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Power geometry in nonlinear partial differential equations

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Abstract. Power Geometry (PG) is a new calculus developing the differential calculus and aimed at nonlinear problems. The main concept of PG is the study of nonlinear problems in logarithms of original coordinates. Then many relations nonlinear in the original coordinates become linear. The algorithms of PG are based on these linear relations. They allow to simplify equations, to resolve their singularities (including singular perturbations), to isolate their first approximations, and to find asymptotic forms and asymptotic expansions of their solutions. In particular, they give simple methods to identify the equations and systems as quasihomogeneous, and then to introduce for them self-similar coordinates. As an application, we consider the stationary spatial axially symmetric flow of the viscous compressible heat conducting gas around a semi-infinite needle. Other application: finding blow-up solutions.

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Introduction

Traditional differential calculus is effective for linear and quasilinear problems. It is less effective for essentially nonlinear problems. A linear problem is the first approximation to a quasilinear problem. The linear problem is usually solved by methods of functional analysis, then the solution to the quasilinear problem is found as a perturbation of the solution to the linear problem. For an essentially nonlinear problem, we need to isolate its first approximations, to find their solutions, and to construct perturbations of these solutions. This is what Power Geometry (PG) is aimed at. For equations and systems of equations (algebraic, ordinary

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differential, and partial differential), PG allows to compute asymptotic forms of solutions as well as asymptotic and local expansions of solutions at infinity and at any singularity of the equation (including boundary layers and singular perturbations) [1].

Elements of plane PG were proposed by Newton for an algebraic equation (1680); and by Briot and Bouquet for an ordinary differential equation (1856). Space PG for a nonlinear autonomous system of ODEs was proposed by the author (1962), and for a linear PDE, by Mikhailov (1963).

In this talk we intend to give basic notions of PG, present some of its algorithms, results, and applications. It is clear that this calculus cannot be mastered during this presentation. This talk consists of three parts: space PG (Section 1), self-similar solutions (Section 2) and boundary layer on a needle (Section 3).

1. The space Power Geometry

Let $X \in \mathbb{C}^m$ be independent and $Y \in \mathbb{C}^n$ be dependent variables. Suppose $Z = (X, Y) \in \mathbb{C}^{m+n}$. A *differential monomial* $a(Z)$ is the product of an ordinary monomial $cZ^R = cz_1^{r_1} \cdots z_{m+n}^{r_{m+n}}$, where $c = \text{const} \in \mathbb{C}$, $R = (r_1, \ldots, r_{m+n}) \in \mathbb{R}^{m+n}$, and a finite number of derivatives of the form

$$\frac{\partial^l y_j}{\partial x_1^{l_1} \cdots \partial x_m^{l_m}} \stackrel{\text{def}}{=} \frac{\partial^l y_j}{\partial X^L}, \quad l_j \geq 0, \qquad \sum_{j=1}^{m} l_j = l, \quad L = (l_1, \ldots, l_m).$$

A differential monomial $a(Z)$ corresponds to its vector *power exponent* $Q(a) \in \mathbb{R}^{m+n}$ formed by the following rules

$$Q(cZ^R) = R, \qquad Q(\partial^l y_j / \partial X^L) = (-L, E_j),$$

where E_j is unit vector. A product of monomials $a \cdot b$ corresponds to the sum of their vector power exponents:

$$Q(ab) = Q(a) + Q(b).$$

A *differential sum* is a sum of differential monomials

$$f(Z) = \sum a_k(Z).$$

A set $\mathbf{S}(f)$ of vector power exponents $Q(a_k)$ is called the *support* of the sum $f(Z)$. The closure of the convex hull $\Gamma(f)$ of the support $\mathbf{S}(f)$ is called the *polyhedron of the sum* $f(Z)$. The boundary $\partial\Gamma(f)$ of the polyhedron $\Gamma(f)$ consists of faces $\Gamma_j^{(d)}$, where $d = \dim \Gamma_j^{(d)}$ and j is its number. Let $\mathbb{R}_*^{m+n}$ be the space dual to the space $\mathbb{R}^{m+n}$ such that the scalar product $\langle P, Q\rangle \stackrel{\text{def}}{=} p_1q_1+\dots+p_{m+n}q_{m+n}$ exists for $P=(p_1,\dots,p_{m+n})\in\mathbb{R}_*^{m+n}$ and $Q=(q_1,\dots,q_{m+n})\in\mathbb{R}^{m+n}$. Each face $\Gamma_j^{(d)}$ has its *normal cone* $\mathbf{U}_j^{(d)}\subset\mathbb{R}_*^{m+n}$. It consists of all vectors P such that the hyperplane supporting Γ and normal to the vector P, intersects the polyhedron Γ exactly along the face $\Gamma_j^{(d)}$. Each face $\Gamma_j^{(d)}$ corresponds to the *truncated sum*

$$\hat{f}_j^{(d)}(Z)=\sum a_k(Z) \text{ over } k:\ Q_k\in\mathbf{S}\cap\Gamma_j^{(d)}.$$

