kl}([u(x,t)]_h)\psi^m(x,t)\eta_r^m(x,t)$$

where $u(x,t)$ is the solution of the problem (1.1), (1.2) satisfying conditions of the Theorem 2.1, $\psi(x,t)$ is a fixed function such that $\psi \in C^\infty(\overline{Q}_T)$, $0 \le \psi(x,t) \le 1$, ψ is equal to one on $\mathcal{D}(\frac{R_0}{2})$ and to zero outside $\mathcal{D}(R_0)$, $m \le p$, $[u(x,t)]_h$ is the Steklov average of $u(x,t)$.

Evaluating the term of (2.4) with the derivative of $[u]_h$ on t and the indicated choice of the test function we have for $t_1 = \theta \ge h,\ t_2 = \tau \le T-h$

$$\int_\theta^\tau \int_\Omega F_{kl}([u]_h)\psi^m \eta_r^m \frac{\partial [u]_h}{\partial t}\, dx\, dt = I_1([u]_h) + \sum_{j=1}^{4} I_{j\theta}([u]_h) \tag{3.4}$$

where

$$\begin{aligned}
I_1([u]_h) &= \int_\Omega G_{kl}([u(x,\tau)]_h)\psi^m(x,\tau)\eta_r^m(x,\tau)\, dx,\\
I_{2\theta}([u]_h) &= -\int_\Omega G_{kl}([u(x,\theta)]_h)\psi^m(x,\theta)\eta_r^m(x,\theta)\, dx,\\
I_{3\theta([u]_h)} &= -\int_\theta^\tau \int_\Omega G_{kl}([u]_h)\frac{\partial \psi^m}{\partial t}\eta_r^m\, dx\, dt,\\
I_{4\theta}([u]_h) &= -\int_\theta^\tau \int_\Omega G_{kl}([u]_h)\psi^m \frac{\partial \eta_r^m}{\partial t}\, dx\, dt.
\end{aligned} \tag{3.5}$$

The function $G_{kl}(u)$ is defined by equalities

$$\begin{aligned}
G_{kl}(u) &= \frac{1}{2(l-\alpha+1)}[1+u^2]^{l-\alpha+1} \quad \text{for} \quad |u| \le k,\\
G_{kl}(u) &= G_{kl}(k) + \frac{(1+k^2)^l}{2(1-\alpha)}\{(1+u^2)^{1-\alpha} - (1+k^2)^{1-\alpha}\} \quad \text{for} \quad |u| > k
\end{aligned}$$

and satisfies the following estimate

$$G_{kl}(u) \le C_1(1+u_k^2)(1+u^2)^{1-\alpha} \le C_2(l+1)G_{kl}(u). \tag{3.6}$$

Here and further we denote by $C_j,\ j = 1,2,\dots$ positive constants depending only on known parameters, γ_1, K_1.

Letting $h \to 0$ in (3.4) we obtain for an arbitrary $\theta > 0$

$$\lim_{h\to 0} \int_\theta^\tau \int_\Omega F_{kl}([u]_h)\psi^m \eta_r^m \frac{\partial [u]_h}{\partial t}\, dx\, dt = I_1(u) + \sum_{j=2}^{4} I_{j\theta}(u) \tag{3.7}$$

and then passing to the limit $\theta \to 0$ we get

$$I_1(u) + \sum_{j=2}^{4} I_{j\theta}(u) + \sum_{j=5}^{7} I_j(u) = 0. \tag{3.8}$$

where

$$\begin{aligned} I_5(u) &= \sum_{i=1}^{n} \int_0^\tau \int_\Omega a_i\Big(x,t,u,\frac{\partial u}{\partial x}\Big)\frac{\partial}{\partial x_i}F_{kl}(u)\psi^m\eta_r^m\,dx\,dt, \\ I_6(u) &= \sum_{i=1}^{n} \int_0^\tau \int_\Omega a_i\Big(x,t,u,\frac{\partial u}{\partial x}\Big)F_{kl}(u)\frac{\partial}{\partial x_i}[\psi^m\eta_r^m]\,dx\,dt, \\ I_7(u) &= \int_0^\tau \int_\Omega a_0\Big(x,t,u,\frac{\partial u}{\partial x}\Big)F_{kl}(u)\psi^m\eta_r^m\,dx\,dt. \end{aligned} \tag{3.9}$$

We estimate terms in (3.8) using inequalities (2.1), (3.6) and Young's inequality and we obtain

$$\begin{aligned} &\frac{1}{l+1}\int_\Omega [1+u_k^2(x,\tau)]^l[1+u^2(x,\tau)]^{1-\alpha}\psi^m(x,\tau)\eta_r^m(x,\tau)\,dx \\ &\qquad + \iint_{Q_r}[1+u_k^2]^l[1+u^3]^{-\alpha}\Big|\frac{\partial u}{\partial x}\Big|^p\psi^m\eta_r^m\,dx\,dt \\ &\le C_3(l+m)^p\bigg\{1+\iint_{Q_r}[1+u_k^2]^l[1+u^2]^{\frac{p}{2}-\alpha}H(x,t)\psi^{m-p}(x,t)\eta_r^m(x,t)\,dx\,dt \\ &+\iint_{Q_r}[1+u_k^2]^l\Big([1+u^2]^{1-\alpha}\Big|\frac{\partial \eta_r}{\partial t}\Big| + [1+u^2]^{\frac{p}{2}-\alpha}\Big|\frac{\partial \eta_r}{\partial x}\Big|^p\Big)\psi^m\eta_r^{m-p}\,dx\,dt\bigg\} \end{aligned} \tag{3.10}$$

where $Q_r = \Omega\times(0,\tau)$ and the function $H(x,t)$ is defined by the condition a_2).

Direct calculations and conditions a_2), (2.6), (3.3) imply that the integral

$$\iint_{Q_T}[1+u^2]^{\frac{p}{2}-\alpha}H(x,t)\psi^{m-p}(x,t)\,dx\,dt$$

is finite and the integral

$$\iint_{Q_T}\Big([1+u^2]^{1-\alpha}\Big|\frac{\partial \eta_r}{\partial t}\Big| + [1+u^2]^{\frac{p}{2}-\alpha}\Big|\frac{\partial \eta_r}{\partial x}\Big|^p\Big)dx\,dt$$

tends to zero as $r \to 0$. Then we may pass to the limit in (3.10) as $r \to 0$ and we get an inequality

$$\sup_{0<\tau<T} \int_\Omega [1 + u_k^2(x,\tau)]^l [1 + u^2(x,\tau)]^{1-\alpha} \psi^m(x,\tau)\,dx + \iint_{Q_T} [1 + u_k^2(x,t)]^l [1 + u^2(x,t)]^{-\alpha} \Big|\frac{\partial u}{\partial x}\Big|^p \psi^m(x,t)\,dx\,dt \le C_3(l+m)^{p+1} \Big\{ 1 + \iint_{Q_T} [1 + u_k^2(x,t)]^l [1 + u^2(x,t)]^{\frac{p}{2}\alpha} \times H(x,t)\psi^{m-p}(x,t)\,dx\,dt \Big\}. \quad (3.11)$$

Now we will show that the integral

$$I(l,m) = \iint_{Q_T} [1 + u^2(x,t)]^{l+\frac{p}{2}-\alpha} H(x,t)\psi^{m-p}(x,t)\,dx\,dt \quad (3.12)$$

is finite for an arbitrary positive l with a suitable choice of m. This assertion follows for

$$l_0 = \frac{1}{12}\Big\{1 - \max\Big[\frac{n-2\gamma_1}{n-\gamma_1}, \frac{2n-\gamma_1 p}{2n-\gamma_1}\Big]\Big\}, \qquad m_0 = p \quad (3.13)$$

by using the inequality (2.6), conditions a_2), (3.3) and direct calculation.

