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ABOUT ONE MODULUS INEQUALITY OF THE ORDER $p \geq 1$

The present paper is devoted to the study of space mappings which are more general than quasiregular. The so-called modulus inequalities of the order p, $p \geq 1$, and it's connections with space mappings are investigated. The analogue of the well-known Poletskii inequality has been proved for the mappings having N, N^{-1} and $L_p^{(2)}$–property

Keywords: *mappings with finite and bounded distortion, modulus of curves families, Poletskii inequality.*

1. Introduction. Here we give some definitions. Everywhere below, D is a domain in $\mathbb{R}^n$, $n \geq 2$, m is the Lebesgue measure in $\mathbb{R}^n$, m_1 is the linear Lebesgue measure in $\mathbb{R}$. The notation $f : D \to \mathbb{R}^n$ assumes that f is continuous.

Recall that a mapping $f : D \to \mathbb{R}^n$ is said to have the *N-property (of Luzin)* iff $m(f(S)) = 0$ whenever $m(S) = 0$ for all measurable sets $S \subset \mathbb{R}^n$. Similarly, f has the *N^{-1}-property* iff $m\left(f^{-1}(S)\right) = 0$ whenever $m(S) = 0$.

A curve γ in $\mathbb{R}^n$ is a continuous mapping $\gamma : \Delta \to \mathbb{R}^n$ where Δ is an interval in $\mathbb{R}$. Its locus $\gamma(\Delta)$ is denoted by $|\gamma|$. Given a family of curves Γ in $\mathbb{R}^n$, a Borel function $\rho : \mathbb{R}^n \to [0, \infty]$ is called *admissible* for Γ, abbr. $\rho \in \operatorname{adm}\Gamma$, if

$$\int\limits_{\gamma} \rho(x)|dx| \geq 1$$

for each (locally rectifiable) $\gamma \in \Gamma$. Let $p \geq 1$. The *p-modulus* $M_p(\Gamma)$ of Γ is defined as

$$M_p(\Gamma) = \inf_{\rho \in \operatorname{adm}\Gamma} \int\limits_{\mathbb{R}^n} \rho^p(x)\, dm(x)$$

interpreted as $+\infty$ if $\operatorname{adm}\Gamma = \varnothing$. Note that $M_p(\varnothing) = 0$; $M_p(\Gamma_1) \leq M_p(\Gamma_2)$ whenever $\Gamma_1 \subset \Gamma_2$, and $M_p\left(\bigcup\limits_{i=1}^{\infty} \Gamma_i\right) \leq \sum\limits_{i=1}^{\infty} M_p(\Gamma_i)$, see Theorem 6.2 in [8].

We say that a property P holds for *p-almost every (p-a.e.)* curves γ in a family Γ if the subfamily of all curves in Γ for which P fails has p-modulus zero.

If $\gamma : \Delta \to \mathbb{R}^n$ is a locally rectifiable curve, then there is the unique nondecreasing length function l_γ of Δ onto a length interval $\Delta_\gamma \subset \mathbb{R}$ with a prescribed normalization $l_\gamma(t_0) = 0 \in \Delta_\gamma$, $t_0 \in \Delta$, such that $l_\gamma(t)$ is equal to the length of the subcurve $\gamma|_{[t_0,t]}$ of γ if $t > t_0$, $t \in \Delta$, and $l_\gamma(t)$ is equal to $-\operatorname{length}\left(\gamma|_{[t,t_0]}\right)$ if $t < t_0$, $t \in \Delta$. Let $g : |\gamma| \to \mathbb{R}^n$ be a continuous mapping, and suppose that the curve $\widetilde{\gamma} = g \circ \gamma$ is also locally rectifiable. Then there is a unique non-decreasing function $L_{\gamma,g} : \Delta_\gamma \to \Delta_{\widetilde{\gamma}}$ such that $L_{\gamma,g}\left(l_\gamma(t)\right) = l_{\widetilde{\gamma}(t)}$ for all $t \in \Delta$. A curve γ in D is called here a (whole) *lifting* of a curve $\widetilde{\gamma}$ in $\mathbb{R}^n$ under $f : D \to \mathbb{R}^n$ if $\widetilde{\gamma} = f \circ \gamma$.