Consider a system of equations

$$f_i(X,Y)=0,\qquad i=1,\dots,n, \tag{1.1}$$

where f_i are differential sums. Each equation $f_i=0$ corresponds to: its support $\mathbf{S}(f_i)$, its polyhedron $\Gamma(f_i)$ with the set of faces $\Gamma_{ij}^{(d_i)}$ in the main space $\mathbb{R}^{m+n}$, the set of their normal cones $\mathbf{U}_{ij}^{(d_i)}$ in the dual space $\mathbb{R}_*^{m+n}$, and the set of truncated equations $\hat{f}_{ij}^{(d_i)}(X,Y)=0$. The set of truncated equations

$$\hat{f}_{ij_i}^{(d_i)}(X,Y)=0,\qquad i=1,\dots,n \tag{1.2}$$

is the *truncated system* if the intersection

$$\mathbf{U}_{1j_1}^{(d_1)}\cap\dots\cap\mathbf{U}_{nj_n}^{(d_n)} \tag{1.3}$$

is not empty. A solution

$$y_i=\varphi_i(X),\qquad i=1,\dots,n \tag{1.4}$$

to the system (1.1) is associated to its normal cone $\mathbf{u}\subset\mathbb{R}_*^{m+n}$.

Theorem 1.1 ([1]). *If the normal cone* $\mathbf{u}$ *intersects with the cone* (1.3), *then the asymptotic form* $y_i=\hat{\varphi}_i(X)$, $i=1,\dots,n$ *of the solution* (1.4) *satisfies the truncated system* (1.2), *which is quasihomogeneous.*

2. Self-similar solutions [2]

Let $\mathbf{S}(f)$ be the support of a differential sum $f(Z)$ and $Q\in\mathbf{S}(f)$. The set

$$\tilde{\mathbf{S}}(f)\stackrel{\text{def}}{=}\mathbf{S}(f)-Q$$

is called *shifted support* of the sum $f(Z)$. Each equation of system (1.1) has its own shifted support $\tilde{\mathbf{S}}(f_i)$. Let $\tilde{\Gamma}$ be the convex hull of their union

$$\tilde{\mathbf{S}}(f_1)\cup\dots\cup\tilde{\mathbf{S}}(f_n)$$

and d be the dimension of $\tilde{\Gamma}$. If $d < m+n$, system (1.1) is *quasihomogeneous.*

Let A be a square real nonsingular $(m+n)$-matrix with the block structure

$$A = \begin{pmatrix} B_{11} & 0 \\ B_{21} & B_{22} \end{pmatrix} \tag{2.1}$$

where B_{11} and B_{22} are square matrices of dimensions m and n respectively. We denote $\log Z = (\log z_1, \ldots, \log z_{m+n})$, and asterisk $*$ means transposition.

The change of variables

$$(\log Z)^* = A(\log \tilde{Z})^* \tag{2.2}$$

is called *power transformation.*

Theorem 2.1 ([2]). *If system* (1.1) *is quasihomogeneous and* $d = \dim \tilde{\Gamma}$, *then there exists a power transformation* (2.2), *which reduces system* (1.1) *to a system containing* $m+n-d$ *variables* $\tilde{z}_j$ *in the form* $\partial(\log \tilde{z}_j)$.

Example 2.1. In [3] the system

$$\begin{aligned} k_t &= \left(\frac{k^2}{\varepsilon} k_x\right)_x - \varepsilon, \\ \varepsilon_t &= \left(\frac{k^2}{\varepsilon} \varepsilon_x\right)_x - \gamma \frac{\varepsilon^2}{k} \end{aligned} \tag{2.3}$$

was considered. Here t and x are independent variables, k and ε are dependent variables, and γ is a real parameter. Here $m = n = 2$, $m+n = 4$, and $x_1 = t$, $x_2 = x$, $y_1 = k$, $y_2 = \varepsilon$. Support $\mathbf{S}_1$ of the first equation consists of three points

$$Q_1 = (-1,\ 0,\ 1,\ 0),$$

$$Q_2 = (0,\ -2,\ 3,\ -1),$$

$$Q_3 = (0,\ 0,\ 0,\ 1).$$

Support $\mathbf{S}_2$ of the second equation consists of three points

$$Q_4 = (-1,\ 0,\ 0,\ 1),$$

$$Q_5 = (0,\ -2,\ 2,\ 0),$$

$$Q_6 = (0,\ 0,\ -1,\ 2).$$

Shifted supports $\tilde{\mathbf{S}}_1 = \mathbf{S}_1 - Q_3$ and $\tilde{\mathbf{S}}_2 = \mathbf{S}_2 - Q_6$ consist of three points

$$P_1 \stackrel{\text{def}}{=} Q_1 - Q_3 = Q_4 - Q_6 = (-1,\ 0,\ 1,\ -1),$$

$$P_2 \stackrel{\text{def}}{=} Q_2 - Q_3 = Q_5 - Q_6 = (0,\ -2,\ 3,\ -2),$$
$$0 = Q_3 - Q_3 = Q_6 - Q_6.$$