Let assume that for some numbers $l_* \ge l_0$, $m_* \ge p$ the integral $I(l_*, m_*)$ is finite. Then from (3.11) and monotone convergence theorem we have

$$\sup_{0<\tau<T} \int_\Omega [1 + u^2(x,\tau)]^{l_*+1-\alpha} \psi^{m_*}(x,\tau)\,dx + \int_{Q_T} [1 + u^2(x,t)]^{l_*-\alpha} \Big|\frac{\partial u}{\partial x}\Big|^p \psi^{m_*}(x,t)\,dx\,dt \le C_4[l_* + m_*]^{p+1}\{1 + I(l_*, m_*)\}. \quad (3.14)$$

Define

$$l^* = (l_* + 1 - \alpha)\Big[\frac{1}{q_0'} - \frac{n-p}{n}\frac{1}{r_0'}\Big] + \Big(l_* + \frac{p}{2} - \alpha\Big)\frac{1}{r_0'} - \frac{p}{2} + \alpha,$$

$$m^* = m_*\Big(\frac{1}{q'} + \frac{p}{nr'}\Big) + p$$

where $q_0' = \frac{q_0}{q_0-1}$, $r_0' = \frac{r_0}{r_0-1}$ and numbers q,r_0 were introduced in the condition a_2).

Using Hölder's inequality and the embedding theorem we have

$$\begin{aligned}
&\iint\limits_{Q_T} [1+u_k^2(x,t)]^{l^*+\frac{p}{2}-\alpha} H(x,t)\psi^{m^*-p}(x,t)\,dx\,dt \\
&\quad \le C_5 \Bigg\{ \int\limits_0^T \Bigg\{ \int\limits_\Omega [1+u_k^2(x,t)]^{(l^*+\frac{p}{2}-\alpha)q_0'} \psi^{(m^*-p)q_0'}(x,t)\,dx \Bigg\}^{\frac{r_0'}{q_0'}} dt \Bigg\}^{\frac{1}{r_0'}} \\
&\qquad \le C_5 \Bigg\{ \int\limits_0^T \Bigg\{ \int\limits_\Omega [1+u_k^2(x,t)]^{l_*+1-\alpha} \psi^{m_*}(x,t)\,dx \Bigg\}^{\frac{r_0'}{q_0'} - \frac{n-p}{n}} \\
&\qquad\quad \times \Bigg\{ \int\limits_\Omega [1+u_k^2(x,t)]^{(l_*+\frac{p}{2}-\alpha)\frac{n}{n-p}} \psi^{m_*}(x,t)\,dx \Bigg\}^{\frac{n-p}{n}} dt \Bigg\} \\
&\quad \le C_6 (l_*+m_*)^p \sup_t \Bigg\{ \int\limits_\Omega [1+u_k^2(x,t)]^{l_*+1-\alpha} \psi^{m_*}(x,t)\,dx \Bigg\}^{\frac{1}{q'0} - \frac{n-p}{n}\frac{1}{r_0'}} \\
&\qquad\quad \times \Bigg\{ \iint\limits_{Q_T} \Big\{ [1+u_k^2(x,t)]^{l_*-\alpha} \Big|\frac{\partial u}{\partial x}\Big|^p \psi^{m_*}(x,t) \\
&\qquad\qquad + [1+u_k^2(x,t)]^{l_*+\frac{p}{2}-\alpha} \psi^{m_*-p}(x,t) \Big\} dx\,dt \Bigg\}^{\frac{1}{r_0'}}. \qquad (3.15)
\end{aligned}$$

Now the assumption on the finiteness of $I(l_*, m_*)$ and the inequality (3.14) imply that the right hand side of (3.15) is estimated by a constant independent on k. Then using monotone convergence theorem we conclude from (3.15) that the integral $I(l^*, m^*)$ is finite and estimate

$$I(l^*, m^*) \le C_7 [l_* + m_*]^{2p+1} \{1 + I(l_*, m_*)\}^{\frac{1}{q_0' + \frac{p}{n}\frac{1}{r_0'}}} \qquad (3.16)$$

holds.

Remark that the assumption on q_0, r_0 implies the following equality

$$\frac{1}{q_0'} + \frac{p}{n}\frac{1}{r_0'} = 1 + \frac{p\delta}{n} + \frac{p-2}{r_0}, \qquad l^* + \frac{p}{2} - \alpha = \Big(l_* + \frac{p}{2} - \alpha\Big)(1+k) - \beta$$

where $\beta = (\frac{p}{2} - 1)(\frac{p\delta}{n} + \frac{p-1}{r_0})$, $k = \frac{p\delta}{n} + \frac{p-2}{r_0}$.

Let us define sequences $\{l_j\}, \{m_j\}$ by the equalities

$$\begin{aligned} &l_j + \frac{p}{2} - \alpha - \frac{\beta}{k} = \Big(l_0 + \frac{p}{2} - \alpha - \frac{\beta}{k}\Big)(1+k)^j, \\ &m_j + \frac{p}{k} = \Big(p + \frac{p}{k}\Big)(1+k)^j, \quad j = 1, 2, \dots \end{aligned} \tag{3.17}$$

It is simple to check that $l_0 + \frac{p}{2} - \alpha - \frac{\beta}{k} > 0$. The inequality (3.16) with $l_* = l_{j-1},\ m_* = m_{j-1}$ implies an estimate

$$I(l_j, m_j) \le C_8 (1+k)^{j(2p+1)} \{1 + I(l_{j-1}, m_{j-1})\}^{1+k}. \tag{3.18}$$

Starting from $j = 1$ and repeating the application of the inequality (3.18) we obtain that $I(l_j, m_j)$ is finite for an arbitrary j. Now the Moser iteration process gives us the boundedness of the function $u(x,t)$ and the estimate (2.7). This is the end of the proof of Theorem 2.1.