We say that a mapping $f : D \to \mathbb{R}^n$ satisfies the $L_p^{(2)}$-property for p-a.e. curve $\widetilde{\gamma}$ in $f(D)$, if each lifting γ of $\widetilde{\gamma}$ is locally rectifiable and the function $L_{\gamma,f}$ has the N^{-1}-property.

Set

$$l\left(f'(x)\right) = \min_{h\in\mathbb{R}^n\setminus\{0\}} \frac{|f'(x)h|}{|h|},$$

$$K_{I,p}(x,f) = \begin{cases} \frac{|J(x,f)|}{l(f'(x))^p}, & J(x,f)\neq 0, \\ 1, & f'(x)=0, \\ \infty, & \text{otherwise} \end{cases}.$$

On of the main results proved in the paper is the following.

Statement 1. *Let a mapping $f : D \to \mathbb{R}^n$ be differentiable a.e. and satisfies N, N^{-1} and $L_p^{(2)}$-properties. Then*

$$M_p(f(\Gamma)) \le \int\limits_D K_{I,p}(x,f)\cdot \rho^p(x)\,dm(x) \tag{1}$$

for every family of curves Γ in D and $\rho\in \operatorname{adm}\Gamma$.

Remark that an analog of the Statement 1 for $p = n$ was proved in [4], see Theorem 8.6 (see also [1] and [3]).

2. Proof of the main result. Let $I = [a, b]$. Given a rectifiable path $\gamma : I \to \mathbb{R}^n$ we define a length function $l_\gamma(t)$ by the rule $l_\gamma(t) = S(\gamma, [a, t])$, where $S(\gamma, [a, t])$ is the length of the path $\gamma|_{[a,t]}$. Let $\alpha : [a, b] \to \mathbb{R}^n$ be a rectifiable curve in $\mathbb{R}^n$, $n \ge 2$, and $l(\alpha)$ be its length. A *normal representation* α^0 of α is defined as a curve $\alpha^0 : [0, l(\alpha)] \to \mathbb{R}^n$ which can be got from α by change of parameter such that $\alpha(t) = \alpha^0(S(\alpha, [a, t]))$ for every $t \in [0, l(\alpha)]$.

Suppose that α and β are curves in $\mathbb{R}^n$. Then a notation $\alpha \subset \beta$ denotes that α is a subpath of β. In what follows, I denotes an open, a closed or a semi-open interval on the real axes. The following definition can be found in the section 5 of Ch. II in [6].

Let $f : D \to \mathbb{R}^n$ be a mapping such that $f^{-1}(y)$ does not contain a non-degenerate curve, $\beta : I_0 \to \mathbb{R}^n$ be a closed rectifiable curve and $\alpha : I \to D$ such that $f \circ \alpha \subset \beta$. If the length function $l_\beta : I_0 \to [0, l(\beta)]$ is a constant on $J \subset I$, then β is a constant on J and consequently a curve α to be a constant on J. Thus, there exists a unique function $\alpha^* : l_\beta(I) \to D$ such that $\alpha = \alpha^* \circ (l_\beta|_I)$. We say that α^* to be a *f-representation of α by the respect to β* if $\beta = f \circ \alpha$.

Remark 1. Given a closed rectifiable curve $\gamma : [a, b] \to \mathbb{R}^n$ and $t_0 \in (a, b)$, let $l_\gamma(t)$ denotes the length of the subcurve $\gamma|_{[t_0,t]}$ of γ if $t > t_0$, $t \in (a, b)$, and $l_\gamma(t)$ is equal to $-l(\gamma|_{[t,t_0]})$ if $t < t_0$, $t \in (a, b)$. Then we observe that properties of the $L_{\gamma,f}$ connected with the length functions $l_\gamma(t)$ and $l_{\widetilde{\gamma}}(t)$, $\widetilde{\gamma} = f \circ \gamma$, do not essentially depend on the choice of $t_0 \in (a, b)$. Moreover, we may consider that in this case $t_0 = a$ because given $t_0 \in (a, b)$, $S(\gamma, [a, t]) = S(\gamma, [a, t_0]) + l_\gamma(t)$. Further, we use the notion $l_\gamma(t)$ for $l_\gamma(t) = S(\gamma, [a, t])$, where $S(\gamma, [a, t])$ is the length of the path $\gamma|_{[a,t]}$, and consider that $t_0 = 0$ whenever a curve γ is closed.