Hence $d = 2$. Let us find basic vectors

$$R_i = \alpha_i P_1 + \beta_i P_2 \qquad (i = 1,\ 2)$$

of the form

$$R_1 = (r_{11},\ r_{12},\ 1,\ 0), \qquad R_2 = (r_{21},\ r_{22},\ 0,\ 1).$$

After evident computation, we obtain

$$R_1 = (2,\ -2,\ 1,\ 0), \qquad R_2 = (3,\ -2,\ 0,\ 1).$$

Now we introduce new dependent variables

$$u = Z^{R_1} = t^2 x^{-2} k, \qquad v = Z^{R_2} = t^3 x^{-2} \varepsilon.$$

Hence

$$k = t^{-2} x^2 u, \qquad \varepsilon = t^{-3} x^2 v. \tag{2.4}$$

It is the power transformation (2.2) with the matrix (2.1) where matrices B_{11} and B_{22} are identical and

$$B_{21} = \begin{pmatrix} -2 & 2 \\ -3 & 2 \end{pmatrix}.$$

Change of variables (2.4) reduces system (2.3) to the form

$$\begin{aligned} u_t t - 2u &= 3\frac{u^2}{v}(2u + u_x x) + x\left[\frac{u^2}{v}(2u + u_x x)\right]_x - v, \\ v_t t - 3v &= 3\frac{u^2}{v}(2v + v_x x) + x\left[\frac{u^2}{v}(2v + v_x x)\right]_x - \gamma\frac{v^2}{u}. \end{aligned} \tag{2.5}$$

Let us consider two cases.

Case 1*:* u, v are constants and system (2.5) is

$$\begin{aligned} -2u &= 6\frac{u^3}{v} - v, \\ -3v &= 6u^2 - \gamma\frac{v^2}{u}. \end{aligned} \tag{2.6}$$

Its nonzero solution is

$$u = \frac{3 - 2\gamma}{6(\gamma - 1)^2}, \qquad v = \frac{3 - 2\gamma}{6(\gamma - 1)^3} \tag{2.7}$$

with two critical values $\gamma = 1$ and $\gamma = 3/2$.

Case 2*:* Let $\zeta = t^\sigma x$, where $\sigma \in \mathbb{R}$. Now u and v are functions of ζ. In this case, in (2.1) the matrix

$$B_{11} = \begin{pmatrix} 1 & 0 \\ -\sigma & 1 \end{pmatrix},$$

but matrices B_{21} and B_{22} are as before. For $u(\zeta)$ and $v(\zeta)$, system (2.5) gives a one-parameter (σ) family of systems of two ordinary differential equations.

If u and v are functions of t only, we obtain the system

$$\begin{aligned} u_t t - 2u &= 6\frac{u^3}{v} - v, \\ v_t t - 3v &= 6u^2 - \gamma\frac{v^2}{u}. \end{aligned} \tag{2.8}$$

For $\gamma \neq 1$ and $\gamma \neq 3/2$, its solutions are

$$u = \frac{(3-2\gamma)t^2}{6w^2 + c_2 w^{1/(\gamma-1)}}, \qquad v = tu/w, \tag{2.9}$$

where $w = (\gamma - 1)t + c_1$ and c_1, c_2 are arbitrary constants.

For $\gamma = 1$,

$$u = \frac{t^2}{6c_1^2 + c_2 \exp(t/c_1)}, \qquad v = tu/c_1. \tag{2.10}$$

For $\gamma = 3/2$,

$$u = -\frac{t^2}{w^2(12\log w + c_2)}, \qquad v = tu/w, \tag{2.11}$$

here $w = t/2 + c_1$.

If in (2.5) u and v are functions of x only, then they satisfy the ODE system

$$\begin{aligned} -2u &= \frac{3u^2}{v}(2u + u_x x) + x\left[\frac{u^2}{v}(2u + u_x x)\right]_x - v, \\ -3v &= \frac{3u^2}{v}(2v + v_x x) + x\left[\frac{u^2}{v}(2v + v_x x)\right]_x - \gamma\frac{v^2}{u}. \end{aligned} \tag{2.12}$$

After the *logarithmic transformation*

$$\tau = \log t, \qquad \xi = \log x \tag{2.13}$$

system (2.5) is

$$
\begin{aligned}
u_\tau - 2u &= 3\frac{u^2}{v}(2u + u_\xi) + \left[\frac{u^2}{v}(2u + u_\xi)\right]_\xi - v,\\
v_\tau - 3v &= 3\frac{u^2}{v}(2v + v_\xi) + \left[\frac{u^2}{v}(2v + v_\xi)\right]_\xi - \gamma\frac{v^2}{u}.
\end{aligned}
\tag{2.14}
$$

Support $\mathbf{S}'_1$ of the first equation consists of six points

$$
\begin{gathered}
Q'_1 = (-1,\ 0,\ 1,\ 0),\\
Q'_2 = (0,\ 0,\ 1,\ 0),\\
Q'_3 = (0,\ 0,\ 3,\ -1),\\
Q'_4 = (0,\ -1,\ 3,\ -1),\\
Q'_5 = (0,\ -2,\ 3,\ -1),\\
Q'_6 = (0,\ 0,\ 0,\ 1).
\end{gathered}
$$