4. Integral estimates of the singular solution

We will assume further that

$$\lim_{r\to 0} M(r) = \infty. \tag{4.1}$$

We fix some number R_1 from the interval $(0, R_0)$ such that

$$M(R_1) \ge 1$$

and denote for $r \in (0, R_1]$

$$M^*(r) = \frac{1}{r^n} \max\{M(\rho)\rho^n : r \le \rho \le R_1\} + 1. \tag{4.2}$$

Define the function $u_R(x,t)$ for $R \in (0, R_1)$ and the set $E(R)$ by equalities

$$\begin{aligned} &u_R(x,t) = \max\{u(x,t) - M(R), 0\} \quad \text{for} \quad (x,t) \in \mathcal{D}(R), \\ &u_R(x,t) = 0 \quad \text{for} \quad (x,t) \in Q_T \setminus \mathcal{D}(R), \\ &E(R) = \{(x,t) \in \mathcal{D}(R) : u(x,t) > M(R)\}. \end{aligned} \tag{4.3}$$

Lemma 4.1. *Assume that conditions of Theorem 2.2 are satisfied. Then there exists a constant K_3 depending only on known parameters such that an estimate*

$$\sup_{0<\tau<T} \int_\Omega u_R^2(x,\tau)\eta_r^2(x,\tau)\,dx + \iint_{E(R)} \Big|\frac{\partial u}{\partial x}\Big|^p \eta_r^p(x,t)\,dx\,dt \le K_3 M(r)[M^*(r)r^n]^{p-1} \tag{4.4}$$

holds with $0 < r < R \le R_1$ and the same function η_r as in Section 3.

Proof. We substitute in the identity (2.4) with $t_1 = \theta > h,\ t_2 = \tau < T-h$ a test function

$$\varphi_2(x,t) = [[u(x,t)]_h - M(R)]_+ \eta_r^p(x,t) \quad \text{with} \quad 0 < r < R < R_1.$$

Here $[[u(x,t)]_h - M(R)]_+ = \max\{[u(x,t)]_h - M(R), 0\}$.

Evaluating the term of (2.4) with the derivative of $[u]_h$ on t and the test function φ_2 we have

$$\begin{aligned}\int_\theta^\tau \int_\Omega \frac{\partial [u]_h}{\partial t} \varphi_2(x,t)\,dx\,dt \\ &= \frac{1}{2}\int_\Omega [[u(x,\tau)]_h - M(R)]_+^2 \eta_r^p(x,\tau)\,dx \\ &- \frac{1}{2}\int_\Omega [[u(x,\theta)]_h - M(R)]_+^2 \eta_r^p(x,\theta)\,dx \\ &- \frac{p}{2}\int_\theta^\tau \int_\Omega [[u(x,\tau)]_h - M(R)]_+^2 \eta_r^{p-1} \frac{\partial \eta_r}{\partial t}\,dx\,dt. \end{aligned} \tag{4.5}$$

Letting $h \to 0$ we obtain for all $\theta \in (0,\tau),\ \tau \in (0,T)$

$$\begin{aligned}\lim_{h\to 0} \int_\theta^\tau \int_\Omega \frac{\partial [u]_h}{\partial t} \varphi_2(x,t)\,dx\,dt \\ &= \frac{1}{2}\int_\Omega u_R^2(x,\tau)\eta_r^p(x,\tau)\,dx - \frac{1}{2}\int_\Omega u_R^2(x,\theta)\eta_r^p(x,\theta)\,dx \\ &- \frac{p}{2}\int_\theta^r \int_\Omega u_R^2(x,t)\eta_r^{p-1}(x,t)\frac{\partial \eta_r(x,t)}{\partial t}\,dx\,dt, \end{aligned} \tag{4.6}$$

using the inequalities (1.3) and (2.1) and Young's inequality, we get

$$\begin{aligned}\lim_{h\to 0} \int_\theta^\tau \int_\Omega \Big\{ \sum_{i=1}^n \Big[a_i\Big(x,t,u,\frac{\partial u}{\partial x}\Big)\Big]_h \frac{\partial \varphi_2}{\partial x_i} + \Big[a_0\Big(x,t,u,\frac{\partial u}{\partial x}\Big)\Big]_h \varphi_2 \Big\} dx\,dt \\ &\geq \int_\theta^\tau \int_\Omega \Big\{ \frac{\nu_1}{2}\Big|\frac{\partial u}{\partial x}\Big| \eta_r^p(x,t) - C_9 u^p(x,t) H(x,t) \eta_r^p(x,t) \\ &- C_9 u_r^p(x,t) \Big|\frac{\partial \eta_r(x,t)}{\partial x}\Big|^p \Big\} \chi(E(R))\,dx\,dt \end{aligned} \tag{4.7}$$

where $\chi(E(R))$ is a characteristic function of the set $E(R)$ and the function $H(x,t)$ was introduced in the condition a_2).

Letting $\theta \to 0$ we deduce from (4.6), (4.7)

$$\sup_{0<\tau<T} \int_\Omega u_R^2(x,\tau)\eta_r^p(x,\tau)\,dx + \iint_{E(R)} \left|\frac{\partial u}{\partial x}\right|^p \eta_r^p(x,t)\,dx\,dt$$
$$\le C_{10} \iint_{E(R)} \left\{u_R^2(x,t)\left|\frac{\partial \eta_r(x,t)}{\partial t}\right| + u^p(x,t)H(x,t)\eta_r^p(x,t)\right.$$
$$\left. + u_R^p(x,t)\left|\frac{\partial \eta_r(x,t)}{\partial x}\right|^p\right\}dx\,dt. \quad (4.8)$$

Using the notation (4.2) and the condition a_2) on the function $H(x,t)$ we obtain

$$\iint_{E(R)} u^{p-1}(x,t)H(x,t)\eta_r^p(x,t)\,dx\,dt \le C_{11}[M^*(r)r^n]^{p-1} \quad (4.9)$$

with the constant C_{11} independent on r.

Now estimating $u(x,t)$ by $M(r)$ under the integral on the right-hand side of (4.8) and using the inequality (4.9) we deduce the estimate (4.4) and the proof of the lemma is completed. $\square$

We will assume further that numbers r, ρ, R satisfy conditions

$$0 < r < \rho < R \le \frac{R_1}{2}, \quad M(\rho) > 2M(R) \quad (4.10)$$

and introduce notations

$$\begin{aligned} &\Phi_{\rho R}(u) = \min\{\max[u - M(R), 0], M(\rho) - M(R)\} \quad \text{for } u \in \mathbb{R}^1, \\ &E(\rho,R) = \{(x,t) \in Q_T : 0 < u_R(x,t) < M(\rho) - M(R)\}, \\ &F(\rho) = \{(x,t) \in Q_T : u(x,t) > M(\rho)\}. \end{aligned} \quad (4.11)$$

It is clear from the definition of $M(\rho)$ that $F(\rho) \subset \mathcal{D}(\rho)$.