The following statement gives the connection between $L_p^{(2)}$-property and some properties of curves meaning above.

Lemma 1. *A mapping $f : D \to \mathbb{R}^n$ has $L_p^{(2)}$-property if and only if $f^{-1}(y)$ does not contain a nondegenerate curve for every $y \in \mathbb{R}^n$, and the f-representation γ^* is rectifiable and absolutely continuous for p-a.e. closed curves $\widetilde{\gamma} = f \circ \gamma$.*

Proof. Suppose that f has $L_p^{(2)}$-property. Then γ^* is rectifiable for p-a.e. closed curves $\widetilde{\gamma}$ whenever $\widetilde{\gamma} = f \circ \gamma$ because $(\gamma^*)^0 = \gamma^0$, see Theorem 2.6 in [8]. Moreover, we observe that $f^{-1}(y)$ does not contain a nondegenerate curve for every $y \in \mathbb{R}^n$ because $L_{\gamma,f}$ is well-defined and has N^{-1}-property for p-a.e. closed curves $\widetilde{\gamma}$ and all γ with $\widetilde{\gamma} = f \circ \gamma$. For such γ and $\widetilde{\gamma}$, we have

$$\gamma(t) = \gamma^* \circ l_{\widetilde{\gamma}}(t) = \gamma^0 \circ l_\gamma(t) = \gamma^0 \circ L_{\gamma,f}^{-1}\left(l_{\widetilde{\gamma}}(t)\right)$$

and, denoting by $s := l_{\widetilde{\gamma}}(t)$ we obtain

$$\gamma^*(s) = \gamma^0 \circ L_{\gamma,f}^{-1}(s)\,.$$

So γ^* is absolutely continuous because $L_{\gamma,f}^{-1}(s)$ is absolutely continuous, see section 2.10.13 in [2], and

$$|\gamma^0(s_1) - \gamma^0(s_2)| \leq |s_1 - s_2|$$

for all $s_1, s_2 \in [0, l(\gamma)]$.

Inversely, let $f^{-1}(y)$ does not contain a nondegenerate curve for every $y \in \mathbb{R}^n$. Then $L_{\gamma,f}^{-1}$ is well-defined for p-a.e. closed curve $\widetilde{\gamma}$ and all γ with $\widetilde{\gamma} = f \circ \gamma$. By assumption curve γ^* is rectifiable for p-a.e. closed curve $\widetilde{\gamma} = f \circ \gamma$; in particular, $\gamma^{*0} = \gamma^0$. Moreover, for all such $\widetilde{\gamma}$, γ and γ^*, $l_{\gamma^*}(s) = L_{\gamma,f}^{-1}(s)$, and absolutely continuity of $L_{\gamma,f}^{-1}(s)$ follows from Theorem 1.3 in [8]. Let Γ_1 be a family of all closed curves $\widetilde{\alpha} = f \circ \alpha$ in $f(D)$ such that α^* either is not rectifiable or $L_{\alpha,f}^{-1}(s)$ is not absolutely continuous. Let Γ be a family of all curves $\widetilde{\gamma} = f \circ \gamma$ in $f(D)$ such that γ either is not locally rectifiable or $L_{\gamma,f}^{-1}(s)$ is not locally absolutely continuous. Then $\Gamma > \Gamma_1$ and, thus, $M_p(\Gamma) \leq M_p(\Gamma_1) = 0$ that implies desired equality $M_p(\Gamma) = 0$. □

A mapping $\varphi : X \to Y$ between metric spaces X and Y is said to be a *Lipschitzian* provided

$$\mathrm{dist}\,(\varphi(x_1), \varphi(x_2)) \leq M \cdot \mathrm{dist}(x_1, x_2)$$

for some $M < \infty$ and for all x_1 and $x_2 \in X$. The mapping φ is called *bi-lipschitz* if, in addition,

$$M^* \mathrm{dist}\,(x_1, x_2) \leq \mathrm{dist}\,(\varphi\,(x_1)\,, \varphi\,(x_2))$$

for some $M^* > 0$ and for all x_1 and $x_2 \in X$. Later on, X and Y are subsets of $\mathbb{R}^n$ with the Euclidean distance.