Its projection to the plane q'_1, q'_2 consists of four points

$$
(-1,\ 0),\ (0,\ 0),\ (0,\ -1),\ (0,\ -2). \tag{2.15}
$$

Support $\mathbf{S}'_2$ of the second equation consists of six points

$$
\begin{gathered}
Q'_7 = (-1,\ 0,\ 0,\ 1),\\
Q'_8 = (0,\ 0,\ 0,\ 1),\\
Q'_9 = (0,\ 0,\ 2,\ 0),\\
Q'_{10} = (0,\ -1,\ 2,\ 0),\\
Q'_{11} = (0,\ -2,\ 2,\ 0),\\
Q'_{12} = (0,\ 0,\ -1,\ 2).
\end{gathered}
$$

Its projection to the plane q'_1, q'_2 consists of four points (2.15). Their convex hull is the triangle Γ' (see Fig. 1) with three vertices

$$
\Gamma_1^{(0)} = (0,\ 0), \quad \Gamma_2^{(0)} = (-1,\ 0), \quad \Gamma_3^{(0)} = (0,\ -2),
$$

and three edges

$$
\Gamma_1^{(1)} = \left[\Gamma_1^{(0)},\ \Gamma_2^{(0)}\right], \quad \Gamma_2^{(1)} = \left[\Gamma_1^{(0)},\ \Gamma_3^{(0)}\right], \quad \Gamma_3^{(1)} = \left[\Gamma_2^{(0)},\ \Gamma_3^{(0)}\right].
$$

Exterior normal vectors N_j to edges $\Gamma_j^{(1)}$ are

$$
N_1 = (0,\ 1), \quad N_2 = (1,\ 0), \quad N_3 = -(2,\ 1).
$$

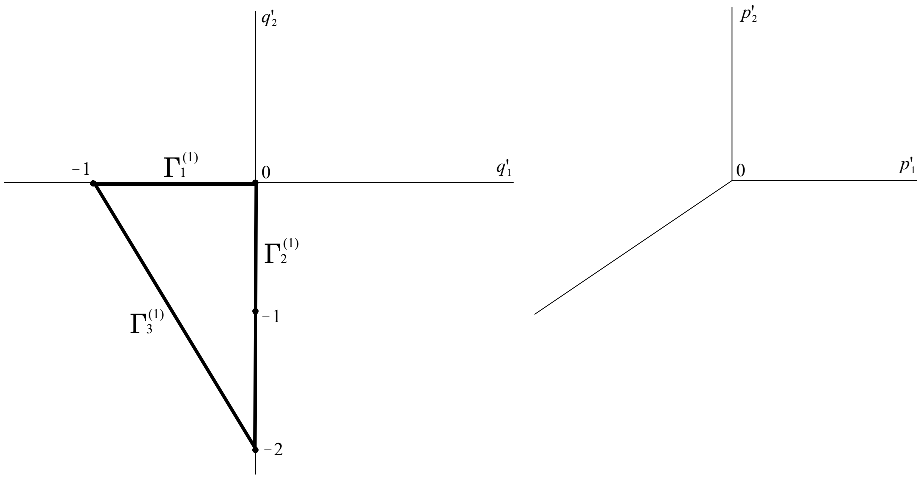

Fig. 1.

So normal cones to edges $\Gamma_1^{(1)}$ and $\Gamma_2^{(1)}$ are rays $(p'_1, p'_2) \in \lambda N_1$ and $(p'_1, p'_2) \in \lambda N_2$, $\lambda > 0$, respectively. And the normal cone to the vertex $\Gamma_1^{(0)}$ is $(p'_1, p'_2) \in \lambda N_1 + \mu N_2$, $0 < \lambda, \mu$; i.e. it is the first quadrant (see Fig. 1).

According to (2.13), if $t \to 0$ or ∞, then $\tau \to \infty$ and the truncated system of system (2.14) contains only terms with $q'_1 = 0$. It is exactly system (2.12) after the logarithmic transformation (2.13). According to Theorem 1.1 (see also [4, Ch. I]), asymptotic forms of solutions to system (2.14) are solutions to the truncated system, i. e. to system (2.12).

If $x \to 0$ or ∞, then $\xi \to \infty$ and the truncated system of system (2.14) contains only terms with $q'_2 = 0$. It is exactly the system (2.8) after the logarithmic transformation (2.13). According to Theorem 1.1 (see also [4, Ch. I]), asymptotic forms of solutions to system (2.14) are solutions to the truncated system, i.e. to system (2.8). Hence, these asymptotic forms are (2.9)–(2.11).

If both t and x tend to zero or infinity, then τ and ξ tend to infinity. If the ratio τ/ξ does not tend to zero or infinity, then the truncated system of system (2.14) contains only terms with $q'_1 = q'_2 = 0$. It is exactly system (2.6). According to Theorem 1.1 (see also [4, Ch. I]), asymptotic forms of solutions to system (2.14) are solutions to system (2.6). Hence these asymptotic forms are (2.7).

Using the power transformation (2.4), we can obtain asymptotic forms of solutions to initial system (2.3).