Lemma 4.2. *Assume that conditions of Theorem* 2.2 *are satisfied. Then there exists a positive constant K_4 depending only on known parameters such that the estimate*

$$\sup_{0<\tau<T} \int_\Omega \Phi_{\rho R}^2(u(x,\tau))\eta_r^p(x,\tau)\,dx + \iint_{E(\rho,R)} \left|\frac{\partial u}{\partial x}\right|^p \eta_r^p(x,t)\,dx\,dt$$
$$\le K_4[M(\rho) - M(R)]\Bigg\{[M(r)r^n[M^*(r)r^n]^{p-1}]^{\frac{p-1}{p}}$$

$$+ [M(\rho) - M(R)]^{-\lambda_1} [M^*(r) r^n]^{p-1+\lambda_1} \\ + \iint\limits_{F(\rho)} \nu_3(x,t) \Big| \frac{\partial u}{\partial x} \Big|^{p-1} \eta_r^p(x,t)\, dx\, dt \Big\} \quad (4.12)$$

holds with $\lambda_1 = \frac{n\delta}{2p}$, *the same function* $\eta_r(x,t)$ *as in* (4.4) *and numbers* r, ρ, R *satisfying conditions* (4.10).

Proof. We substitute in the integral identity (2.4) with $t_1 = \theta > h,\ t_3 = \tau < T - h$ a test function

$$\varphi_3(x,t) = \min\{[[u(x,t)]_h - M(R)]_+, M(\rho) - M(R)\} \eta_r^p(x,t).$$

Evaluating the term of (2.4) with the derivative $[u]_h$ on t and $\varphi = \varphi_3$ and letting $h \to 0$ we obtain for all $\theta \in (0,\tau),\ \tau \in (0,T)$

$$\lim_{h\to 0} \int\limits_\theta^\tau \int\limits_\Omega \frac{\partial [u]_h}{\partial t} \varphi_3(x,t)\, dx\, dt \\ = \int\limits_\Omega G_{\rho,R}(u(x,\tau)) \eta_r^p(x,\tau)\, dx - \int\limits_\Omega G_{\rho,R}(u(x,\theta)) \eta_r^p(x,\theta)\, dx \\ - p \int\limits_\theta^\tau \int\limits_\Omega G_{\rho,R}(u(x,t)) \eta_r^{p-1}(x,t) \frac{\partial \eta_r(x,t)}{\partial t}\, dx\, dt \quad (4.13)$$

with the function $G_{\rho,R}(u)$ defined by the equality

$$G_{\rho,R}(u) = \frac{1}{2} \min\{u_R^2, [M(\rho) - M(R)]^2\} + [M(\rho) - M(R)] u_R. \quad (4.14)$$

Using the inequalities (1.3), (2.1) and Young's inequality we obtain

$$\lim_{h\to 0} \int\limits_\theta^\tau \int\limits_\Omega \Big\{ \sum_{i=1}^n \Big[a_i\Big(x,t,u,\frac{\partial u}{\partial x}\Big)\Big]_h \frac{\partial \varphi_3}{\partial x_i} + \Big[a_0\Big(x,t,u,\frac{\partial u}{\partial x}\Big)\Big]_h \varphi_3 \Big\} dx\, dt \\ \geq \int\limits_\theta^\tau \int\limits_\Omega \Big\{ \Big[\frac{\nu_1}{2} \Big| \frac{\partial u}{\partial x} \Big|^p \eta_r^p(x,t) - C_{12} H(x,t) u^p(x,t) \eta_r^p(x,t) \Big] \chi(E(\rho,R)) \\ - C_{12} [M(\rho) - M(R)] \Big[\Big| \frac{\partial u}{\partial x} \Big|^{p-1} + [H(x,t)]^{\frac{p-1}{p}} u^{p-1}(x,t) \Big] \\ \times \eta_r^{p-1}(x,t) \Big| \frac{\partial \eta_r}{\partial x} \Big| \chi(E(R)) \Big\} dx\, dt$$

$$- C_{12}[M(\rho) - M(R)] \int\limits_{\theta}^{\tau} \int\limits_{\Omega} \Big[\nu_3(x,t)\Big|\frac{\partial u}{\partial x}\Big|^{p-1} + H(x,t)u^{p-1}(x,t)\Big] \times \eta_r^p(x,t)\chi(F(\rho))\,dx\,dt \quad (4.15)$$

where $\chi(E(\rho,R))$, $\chi(F(\rho))$ are characteristic functions of sets $E(\rho,R)$, $F(\rho)$ and $H(x,t)$ is the function introduced in the condition a_2).

Letting $\theta \to 0$ and estimating terms of (4.13) we obtain from (4.13), (4.15)

$$\sup_{0<\tau<T} \int\limits_{\Omega} \Phi^2_{\rho R}(u(x,\tau))\eta_r^p(x,\tau)\,dx + \iint\limits_{E(\rho,R)} \Big|\frac{\partial u}{\partial x}\Big|^p \eta_r^p(x,t)\,dx\,dt$$
$$\le C_{13}\Bigg\{[M(\rho) - M(R)]\Bigg(M(r)r^n + \iint\limits_{F(\rho)} \nu_3(x,t)\Big|\frac{\partial u}{\partial x}\Big|^{p-1}\eta_r^p(x,t)\,dx\,dt + I(1) + I(2)\Bigg) + I(3)\Bigg\} \quad (4.16)$$

where

$$I(1) = \iint\limits_{E(R)} \Big\{\Big|\frac{\partial u}{\partial x}\Big|^{p-1} + [H(x,t)]^{\frac{p-1}{p}} u^{p-1}(x,t)\Big\}\eta_r^{p-1}(x,t)\Big|\frac{\partial \eta_r}{\partial x}\Big|\,dx\,dt,$$

$$I(2) = \iint\limits_{F(\rho)} H(x,t)u^{p-1}(x,t)\eta_r^p(x,t)\,dx\,dt,$$

$$I(3) = \iint\limits_{E(\rho,R)} H(x,t)u^p(x,t)\eta_r^p(x,t)\,dx\,dt.$$

Next estimates follow form the condition a_2), the notation (4.2) and direct calculation

$$\iint\limits_{\mathcal{D}(R_0)} H(x,t)u^{p-1+\lambda_1}(x,t)\,dx\,dt \le C_{14}[M^*(r)r^n]^{p-1+\lambda_1}, \quad (4.17)$$

$$\iint\limits_{E(R)} [H(x,t)]^{\frac{p-1}{p}}\Big|\frac{\partial \eta_r}{\partial x}\Big|\,dx\,dt \le C_{14}r^{(n+\delta)(p-1)+1} \quad (4.18)$$

with the constant c_{14} independent on r and $\lambda_1 = \frac{n\delta}{2p}$. Using (4.17) we have immediately

$$\begin{aligned} I(2) &\le C_{15}[M(\rho) - M(R)]^{-\lambda_1}[M^*(r)r^n]^{p-1+\lambda_1}, \\ I(3) &\le C_{15}[M(\rho) - M(R)]^{1-\lambda_1}[M^*(r)r^n]^{p-1+\lambda_1}. \end{aligned} \quad (4.19)$$