The following proposition can be found in [3], see Lemma 3.20, see also Lemma 8.3 Ch. VIII in [4].

Lemma 2. *Let $f : D \to \mathbb{R}^n$ be a differentiable a.e. in D, and have N- and N^{-1}-properties. Then there is a countable collection of compact sets $C_k^* \subset D$ such that $m(B_0) = 0$*

where $B_0 = D \setminus \bigcup\limits_{k=1}^{\infty} C_k^*$ *and* $f|_{C_k^*}$ *is one-to-one and bi-lipschitz for every* $k = 1, 2, \ldots$. *Moreover,* f *is differentiable at* C_k^* *and* $J(x, f) \neq 0$.

Given a set E in $\mathbb{R}^n$ and a curve $\gamma : \Delta \to \mathbb{R}^n$, we identify $\gamma \cap E$ with $\gamma(\Delta) \cap E$. If γ is locally rectifiable, then we set

$$l(\gamma \cap E) = m_1(E_\gamma),$$

where $E_\gamma = l_\gamma\left(\gamma^{-1}(E)\right)$; here $l_\gamma : \Delta \to \Delta_\gamma$ as in the previous section. Note that $E_\gamma = \gamma_0^{-1}(E)$, where $\gamma_0 : \Delta_\gamma \to \mathbb{R}^n$ is the natural parametrization of γ and

$$l(\gamma \cap E) = \int\limits_{\Delta} \chi_E(\gamma(t))\, |dx| := \int\limits_{\Delta_\gamma} \chi_{E_\gamma}(s) ds\,.$$

The bellow statement can be found in Chapter IX of [4], see Theorem 9.1.

Lemma 3. *Let* E *be a set in a domain* $D \subset \mathbb{R}^n$, $n \geq 2$, $p \geq 1$. *Then* E *is measurable if and only if* $\gamma \cap E$ *is measurable for* p*-a.e. curve* γ *in* D. *Moreover,* $m(E) = 0$ *if and only if*

$$l(\gamma \cap E) = 0$$

on p*-a.e. curve* γ *in* D.

The following result is a generalization of the known Poletskii inequality for quasiregular mappings, see Theorem 1 in [5] and Theorem 8.1 Ch. II in [6]. It's analog was also proved in [3-4] for the case $p = n$, see also [1].

Theorem 1. *Let a mapping* $f : D \to \mathbb{R}^n$ *be a differentiable a.e. in* D, *have* N*- and* N^{-1}*-properties, and* $L_p^{(2)}$*-property, too. Then the relation (1) holds for every curve family* Γ *in* D *and a function* $\rho \in \operatorname{adm}\Gamma$.

Proof. Let B_0 and C_k^*, $k = 1, 2, \ldots$, be as in Lemma 2. Setting by induction $B_1 = C_1^*$, $B_2 = C_2^* \setminus B_1, \ldots$, and

$$B_k = C_k^* \setminus \bigcup_{l=1}^{k-1} B_l \tag{2}$$

we obtain the countable covering of D consisting of mutually disjoint Borel sets $B_k, k = 0, 1, 2, \ldots$ with $m(B_0) = 0$, $B_0 = D \setminus \bigcup\limits_{k=1}^{\infty} B_k$. By the assumption, f has N-property in D and, consequently, $m(f(B_0)) = 0$. Let $\rho \in \operatorname{adm}\Gamma$ and

$$\widetilde{\rho}(y) = \chi_{f(D \setminus B_0)} \cdot \sup_{x \in f^{-1}(y) \cap D \setminus B_0} \rho^*(x)\,,$$

where

$$\rho^*(x) = \begin{cases} \rho(x)/l\left(f'(x)\right), & \text{for} \quad x \in D \setminus B_0, \\ 0, & \text{otherwise.} \end{cases}$$

Note that $\widetilde{\rho}(y) = \sup\limits_{k \in \mathbb{N}} \rho_k(y)$ where

$$\rho_k(y) = \begin{cases} \rho^*(f_k^{-1}(y)), & \text{for } y \in f(B_k), \\ 0, & \text{otherwise,} \end{cases}$$

and every $f_k = f|_{B_k}$, $k = 1, 2, \ldots$, is injective. Thus, the function $\widetilde{\rho}$ is Borel, see section 2.3.2 in [2].