3. Boundary layer on a needle [4]

The theory of the boundary layer on the plate for a stream of viscous incompressible fluid was developed by Prandtl (1904) and Blasius (1908). However a similar theory for the boundary layer on the needle was not known untill recently, since no-slip conditions on the needle correspond to a more strong singularity as for the plate. This theory was developed with the help of Power Geometry (2004).

Let x be an axis in three-dimensional space, r be the distance from the axis, and semi-infinite needle be placed on the half-axis $x \geq 0$, $r = 0$. We studied stationary axisymmetric flows of viscous fluid which had constant velocity at $x = -\infty$ parallel to the axis x, and which satisfied no-slip conditions on the needle (Fig. 2). We considered two cases.

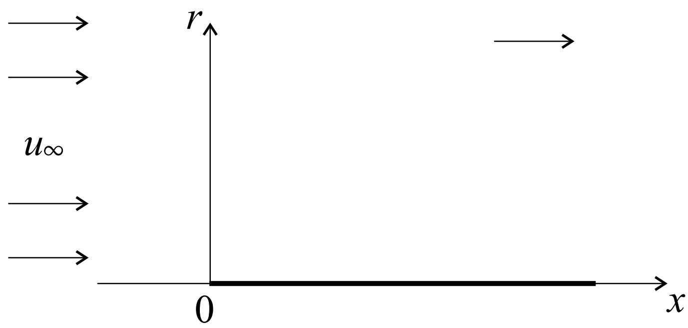

Fig. 2.

First case: incompressible fluid. For it, the Navier–Stokes equations in independent variables x, r are equivalent to the system of two equations for the stream function ψ and the pressure p

$$\begin{aligned} g_1 \stackrel{\text{def}}{=} & -\frac{1}{r}\frac{\partial\psi}{\partial x}\frac{\partial}{\partial r}\Big(\frac{1}{r}\frac{\partial\psi}{\partial r}\Big) + \frac{1}{r}\frac{\partial\psi}{\partial r}\frac{\partial}{\partial x}\Big(\frac{1}{r}\frac{\partial\psi}{\partial r}\Big) + \frac{1}{\rho}\frac{\partial p}{\partial x} \\ & \qquad -\nu\Big(\frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial}{\partial r}\Big(\frac{1}{r}\frac{\partial\psi}{\partial r}\Big)\Big) + \frac{\partial^2}{\partial x^2}\Big(\frac{1}{r}\frac{\partial\psi}{\partial r}\Big)\Big) = 0, \\ g_2 \stackrel{\text{def}}{=} & \frac{1}{r}\frac{\partial\psi}{\partial x}\frac{\partial}{\partial r}\Big(\frac{1}{r}\frac{\partial\psi}{\partial x}\Big) - \frac{1}{r}\frac{\partial\psi}{\partial r}\frac{\partial}{\partial x}\Big(\frac{1}{r}\frac{\partial\psi}{\partial x}\Big) + \frac{1}{\rho}\frac{\partial p}{\partial r} \\ & \qquad + \nu\Big(\frac{\partial}{\partial r}\Big(\frac{1}{r}\frac{\partial^2\psi}{\partial x\partial r}\Big) + \frac{\partial^2}{\partial x^2}\Big(\frac{1}{r}\frac{\partial\psi}{\partial x}\Big)\Big) = 0, \end{aligned} \tag{3.1}$$

where $\rho, \nu = \text{const}$, with the boundary conditions

$$\psi = \psi_0 r^2 \quad \text{for} \;\; x = -\infty, \; \psi_0 = \text{const}; \tag{3.2}$$

$$\begin{gathered}\partial\psi/\partial x = \partial\psi/\partial r = \partial^2\psi/\partial x\partial r = \partial^2\psi/\partial r^2 = 0\\ \text{for } x \geq 0, \quad r = 0.\end{gathered} \tag{3.3}$$

The system (3.1) has the form (1.1) with $m = n = 2$ and $m + n = 4$. Hence the supports of the equations (3.1) must be considered in $\mathbb{R}^4$. It turned out that polyhedrons $\Gamma(g_1)$ and $\Gamma(g_2)$ of the equations (3.1) are three-dimensional tetrahedrons, which can be moved by translation in one linear three-dimensional subspace, that simplified the isolation of the truncated systems. An analysis of truncated systems and of the results of their matching revealed [4, Ch. II] that the system (3.1) had no solution with $p \geq 0$ satisfying both boundary conditions (3.2), (3.3).