The term $I(1)$ can be estimated by using Hölder inequality together with Lemma 4.1 and the inequalities (4.18). Thus

$$I(1) \leq \Big\{ \iint\limits_{E(R)} \Big|\frac{\partial u}{\partial x}\Big|^p \eta_r^p(x,t)\,dx\,dt \Big\}^{\frac{p-1}{p}} \Big\{ \iint\limits_{E(R)} \Big|\frac{\partial \eta_r}{\partial x}\Big|^p \,dx\,dt \Big\}^{\frac{1}{p}}$$
$$+ C_{16} M^{p-1}(r) r^{(n+\delta)(p-1)} \leq C_{17}\{[M(r)r^n][M^*(r)r^n]^{p-1}\}^{\frac{p-1}{p}}. \quad (4.20)$$

If the three previous estimates are inserted into the right-hand side of (4.16) we have the inequality (4.12) an Lemma 4.2 is therefore proved. $\square$

Lemma 4.3. *Assume that conditions of Theorem* 2.2 *are satisfied. Then there exists a positive constant K_5 depending only on known parameters such that the estimate*

$$\iint\limits_{F(\rho)} u_R^{-q}(x,t)\Big|\frac{\partial u}{\partial x}\Big|^p \eta_r^p(x,t)\,dx\,dt$$
$$\leq K_5 [M(\rho) - M(R)]^{-\lambda_2} \big\{ [M(r)r^n [M^*(r)r^n]^{p-1}]^{\frac{p-1}{p}}$$
$$+ [M(\rho)-M(R)]^{-\lambda_1} [M^*(r)r^n]^{p-1+\lambda_1} \big\} \quad (4.21)$$

holds with $q = 1+\lambda_2$, $\lambda_2 = \frac{n\delta}{2p^2}$, the same function $\eta_r(x,t)$ as in (4.4) *and numbers r, ρ, R satisfying conditions* (4.10).

Proof. We substitute in the integral identity (2.4) with $t_1 = \theta > h$, $t_2 = \tau < T_h$ a test function

$$\varphi_4(x,t) = \{[M(\rho) - M(R)]^{1-q} - \max\{[[u(x,t)]_h - M(R)]_+, $$
$$M(\rho) - M(R)\}^{1-q}\}\eta_r^p(x,t).$$

Evaluating the term of (2.4) with the derivative of t and $\varphi = \varphi_4$ and letting $h \to 0$ we obtain for all $\theta \in (0,\tau)$, $\tau \in (0,T)$

$$\lim_{h\to 0} \int\limits_\theta^\tau \int\limits_\Omega \frac{\partial [u]_h}{\partial t} \varphi_4(x,t)\,dx\,dt \geq -C_{18} M(r)[M(\rho)-M(R)]^{1-q} r^n. \quad (4.22)$$

The remaining terms of (2.4) with $\varphi = \varphi_4$ are then estimated as follows

$$\lim_{h\to 0} \int\limits_\theta^\tau \int\limits_\Omega \Big\{ \sum_{i=1}^n \Big[a_i\Big(x,t,u,\frac{\partial u}{\partial x}\Big)\Big]_h \frac{\partial \varphi_4}{\partial x_i} + \Big[a_0\Big(x,t,u,\frac{\partial u}{\partial x}\Big)\Big]_h \varphi_4 \Big\}\,dx\,dt$$

$$\geq \int_\theta^\tau \int_\Omega \Big\{ \Big[\frac{\nu_1}{2} u_R^{-q}(x,t) \Big| \frac{\partial u}{\partial x} \Big|^p - C_{19} u^{p-q}(x,t) H(x,t)$$

$$- C_{19}[M(\rho) - M(R)]^{p-pq} u^{q(p-1)}(x,t) H(x,t) \Big] \eta_r^p(x,t)$$

$$- C_{19}[M(\rho) - M(R)]^{1-q} \Big[\Big| \frac{\partial u}{\partial x} \Big|^{p-1}$$

$$+ [H(x,t)]^{\frac{p-1}{p}} u^{p-1}(x,t) \Big] \eta_r^{p-1}(x,t) \Big| \frac{\partial \eta_r}{\partial x} \Big| \Big\} \chi(F(\rho))\, dx\, dt. \quad (4.23)$$

It is simple to continue to estimation of terms on the right-hand side of (4.23) by using inequalities (4.17), (4.18). In particular we have

$$\iint_{F(\rho)} u^{q(p-1)}(x,t) H(x,t) \eta_r^p(x,t)\, dx\, dt$$

$$= \iint_{F(\rho)} u^{p-1+\lambda_1+\lambda_2}(x,t) H(x,t) \eta_r^p(x,t)\, dx\, dt$$

$$\leq C_{14}[M(\rho) - M(R)]^{-\lambda_2} [M^*(r) r^n]^{p-1+\lambda_1}. \quad (4.24)$$

Letting $\theta \to 0$ in (4.22), (4.23) and using inequalities (4.20), (4.24) we obtain the estimate (4.21). This is the end of the proof of Lemma 4.3. □

The main result of the integral estimate of the solution with isolated singularity is given in the following theorem.

Theorem 4.1. *Assume that conditions of Theorem* 2.1 *are satisfied. Then there exist a positive constant* K_6, λ *depending only on known parameters such that the estimate*

$$\sup_{0<\tau<T} \int_\Omega \Phi_{\rho R}^2(u(x,\tau)) \eta_r^p(x,\tau)\, dx + \iint_{E(\rho,R)} \Big| \frac{\partial u}{\partial x} \Big|^p \eta_r^p(x,t)\, dx\, dt$$

$$\leq K_6 [M(\rho) - M(R)] \big\{ [M(r) r^n [M^*(r) r^n]^{p-1}]^{\frac{p-1}{p}}$$

$$+ [M(\rho) - M(R)]^{-\lambda} [M^*(r) r^n]^{p-1+\lambda} \big\} \quad (4.25)$$

holds with $\Phi_{\rho R}(u)$, $E(\rho,R)$ *defined by* (4.11), *the same function* $\eta_r^{x,t}$ *as in* (4.4) *and* r, ρ, R *satisfying condition* (4.10).

Proof. We estimate the integral on the right-hand side of (4.12) by using Young's inequality and inequalities (4.21), (4.24). We have with $q = 1 + \frac{n\delta}{2p^2}$

$$
\begin{aligned}
\iint\limits_{F(\rho)} \nu_3(x,t)\Big|\frac{\partial u}{\partial x}\Big|^{p-1} \eta_r^p(x,t)\,dx\,dt \\
\le \iint\limits_{F(\rho)} \Big\{[M(\rho)-M(R)]^{\lambda_2} u_R^{-q}(x,t)\Big|\frac{\partial u}{\partial x}\Big|^p \\
+ [M(\rho)-M(R)]^{\lambda_2-\lambda_1} H(x,t) u_R^{q(p-1)}(x,t)\Big\}\eta_r^p(x,t)\,dx\,dt \\
\le C_{20}\big\{[M(r)r^n[M^*(r)r^n]^{p-1}]^{\frac{p-1}{p}} \\
+ [M(\rho)M(R)]^{-\lambda_1}[M^*(r)r^n]^{p-1+\lambda_1}\big\}. \quad (4.26)
\end{aligned}
$$

Now the estimate (4.25) follows from inequalities (4.12), (4.26) and the proof of Theorem 4.1 is completed. □

Remark 4.1. Changing in the equation (1.1) the function $u(x,t)$ on the function $v(x,t) = -u(x,t)$ we obtain immediately that all estimates of this section are true for $v(x,t)$ instead of $u(x,t)$.