Let $\widetilde{\gamma}$ be a closed rectifiable curve such that $\widetilde{\gamma} = f \circ \gamma$, $\widetilde{\gamma}^0$ be a normal representation of $\widetilde{\gamma}$ and γ^* be f-representation of γ by the respect to $\widetilde{\gamma}$, see above. Since $m(f(B_0)) = 0$, $\widetilde{\gamma}^0(s) \notin f(B_0)$ for p-a.e. curve $\widetilde{\gamma}$ and a.e. $s \in [0, l(\widetilde{\gamma})]$, see Lemma 3. For p-a.e. paths $\widetilde{\gamma}$ and all γ with $\widetilde{\gamma} = f \circ \gamma$, we have that

$$\int\limits_{\widetilde{\gamma}} \widetilde{\rho}(y)|dy| = \int\limits_0^{l(\widetilde{\gamma})} \widetilde{\rho}(\widetilde{\gamma}^0(s))\, ds =$$

$$= \int\limits_0^{l(\widetilde{\gamma})} \sup\limits_{x \in f^{-1}(\widetilde{\gamma}^0(s)) \cap D \setminus B_0} \rho^*(x)\; ds \geq \int\limits_0^{l(\widetilde{\gamma})} \frac{\rho(\gamma^*(s))}{l\,(f'(\gamma^*(s)))}\, ds\,. \tag{3}$$

Since $\widetilde{\gamma}^0$ is rectifiable, $\widetilde{\gamma}^0(s)$ is differentiable a.e. Besides that, a curve γ^* is absolutely continuous for p-a.e. $\widetilde{\gamma}$ by Lemma 1. Since $\widetilde{\gamma}^0(s) \notin f(B_0)$ for a.e. $s \in [0, l(\widetilde{\gamma})]$ and p-a.e. curves $\widetilde{\gamma}$, we have $\gamma^*(s) \notin B_0$ at a.e. $s \in [0, l(\widetilde{\gamma})]$. Thus, the derivatives $f'(\gamma^*(s))$ and $\gamma^{*\prime}(s)$ exist for a.e. s. Taking into account the formula of the derivative of the superposition of functions, and that the modulus of the derivative of the curve by the natural parameter equals to 1, we have

$$1 = \left|(f \circ \gamma^*)'(s)\right| = \left|f'(\gamma^*(s))\, \gamma^{*\prime}(s)\right| =$$

$$= \left|f'(\gamma^*(s)) \cdot \frac{\gamma^{*\prime}(s)}{|\gamma^{*\prime}(s)|}\right| \cdot |\gamma^{*\prime}(s)| \geq l\left(f'(\gamma^*(s))\right) \cdot |\gamma^{*\prime}(s)|\,. \tag{4}$$

It follows from (4) that a.e.

$$\frac{\rho(\gamma^*(s))}{l\,(f'(\gamma^*(s)))} \geq \rho(\gamma^*(s)) \cdot |\gamma^{*\prime}(s)|\,. \tag{5}$$

By absolutely continuity of γ^*, definition of ρ and Theorem 4.1 in [8] we obtain

$$1 \leq \int\limits_{\gamma} \rho(x)|dx| = \int\limits_0^{l(\widetilde{\gamma})} \rho\,(\gamma^*(s)) \cdot |\gamma^{*\prime}(s)|\, ds\,. \tag{6}$$

It follows from (3), (5) and (6) that $\int\limits_{\widetilde{\gamma}} \widetilde{\rho}(y)|dy| \geq 1$ for p-a.e. closed curve $\widetilde{\gamma}$ in $f(\Gamma)$. The case of the arbitrary path $\widetilde{\gamma}$ can be got from the taking of sup in $\int\limits_{\widetilde{\gamma}'} \widetilde{\rho}(y)|dy| \geq 1$ over all closed subpaths $\widetilde{\gamma}'$ of $\widetilde{\gamma}$. Thus, $\widetilde{\rho}(y) \in \operatorname{adm} f(\Gamma) \setminus \Gamma_0$, where $M_p(\Gamma_0) = 0$. Hence

$$M_p\left(f\left(\Gamma\right)\right) \le \int\limits_{f(D)} \widetilde{\rho}^p(y)dm(y)\,. \tag{7}$$