Second case: compressible heat-conducting gas. For this case, the Navier–Stokes equations in independent variables x, r are equivalent to the system of three equations for the stream function ψ, the density ρ, and the enthalpy h (an analog of the temperature)

$$\begin{aligned} f_1 \overset{\text{def}}{=} & -\frac{1}{r}\frac{\partial\psi}{\partial x}\frac{\partial}{\partial r}\Big(\frac{1}{\rho r}\frac{\partial\psi}{\partial x}\Big) + \frac{1}{r}\frac{\partial\psi}{\partial r}\frac{\partial}{\partial x}\Big(\frac{1}{\rho r}\frac{\partial\psi}{\partial x}\Big) - A\frac{\partial}{\partial r}(\rho h)\\ & + \frac{2}{3}C^n\frac{\partial}{\partial r}\Big(\frac{h^n}{r}\frac{\partial}{\partial r}\Big(\frac{1}{\rho}\frac{\partial\psi}{\partial x}\Big)\Big) - \frac{2}{3}C^n\frac{\partial}{\partial r}\Big(\frac{h^n}{r}\frac{\partial}{\partial x}\Big(\frac{1}{\rho}\frac{\partial\psi}{\partial r}\Big)\Big)\\ & - \frac{2C^n}{r}\frac{\partial}{\partial r}\Big(h^n r\frac{\partial}{\partial r}\Big(\frac{1}{\rho r}\frac{\partial\psi}{\partial x}\Big)\Big) + C^n\frac{\partial}{\partial x}\Big(h^n\frac{\partial}{\partial r}\Big(\frac{1}{\rho r}\frac{\partial\psi}{\partial r}\Big)\Big)\\ & \qquad - C^n\frac{\partial}{\partial x}\Big(h^n\frac{\partial}{\partial x}\Big(\frac{1}{\rho r}\frac{\partial\psi}{\partial x}\Big)\Big) + \frac{2C^n h^n}{\rho r^3}\frac{\partial\psi}{\partial x} = 0,\end{aligned}$$

$$\begin{aligned} f_2 \overset{\text{def}}{=} & \frac{1}{r}\frac{\partial\psi}{\partial x}\frac{\partial}{\partial r}\Big(\frac{1}{\rho r}\frac{\partial\psi}{\partial r}\Big) - \frac{1}{r}\frac{\partial\psi}{\partial r}\frac{\partial}{\partial x}\Big(\frac{1}{\rho r}\frac{\partial\psi}{\partial r}\Big) - A\frac{\partial}{\partial x}(\rho h)\\ & + \frac{2}{3}C^n\frac{\partial}{\partial x}\Big(\frac{h^n}{r}\frac{\partial}{\partial r}\Big(\frac{1}{\rho}\frac{\partial\psi}{\partial x}\Big)\Big) - \frac{2}{3}C^n\frac{\partial}{\partial x}\Big(\frac{h^n}{r}\frac{\partial}{\partial x}\Big(\frac{1}{\rho}\frac{\partial\psi}{\partial r}\Big)\Big)\\ & + \frac{C^n}{r}\frac{\partial}{\partial r}\Big(h^n r\frac{\partial}{\partial r}\Big(\frac{1}{\rho r}\frac{\partial\psi}{\partial r}\Big)\Big) - \frac{C^n}{r}\frac{\partial}{\partial r}\Big(h^n r\frac{\partial}{\partial x}\Big(\frac{1}{\rho r}\frac{\partial\psi}{\partial x}\Big)\Big)\\ & \qquad + 2C^n\frac{\partial}{\partial x}\Big(h^n\frac{\partial}{\partial x}\Big(\frac{1}{\rho r}\frac{\partial\psi}{\partial r}\Big)\Big) = 0, \end{aligned} \tag{3.4}$$

$$\begin{aligned} f_3 \overset{\text{def}}{=} & \frac{1}{r}\frac{\partial\psi}{\partial x}\frac{\partial h}{\partial r} - \frac{1}{r}\frac{\partial\psi}{\partial r}\frac{\partial h}{\partial x} - \frac{A}{\rho r}\frac{\partial\psi}{\partial x}\frac{\partial(\rho h)}{\partial r} + \frac{A}{\rho r}\frac{\partial\psi}{\partial r}\frac{\partial(\rho h)}{\partial x}\\ & + 2C^n h^n\Big(\frac{\partial}{\partial r}\Big(\frac{1}{\rho r}\frac{\partial\psi}{\partial x}\Big)\Big)^2 + 2C^n h^n\Big(\frac{1}{r^2\rho}\frac{\partial\psi}{\partial x}\Big)^2 + 2C^n h^n\Big(\frac{\partial}{\partial x}\Big(\frac{1}{\rho r}\frac{\partial\psi}{\partial r}\Big)\Big)^2\\ & + C^n h^n\Big(\frac{\partial}{\partial x}\Big(\frac{1}{\rho r}\frac{\partial\psi}{\partial x}\Big)\Big)^2 - C^n h^n\frac{\partial}{\partial x}\Big(\frac{1}{\rho r}\frac{\partial\psi}{\partial x}\Big)\frac{\partial}{\partial r}\Big(\frac{1}{\rho r}\frac{\partial\psi}{\partial r}\Big)\\ & + C^n h^n\Big(\frac{\partial}{\partial r}\Big(\frac{1}{\rho r}\frac{\partial\psi}{\partial r}\Big)\Big)^2 - \frac{2}{3}C^n h^n\Big(\frac{1}{r}\frac{\partial}{\partial r}\Big(\frac{1}{\rho}\frac{\partial\psi}{\partial x}\Big)\Big)^2 \end{aligned}$$