Corollary 4.1. *Taking into account the condition* (2.9) *and Remark* 2.1 *we can pass to the limit in* (4.25) *as* $r \to 0$ *and we get an estimate*

$$
\sup_{0<\tau<T} \int\limits_{\Omega} \Phi_{\rho R}^2(u(x,\tau))\,dx + \iint\limits_{E(\rho,R)} \Big|\frac{\partial u}{\partial x}\Big|^p dx\,dt \le K_7[M(\rho)-M(R)]^{1-\lambda} \tag{4.27}
$$

with the same positive λ *as in* (4.25).

5. Point-wise estimate of singular solution

In this section we prove the fundamental result on the behavior of the solution of the equation (1.1) with the singularity at the point.

Theorem 5.1. *Let conditions of the Theorem* 2.2 *be satisfied. Then there exist a positive constants* K_8, γ *depending only on known parameters such that the estimates*

$$
\begin{aligned}
|u(x,t)| \le K_8\{|x| + t^{\frac{1}{p+n(p-2)}}\}^{-n+\gamma}, \\
\sup_{0<\tau<T} \int\limits_{\Omega} u_R^2(x,\tau)\,dx + \iint\limits_{E(R)} \Big|\frac{\partial u}{\partial x}\Big|^p dx\,dt \le K_8
\end{aligned} \tag{5.1}
$$

holds for $(x,t) \in \mathcal{D}(R)$.

Proof. Let r, ρ, R be numbers satisfying conditions (4.10). We can assume additionally that $\rho > 4r$ since the inequality (5.1) is trivial in the opposite case.

We define numerical sequences $\{\rho_i\}$, $\{\alpha_i\}$

$$\rho_i = \frac{\rho}{2}(1 + 2^{-i}), \qquad \alpha_i = \frac{2^{-i-1}}{1 + 2^{-i}}$$

and a sequence of functions $\{\varphi_i(x,t)\}$

$$\varphi_i(x,t) = \omega_i\Big(\Big(\frac{|x|}{\rho_i}\Big)^p + \frac{t}{\rho_i^{p+n(p-2)}}\Big)[1 - \eta_R(x,t)]. \tag{5.2}$$

Here η_R is the function introduced in Section 3, the function $\omega_i : \mathbb{R}^1 \to \mathbb{R}^1$ is defined by the equality $\omega_i(s) = \alpha_i^{-p} \min\{[s - (1-\alpha_i)^p]_+, 1\}$. Such defined function φ_i has following properties: $\varphi_i(x,t) = 0$ for $(x,t) \in \mathcal{D}(\rho)$, $\varphi_i(x,t) = 1$ for $(x,t) \in \mathcal{D}(R) \backslash \mathcal{D}(\rho_i)$, $|\frac{\partial \varphi_i}{\partial x}| \le \frac{C2^{ip}}{\rho}$, $|\frac{\partial \varphi_i}{\partial t}| \le \frac{C2^{ip}}{\rho^{p+n(p-2)}}$.

We substitute in the identity (2.4) with $t_1 = \theta > h$, $t_2 = \tau > T - h$ a test function

$$\varphi_5(x,t) = \big\{[[u(x,t)]_h - M(R)]_+^2 + 1\big\}^l [[u(x,t)]_h - M(R)]_+ \varphi_i^{m+p}(x,t)$$

where l, m are arbitrary nonnegative numbers.

Evaluating the term of (2.4) with the derivative on t and $\varphi = \varphi_5$ and letting $h \to 0$ we obtain for $0 < \theta < \tau < T$

$$\begin{aligned}
\lim_{h\to 0} \int_\theta^\tau \int_\Omega & \frac{\partial [u]_h}{\partial t} \varphi_5(x,t)\, dx\, dt \\
&\ge \frac{1}{2(l+1)} \int_\Omega [u_R^2(x,\tau) + 1]^{l+1} \varphi_i^{m+p}(x,\tau)\, dx \\
&\quad - \frac{1}{2(l+1)} \int_\Omega [u_R^2(x,\theta) + 1]^{l+1} \varphi_i^{m+p}(x,\theta)\, dx \\
- C_{21}(m+1) & \iint_{Q_T} [u_R^2(x,t) + 1]^{l+1} \varphi_i^{m+p-1}(x,t) \Big|\frac{\partial \varphi_i}{\partial t}\Big|\, dx\, dt. \qquad (5.3)
\end{aligned}$$

Letting $h \to 0$ in the remaining terms of (2.4) with $\varphi = \varphi_5$ and estimating these terms we have

$$\lim_{h\to 0} \int_\theta^\tau \int_\Omega \Big\{ \sum_{i=1}^n \Big[a_i\Big(x,t,u,\frac{\partial u}{\partial x}\Big)\Big]_h \frac{\partial \varphi_5}{\partial x_i} + \Big[a_0\Big(x,t,u,\frac{\partial u}{\partial x}\Big)\Big]_h \varphi_5 \Big\} dx\, dt$$

$$\geq \frac{\nu_1}{2}\int\limits_{\theta}^{\tau}\int\limits_{\Omega}[u_R^2(x,t)+1]^l\Big|\frac{\partial u}{\partial x}\Big|^p\varphi_i^{m+p}(x,t)\,dx\,dt$$
$$- C_{22}(l+1)^p\iint\limits_{Q_T}[u_R^2(x,t)+1]^{l+\frac{p}{2}}\Big\{\Big|\frac{\partial\varphi_i}{\partial x}\Big|^p+H(x,t)\Big\}\varphi_i^m(x,t)\,dx\,dt. \tag{5.4}$$

Inserting estimates (5.3), (5.4) into (2.3) yields

$$\sup_{0<\tau<T}\int\limits_{\Omega}[u_R^2(x,\tau)+1]^{l+1}\varphi_i^{m+p}(x,\tau)\,dx$$
$$+\iint\limits_{Q_T}[u_R^2(x,t)+1]^l\Big|\frac{\partial u}{\partial x}\Big|^p\varphi_i^{m+p}(x,t)\,dx\,dt$$
$$\leq C_{23}(l+m+1)^{p+1}\iint\limits_{Q_T}\Big\{[u_R^2(x,t)+1]^{l+1}\Big|\frac{\partial\varphi_i}{\partial t}\Big|$$
$$+[u_R^2(x,t)+1]^{l+\frac{p}{2}}\Big(\Big|\frac{\partial\varphi_i}{\partial x}\Big|^p+H(x,t)\Big)\Big\}\varphi_i^m(x,t)\,dx\,dt. \tag{5.5}$$