Further, by 3.2.5 for $m = n$ in [2] we have that

$$\int\limits_{B_k} K_{I,p}(x,f) \cdot \rho^p(x)dm(x) = \int\limits_{B_k} \frac{|J(x,f)|}{\left(l\left(f'(x)\right)\right)^p} \cdot \rho^p(x)dm(x) =$$

$$= \int\limits_{f(B_k)} \frac{\rho^p\left(f_k^{-1}(y)\right)}{\left(l\left(f'\left(f_k^{-1}(y)\right)\right)\right)^p}\,dm(y) = \int\limits_{f(D)} \rho_k^p(y)dm(y). \tag{8}$$

Finally, by the Lebesgue theorem, see Theorem 12.3 § 12 of Ch. I in [7], we obtain from (7) and (8) the desired inequality

$$\int\limits_{D} K_{I,p}(x,f) \cdot \rho^p(x)dm(x) = \sum\limits_{k=1}^{\infty} \int\limits_{B_k} K_{I,p}(x,f) \cdot \rho^p(x)dm(x) =$$

$$= \int\limits_{f(D)} \sum\limits_{k=1}^{\infty} \rho_k^p(y)dm(y) \ge \int\limits_{f(D)} \sup\limits_{k\in\mathbb{N}} \rho_k^p(y)dm(y) =$$

$$= \int\limits_{f(D)} \widetilde{\rho}^p(y)dm(y) = M_p\left(f(\Gamma)\right).$$

□

1. *Bishop C.J., Gutlyanskii V.Ya., Martio O., Vuorinen M.* On conformal dilatation in space // Intern. J. Math. and Math. Scie. – 2003. – V. 22. – P. 1397-1420.
2. *Federer H.* Geometric Measure Theory. – Berlin etc.: Springer, 1969.
3. *Martio O., Ryazanov V., Srebro U., Yakubov E.* Mappings with finite length distortion // J. Anal. Math. – 2004. – V. 93. – P. 215-236.
4. *Martio O., Ryazanov V., Srebro U., Yakubov E.* Moduli in Modern Mapping Theory. – New York: Springer Science + Business Media, LLC, 2009.
5. *Poletskii E.A.* The modulus method for non–homeomorphic quasiconformal mappings // Mat. Sb. – 1970. – V. 83, no. 2. – P. 261-272 (in Russian).
6. *Rickman S.* Quasiregular mappings. – Berlin etc.: Springer-Verlag, 1993.
7. *Saks S.* Theory of the Integral. – New York: Dover Publ. Inc., 1964.
8. *Väisälä J.* Lectures on n-Dimensional Quasiconformal Mappings. Lecture Notes in Math., V. 229. – Berlin etc.: Springer-Verlag, 1971.

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Об одном модульном неравенстве порядка $p \geq 1$.

Работа посвящена изучению пространственных отображений более общих, чем квазирегулярные. Предметом изучения работы являются так называемые модульные неравенства порядка $p \geq 1$ и их взаимосвязь с пространственными отображениями. Для отображений, имеющих N, N^{-1} и $L_p^{(2)}$-свойства доказано хорошо известное неравенство Полецкого.

Ключевые слова: *отображения с конечным и ограниченным искажением, модуль семейств кривых, неравенство Полецкого.*

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Про одну модульну нерівність порядку $p \geq 1$.

Роботу присвячено вивченню просторових відображень, більш загальних, ніж квазірегулярні. Предметом дослідження статті є так звані модульні нерівності порядку $p \geq 1$, та їх взаємозв'язок з просторовими відображеннями. Для відображень, що мають N, N^{-1} і $L_p^{(2)}$-властивості, доведено аналог добре відомої нерівності типу Полецького.

Ключові слова: *відображення зі скінченним і обмеженим спотворенням, модуль сім'ї кривих, нерівність Полецького.*

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