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ENERGY ESTIMATES OF BOUNDED SOLUTIONS OF THE DIRICHLET PROBLEM FOR A CLASS OF NONLINEAR FOURTH-ORDER ELLIPTIC EQUATIONS

We consider a class of nonlinear elliptic fourth-order equations with the principal part satisfying a strengthened coercivity condition, absorption and a lower-order term. It is supposed that the lower-order term of the equations admits the growth rates of derivatives of unknown function coinciding with the exponents of the corresponding energy space. At the same time, we do not suppose that the lower-order term satisfies a sign condition. Energy estimates of bounded generalized solutions of the Dirichlet problem for equations of the given class are established.

Keywords: *nonlinear elliptic fourth-order equations, strengthened coercivity, Dirichlet problem, bounded solutions, energy estimates.*

1. Preliminaries and the statement of the main result. Let $n \in \mathbb{N}$, $n > 2$, and let Ω be a bounded open set of $\mathbb{R}^n$.

We shall use the following notation: Λ is the set of all n-dimensional multi-indices α such that $|\alpha| = 1$ or $|\alpha| = 2$; $\mathbb{R}^{n,2}$ is the space of all mappings $\xi : \Lambda \to \mathbb{R}$; if $u \in W^{2,1}(\Omega)$, then $\nabla_2 u : \Omega \to \mathbb{R}^{n,2}$, and for every $x \in \Omega$ and for every $\alpha \in \Lambda$, $(\nabla_2 u(x))_\alpha = D^\alpha u(x)$.

Let $p \in (1, n/2)$ and $q \in (2p, n)$. We denote by $W^{1,q}_{2,p}(\Omega)$ the set of all functions in $W^{1,q}(\Omega)$ that have the second-order generalized derivatives in $L^p(\Omega)$. The set $W^{1,q}_{2,p}(\Omega)$ is a Banach space with the norm

$$\|u\| = \|u\|_{W^{1,q}(\Omega)} + \left(\sum_{|\alpha|=2} \int_\Omega |D^\alpha u|^p dx \right)^{1/p}.$$

We denote by $\overset{\circ}{W}{}^{1,q}_{2,p}(\Omega)$ the closure of the set $C_0^\infty(\Omega)$ in $W^{1,q}_{2,p}(\Omega)$.

We set $q^* = nq/(n-q)$. As is known (see for instance [1, Chapter 7]),

$$\overset{\circ}{W}{}^{1,q}(\Omega) \subset L^{q*}(\Omega), \tag{1}$$

and there exists a positive constant c depending only on n and q such that for every function $u \in \overset{\circ}{W}{}^{1,q}(\Omega)$,

$$\left(\int_\Omega |u|^{q^*} dx \right)^{1/q^*} \leqslant c \left(\sum_{|\alpha|=1} \int_\Omega |D^\alpha u|^q \, dx \right)^{1/q}. \tag{2}$$

Next, let $c_1, c_2, c_3, c_4, c_5 > 0$, let g_1, g_2, g_3, g_4, g_5 be nonnegative summable functions on Ω, and let $A_0 : \Omega \times \mathbb{R} \to \mathbb{R}$, $B : \Omega \times \mathbb{R} \times \mathbb{R}^{n,2} \to \mathbb{R}$ and $A_\alpha : \Omega \times \mathbb{R}^{n,2} \to \mathbb{R}$, where

$\alpha \in \Lambda$, are Carathéodory functions. We assume that for almost every $x \in \Omega$, for every $s \in \mathbb{R}$ and for every $\xi \in \mathbb{R}^{n,2}$ the following inequalities hold:

$$\sum_{|\alpha|=1} |A_\alpha(x,\xi)|^{q/(q-1)} \leqslant c_1 \Big\{ \sum_{|\alpha|=1} |\xi_\alpha|^q + \sum_{|\alpha|=2} |\xi_\alpha|^p \Big\} + g_1(x), \tag{3}$$

$$\sum_{|\alpha|=2} |A_\alpha(x,\xi)|^{p/(p-1)} \leqslant c_2 \Big\{ \sum_{|\alpha|=1} |\xi_\alpha|^q + \sum_{|\alpha|=2} |\xi_\alpha|^p \Big\} + g_2(x), \tag{4}$$