Now consider the term that appears in the right-hand side of (5.5). Using Hölder inequality and the notation (4.2) we obtain

$$\iint\limits_{Q_T}\Big\{[u_R^2(x,t)+1]^{l+1}\Big|\frac{\partial\varphi_i}{\partial t}\Big|$$
$$+[u_R^2(x,t)+1]^{l+\frac{p}{2}}\Big(\Big|\frac{\partial\varphi_i}{\partial x}\Big|^p+H(x,t)\Big)\Big\}\varphi_i^m(x,t)\,dx\,dt$$
$$\leq C_{24}2^{ip^2}\Big[M^*\Big(\frac{p}{2}\Big)\Big]^{p-2}\rho^{-\delta p}\Big\{\int\limits_0^T\Big[\int\limits_{\Omega}([u_R^2(x,t)+1]^{l+1}$$
$$\times\varphi_i^m(x,t))^{q_0'}\,dx\Big]^{\frac{r_0'}{q_0'}}dt\Big\}^{\frac{1}{r_0'}} \tag{5.6}$$

with numbers δ, q_0, r_0 given by the condition a_2).

We denote

$$I_i(l,m)=\Big\{\int\limits_0^T\Big[\int\limits_{\Omega}([u_R^2(x,t)+1]^{l+1}\varphi_i^m(x,t))^{q_0'}\,dx\Big]^{\frac{r_0'}{q_0'}}dx\Big\}^{\frac{1}{r_0'}} \tag{5.7}$$

and estimate the last integral by using embedding theorem and Hölder inequality. Define

$$k_1=\Big(p\delta+\frac{n(p-1)}{r_0}\Big)\Big[n+p\delta+\frac{n(p-2)}{r_0}\Big]^{-1}=\Big(\frac{r_0'}{q_0'}-\frac{n-p}{n}\Big)\Big(\frac{r_0'}{q_0'}+\frac{p}{n}\Big)^{-1},$$

$$l_1=(l+1)k_1-\frac{p-2}{2r_0'}k_1, \quad l_2=(l+1)(1-k_1)+\frac{p-2}{2r_0'}k_1,$$

$$p_1=\frac{p_2}{p_2-1}, \quad p_2=\frac{nr_0'}{q_0'(n-p)}, \quad m_1=mk_1, \quad m_2=m(1-k_1).$$

Then we have

$$\begin{aligned} I_i(l,m) &\le C_{25}2^{\frac{ip^2}{r_0'}}(l+m+1)^{\frac{p}{r_0'}} \\ &\quad\times \sup_{0<\tau<T}\Big\{\int\limits_{\Omega}\{[u_R^2(x,t)+1]^{l_1}\varphi_i^{m_1}(x,t)\}^{p_1q_0'}\,dx\Big\}^{\frac{1}{p_1q_0'}} \\ &\quad\times\Big\{\iint\limits_{Q_T}\Big\{[u_R^2(x,t)+1]^{l_2r_0'-\frac{p}{2}}\Big|\frac{\partial u}{\partial x}\Big|^p\varphi_i^{m_2r_0'}(x,t) \\ &\qquad\qquad+\frac{1}{\rho^p}[u_R^2(x,t)+1]^{l_2r_0'}\varphi_i^{m_2r_0'-p}(x,t)\Big\}dx\,dt\Big\}^{\frac{1}{r_0'}}. \end{aligned} \tag{5.8}$$

Estimating the two last integrals by virtue of inequalities (5.5), (5.6) we get

$$\begin{aligned} I_i(l,m) &\le C_{26}^i(l+m+1)^{p_3} \\ &\quad\times\Big\{\Big[M^*\Big(\frac{\rho}{2}\Big)\Big]^{p-2}\frac{1}{\rho^{\delta p}}I_i(l_1p_1q_0'-1,m_1p_1q_0'-p)\Big\}^{\frac{1}{p_1q_0'}+\frac{1}{r_0'}} \end{aligned} \tag{5.9}$$

with $p_3=\frac{p}{r_0'}+(p+1)[\frac{1}{p_1q_0'}+\frac{1}{r_0'}]$.

Define $\bar{k}=r_0'(1-k_1)=\frac{n}{n+p\delta+\frac{(p-2)n}{r_0}}$ and sequences $\{l_j\}$, $\{m_j\}$ by equalities

$$\begin{aligned} l_j &= \Big[l_0+1+\frac{p-2}{2r_0'}\frac{\bar{k}}{1-\bar{k}}\Big]\bar{k}^{-j}-\frac{p-2}{2r_0'}\frac{\bar{k}}{1-\bar{k}}-1, \\ l_0 &= \frac{p}{2}-1+\frac{p\delta}{n}+\frac{p-2}{2r_0}, \\ m_j &= \Big[m_0+\frac{p}{1-\bar{k}}\Big]\bar{k}^{-j}-\frac{p}{1-\bar{k}}, \\ m_0 &= p+\frac{p^2\delta}{n}+\frac{p(p-2)}{r_0}. \end{aligned}$$

Then we rewrite the inequality (5.9) with $l = l_j,\ m = m_j$ as follows

$$I_i(l_j, m_j) \le C_{27}^i C_{28}^j \Big\{ \Big[M^*\Big(\frac{\rho}{2}\Big)\Big]^{p-2} \frac{1}{\rho^{\delta p}} I_i(l_{j-1}, m_{j-1}) \Big\}^{\bar{k}}. \tag{5.10}$$

The iteration by j of the last inequality yields the estimate

$$[M(\rho_i) - M(R)]^{2(l_0+1+\frac{p-2}{2r_0'}\frac{\bar{k}}{1-\bar{k}})} \le C_{29}^i \Big\{ \Big[M^*\Big(\frac{\rho}{2}\Big)\Big]^{p-2} \frac{1}{\rho\delta p} \Big\}^{\frac{1}{1-\bar{k}}} I_i(l_0, m_0). \tag{5.11}$$

Now we estimate $I_i(l_0, m_0)$ by using Hölder inequality, embedding theorem and the inequality (4.27). We have for

$$v_{i+1}(x,t) = \min\{u_R(x,t), M(\rho_{i+1}) - M(R)\}$$

the following estimate

$$\begin{aligned}
&I_i(l_0, m_0) \\
&= \Big\{ \int_0^T \Big[\int_\Omega \big[(v_{i+1}^2 + 1)^{\frac{p}{2}+\frac{p}{n}\delta+\frac{p-2}{2r_0}} \varphi_i^{p+\frac{p^2}{n}\delta+\frac{p(p-2)}{r_0}}\big]^{q_0'}\, dx \Big]^{\frac{r_0'}{q_0'}} dt \Big\}^{\frac{1}{r_0'}} \\
&\le \sup_t \Big\{ \int_\Omega [v_{i+1}^2(x,t) + 1] \varphi_i^p(x,t)\, dx \Big\}^{\frac{1}{q_0'} - \frac{n-p}{nr_0'}} \\
&\times \Big\{ \iint_{Q_T} \Big[\Big| \frac{\partial v_{i+1}}{\partial x} \Big|^p \varphi_i^p + (1 + v_{i+1})^p \Big| \frac{\partial \varphi_i}{\partial x} \Big|^p \Big]\, dx\, dt \Big\}^{\frac{1}{r_0'}} \\
&\le C_{30} 2^{\frac{ip^2}{r_0'}} [M(\rho_{i+1}) - M(R)]^{(1-\lambda)(\frac{1}{q_0'} + \frac{p}{nr_0'})}.
\end{aligned} \tag{5.12}$$