$$+\frac{4C^nh^n}{3r}\frac{\partial}{\partial r}\Big(\frac{1}{\rho}\frac{\partial\psi}{\partial x}\Big)\frac{\partial}{\partial x}\Big(\frac{1}{\rho r}\frac{\partial\psi}{\partial r}\Big)-\frac{2}{3}C^nh^n\Big(\frac{\partial}{\partial x}\Big(\frac{1}{\rho r}\frac{\partial\psi}{\partial r}\Big)\Big)^2 + \frac{C^n}{\sigma r}\frac{\partial}{\partial r}\Big(rh^n\frac{\partial h}{\partial r}\Big)+\frac{C^n}{\sigma}\frac{\partial}{\partial x}\Big(h^n\frac{\partial h}{\partial x}\Big)=0,$$

where parameters $A, C, \sigma > 0$ and $n \in [0,1]$, with the boundary conditions

$$\begin{gathered}\psi=\psi_0 r^2,\quad \rho=\rho_0,\quad h=h_0 \quad \text{for}\ \ x=-\infty,\\ \psi_0,\ \rho_0,\ h_0=\text{const}\end{gathered}\tag{3.5}$$

and (3.3). Here $m=2$, $n=3$, and $m+n=5$. In the space $\mathbb{R}^5$, all polyhedrons $\Gamma(f_1)$, $\Gamma(f_2)$, $\Gamma(f_3)$ of the equations (3.4) are three-dimensional, and they can be moved into one linear subspace. In coordinates $(\tilde{q}_1', \tilde{q}_2', \tilde{q}_3')$ of this three-dimensional space, they are shown in Figs. 3, 4, 5 respectively. This simplified the isolation of the truncated system corresponding to the boundary layer on the needle

$$\hat{f}_{12}^{(0)} \stackrel{\text{def}}{=} -A\partial(\rho h)/\partial r=0 \quad (\text{or } \partial(\rho h)/\partial r=0),$$

$$\begin{aligned}\hat{f}_{22}^{(2)} \stackrel{\text{def}}{=}\ & \frac{1}{r}\frac{\partial\psi}{\partial x}\frac{\partial}{\partial r}\Big(\frac{1}{\rho r}\frac{\partial\psi}{\partial r}\Big)-\frac{1}{r}\frac{\partial\psi}{\partial r}\frac{\partial}{\partial x}\Big(\frac{1}{\rho r}\frac{\partial\psi}{\partial r}\Big)\\ & -A\frac{\partial}{\partial x}(\rho h)+\frac{C^n}{r}\frac{\partial}{\partial r}\Big(h^n r\frac{\partial}{\partial r}\Big(\frac{1}{\rho r}\frac{\partial\psi}{\partial r}\Big)\Big)=0,\end{aligned}\tag{3.6}$$

$$\begin{aligned}\hat{f}_{32}^{(2)} \stackrel{\text{def}}{=}\ & \frac{1}{r}\frac{\partial\psi}{\partial x}\frac{\partial h}{\partial r}-\frac{1}{r}\frac{\partial\psi}{\partial r}\frac{\partial h}{\partial x}-\frac{A}{\rho r}\frac{\partial\psi}{\partial x}\frac{\partial(\rho h)}{\partial r}+\frac{A}{\rho r}\frac{\partial\psi}{\partial r}\frac{\partial(\rho h)}{\partial x}\\ & +C^nh^n\Big(\frac{\partial}{\partial r}\Big(\frac{1}{\rho r}\frac{\partial\psi}{\partial r}\Big)\Big)^2+\frac{C^n}{\sigma r}\frac{\partial}{\partial r}\Big(rh^n\frac{\partial h}{\partial r}\Big)=0,\end{aligned}$$

with self-similar variables

$$\psi=xG(\xi),\quad \rho=\mathrm{P}(\xi),\quad h=H(\xi),\quad \xi=r^2/x,\tag{3.7}$$

and with the boundary conditions

$$\psi=\psi_0 r^2,\quad \rho=\rho_0,\quad h=h_0;\qquad \psi_0,\ \rho_0,\ h_0=\text{const},\ r\to\infty\tag{3.8}$$

and (3.3). In Figs. 3–5, the faces corresponding to the truncated system (3.6) are shown in bold. According to the first equation (3.6) and the equalities (3.7), (3.8), the product $P(\xi)H(\xi)=\text{const}=C_0\stackrel{\text{def}}{=}\rho_0 h_0$. Hence $P(\xi)=C_0/H(\xi)$, and the system (3.6), for the variables (3.7), is equivalent to the system of two ordinary differential equations

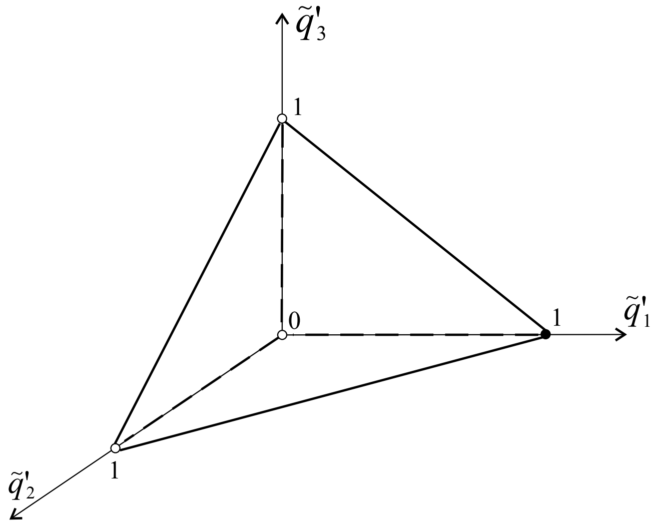

Fig. 3.