Inequalities (5.11), (5.12) and Remark 2.1 imply

$$\begin{aligned}
&[M(\rho_i) - M(R)]^{2(l_0+1+\frac{p-2}{2r_0'}\frac{\bar{k}}{1-\bar{k}})} \\
&\qquad \le C_{31} \rho^{-\frac{n(p-2)+\delta p}{1-\bar{k}}} [M(\rho_{i+1}) - M(R)]^{(1-\lambda)(\frac{1}{q_0'} + \frac{p}{nr_0'})}.
\end{aligned} \tag{5.13}$$

Rewrite this inequality in the form

$$M(\rho_i) - M(R) \le C_{32}^i \rho^{-a_1} [M(\rho_{i+1}) - M(R)]^{a_2} \tag{5.14}$$

with

$$a_1 = \frac{[n(p-2) + \delta p] r_0'}{2(l_0+1) r_0'(1-\bar{k}) + (p-2)\bar{k}},$$

$$a_2 = \frac{1-\lambda}{2}\Big(\frac{1}{q_0'} + \frac{p}{nr_0'}\Big)\Big(l_0 + 1 + \frac{p-2}{2r_0'}\frac{\bar{k}}{1-\bar{k}}\Big)^{-1}.$$

Direct calculations show that

$$\begin{gathered} l_0 + 1 + \frac{p-2}{2r_0'}\frac{\bar{k}}{1-\bar{k}} = 1 + \frac{p\delta}{n} + \frac{p-2}{r_0} + \frac{p-2}{2r_0'}\frac{1}{1-\bar{k}}, \\ \frac{1}{q_0'} + \frac{p}{nr_0'} = 1 + \frac{p\delta}{n} + \frac{p-2}{r_0} \end{gathered} \tag{5.15}$$

and therefore $a_2 < 1$. Iterating the inequality (5.14) by i and using bondedness of the sequence $\{M(\rho_i) - M(R)\}$ we obtain from (5.14)

$$M(\rho_1) - M(R) \le C_{33}\rho^{-\frac{a_1}{1-a_2}}. \tag{5.16}$$

Denote $\Delta = \frac{p\delta}{n} + \frac{p-2}{r_0}$ and calculate $\frac{a_1(1-\lambda)}{1-\lambda-a_2}$. Using equalities (5.15) we get

$$\frac{a_1(1-\lambda)}{1-\lambda-a_2} = \frac{n(\Delta + \frac{p-2}{r_0'})(1+\Delta)}{\Delta[1_\Delta + \frac{p-2}{r_0'}\frac{1+\Delta}{\delta}]} = n. \tag{5.17}$$

It means that

$$\frac{a_1}{1-a_2} \le n - \gamma \tag{5.18}$$

with the constant γ depending only on known parameters, Now inequalities (5.16), (5.18) imply the first estimate in (5.1).

The second inequality in (5.1) follows now from the proof of Theorem 2.1. The inequality (5.1) implies that the condition (2.6) is satisfied. Then the second inequality in (5.1) follows from the estimate (3.14) that is true for an arbitrary positive l_*. Therefore the proof of Theorem 5.1 is completed. □

6. Proof of Theorem 2.2

The inequality (5.1) and Theorem 2.1 implies the boundedness un $\mathcal{D}(\frac{R_0}{2})$ of the solution $u(x,t)$ satisfying the conditions of Theorem 2.2.

We need to establish an equality (2.2) for an arbitrary function $\varphi(x,t) = \overline{\varphi}(x,t) = \psi(x,t)\overline{\zeta}(x,t)$ where $\psi \in L^p(0,T;W^{1,p}(\Omega))$ is such function that $\frac{\partial\psi}{\partial t} \in L^2(Q_T)$ and $\overline{\zeta} \in C^\infty(\overline{Q}_T)$ and $\overline{\zeta}$ is equal to zero near $\partial\Omega \times (0,T)$.

Let us substitute in (2.2) $\varphi(x,t) = \overline{\varphi}(x,t)\eta_r(x,t)$ where η_r is the function defined at the beginning of Section 3. We obtain

$$\int_\Omega u(x,\tau)\overline{\varphi}(x,\tau)\eta_r(x,\tau)\,dx$$
$$+\int_0^\tau\int_\Omega\Big\{-u\frac{\partial\overline{\varphi}}{\partial t}+\sum_{i=1}^n a_i\Big(x,t,u,\frac{\partial u}{\partial x}\Big)\frac{\partial\overline{\varphi}}{\partial x_i}+a_0\Big(x,t,u,\frac{\partial u}{\partial x}\Big)\overline{\varphi}\Big\}\eta_r\,dx\,dt$$
$$=\int_0^\tau\int_\Omega\Big\{u\frac{\partial\eta_r}{\partial t}-\sum_{i=1}^n a_i\Big(x,t,u,\frac{\partial u}{\partial x}\Big)\frac{\partial\eta_r}{\partial x_i}\Big\}\overline{\varphi}(x,t)\,dx\,dt. \quad (6.1)$$

Taking into account estimates (5.1) we can estimate the integral on the right of (6.1) as follows

$$\Big|\int_0^\tau\int_\Omega\Big\{u\frac{\partial\eta_r}{\partial t}-\sum_{i=1}^n a_i\Big(x,t,u,\frac{\partial u}{\partial x}\Big)\frac{\partial\eta_r}{\partial x_i}\Big\}\overline{\varphi}(x,t)\,dx\,dt\Big|$$
$$\le C_{34}\iint_{Q_T}\Big\{\Big|\frac{\partial\eta_r}{\partial t}\Big||\overline{\varphi}(x,t)|+\Big|\frac{\partial\eta_r}{\partial x}\Big|^p|\overline{\varphi}(x,t)|^p\Big\}dx\,dt. \quad (6.2)$$

Let us assume at first that $\psi\in L^\infty(Q_T)$. Applying direct calculation we have

$$\lim_{r\to0}\iint_{Q_T}\Big\{\Big|\frac{\partial\eta_r}{\partial t}\Big|+\Big|\frac{\partial\eta_r}{\partial x}\Big|^p\Big\}\,dx\,dt=0$$

and therefore the right-hand side of (6.1) tends to zero as $r\to0$ for founded function ψ. Approximating the function $\psi(x,t)$ by the sequence of bounded functions we end the proof of Theorem 2.2.

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