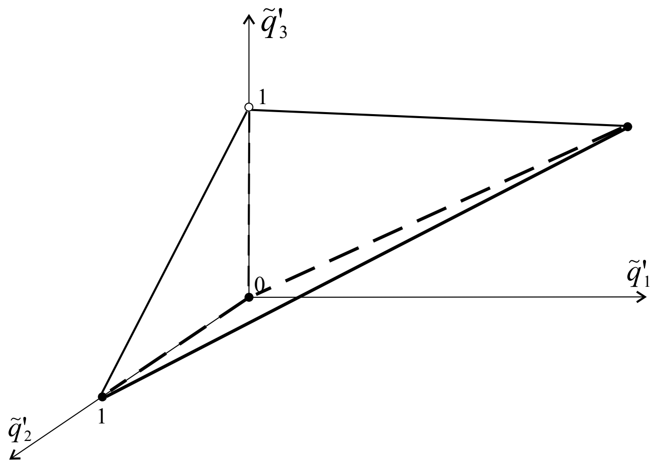

Fig. 4.

$$
\begin{gathered}
F_2 \stackrel{\text{def}}{=} G\left(G'H\right)' + 2C^n\left[\xi H^n (G'H)'\right]' = 0, \\
F_3 \stackrel{\text{def}}{=} 2GH' + 16C^n C_0^{-2} \xi H^n ((G'H)')^2 + 4C^n \sigma^{-1} (\xi H^n H')' = 0,
\end{gathered} \tag{3.9}
$$

where $' \stackrel{\text{def}}{=} d/d\xi$, with the boundary conditions

$$
G = \psi_0 \xi, \quad H = h_0 \text{ as } \xi \to +\infty, \tag{3.10}
$$

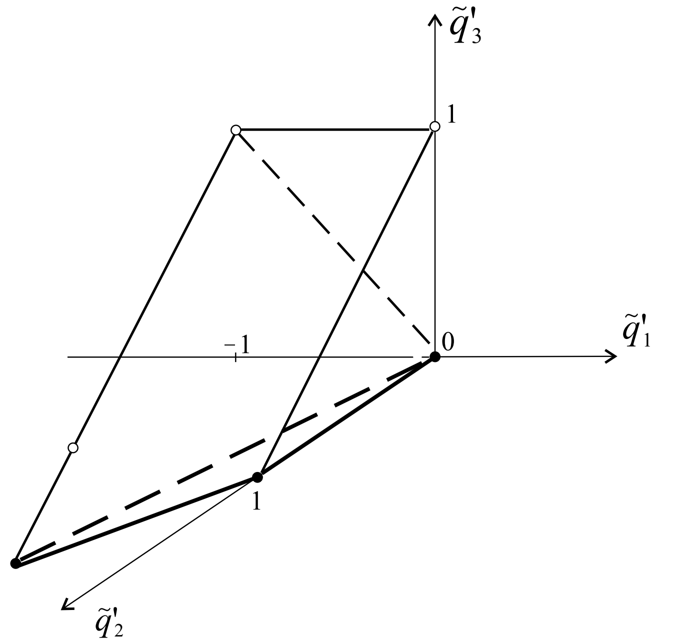

Fig. 5.

$$G = dG/d\xi = 0 \quad \text{as } \xi = 0. \tag{3.11}$$

The problem (3.9)–(3.11) has an invariant manifold $(G'H)' = 0$ on which it is reduced to one equation

$$\Delta \stackrel{\text{def}}{=} 2(\xi H^n H') - 2\xi H^n H'^2 + (\xi + c_2)H' = 0,$$

where c_2 is an arbitrary constant, with the boundary conditions

$$H \to 1 \quad \text{as } \xi \to +\infty,$$
$$H \to +\infty \quad \text{as } \xi \to +0.$$

An analysis of solutions to the latter problem by methods of PG revealed that for $n \in (0,1)$ it has solutions of the form

$$H \sim c_3 |\ln \xi|^{1/n}, \quad \xi \to 0,$$

where c_3 is an arbitrary constant.

Thus, for $n \in (0,1)$, in the boundary layer $r^2/x <$ const, as $x \to +\infty$ and $\xi = r^2/x \to 0$, we obtained the asymptotic form of the flow

$$\psi \sim c_1 r^2 |\ln \xi|^{-1/n}, \quad \rho \sim c_2 |\ln \xi|^{-1/n}, \quad h \sim c_3 |\ln \xi|^{1/n},$$

i.e. near the needle, the density tends to zero, and the temperature increases to infinity as the distance to the point of the needle tends to $+\infty$.

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