$$\sum_{\alpha\in\Lambda} A_\alpha(x,\xi)\xi_\alpha \geqslant c_3 \Big\{ \sum_{|\alpha|=1} |\xi_\alpha|^q + \sum_{|\alpha|=2} |\xi_\alpha|^p \Big\} - g_3(x), \tag{5}$$

$$|B(x,s,\xi)| \leqslant c_4 \Big\{ \sum_{|\alpha|=1} |\xi_\alpha|^q + \sum_{|\alpha|=2} |\xi_\alpha|^p \Big\} + g_4(x), \tag{6}$$

$$A_0(x,s)s \geqslant c_0|s|^q, \tag{7}$$

$$|A_0(x,s)| \leqslant c_5|s|^{q-1} + g_5(x). \tag{8}$$

Further, let

$$f \in L^{q^*/(q^*-1)}(\Omega). \tag{9}$$

We consider the following Dirichlet problem:

$$\sum_{\alpha\in\Lambda} (-1)^{|\alpha|} D^\alpha A_\alpha(x,\nabla_2 u) + A_0(x,u) + B(x,u,\nabla_2 u) = f \quad \text{in} \quad \Omega, \tag{10}$$

$$D^\alpha u = 0, \quad |\alpha| = 0,1, \quad \text{on} \quad \partial\Omega. \tag{11}$$

Observe that, by virtue of (3) and (4), for every u, $v \in \mathring{W}^{1,q}_{2,p}(\Omega)$ and for every $\alpha \in \Lambda$ the function $A_\alpha(x,\nabla_2 u)D^\alpha v$ is summable on Ω, by (8), for every $u, v \in \mathring{W}^{1,q}_{2,p}(\Omega)$ the function $A_0(x,u)v$ belongs to $L^1(\Omega)$, and finally, by (6), for every $u \in \mathring{W}^{1,q}_{2,p}(\Omega)$ and for every $v \in \mathring{W}^{1,q}_{2,p}(\Omega) \cap L^\infty(\Omega)$ the function $B(x,u,\nabla_2 u)\, v$ is summable on Ω. Moreover, it follows from (1) and (9) that for every $v \in \mathring{W}^{1,q}_{2,p}(\Omega)$ the function fv is summable on Ω.

Definition 1. A generalized solution of problem (10), (11) is a function $u \in \mathring{W}^{1,q}_{2,p}(\Omega)$ such that for every function $v \in \mathring{W}^{1,q}_{2,p}(\Omega) \cap L^\infty(\Omega)$,

$$\int_\Omega \Big\{ \sum_{\alpha\in\Lambda} A_\alpha(x,\nabla_2 u)D^\alpha v + A_0(x,u)v + B(x,u,\nabla_2 u)v \Big\} dx = \int_\Omega fv dx. \tag{12}$$

Before stating the main result of the work we give some comments on the equations of the given class.

If in (10) $B \equiv 0$, then equation (10) belongs to the class of quasi-linear divergent high-order equations with strengthened coercivity condition on the leading coefficients (for the fourth-order equations, the condition has the form (5)). This class was introduced in [2], where, by modifying Moser's method [3], the boundedness and Holder continuity of generalized solutions of equations of the given class with sufficiently regular data were established at first.

Using an analogue of Stampacchia's method [4-6], a weaker (exact) condition on integrability of data was established in [7] to guarantee the boundedness of generalized solutions of nonlinear fourth-order equations with a strengthened coercivity. Moreover, in [7] a dependence of summability of generalized solutions of these equations on integrability of data was described. Analogous results for nonlinear high-order equations with a strengthened coercivity were obtained in [8].

Condition (6) means that the lower-order term B in equation (10) may have growth of the derivatives of unknown function of orders q and p, which coincide with the exponents of the energy space $\overset{\circ}{W}{}^{1,q}_{2,p}(\Omega)$ (the natural growth of the derivatives of unknown function). The question of the solvability of the Dirichlet problem for elliptic second-order equations with natural growth lower-order terms was considered by many authors (see for instance [9-12]).

Usually, to solve this problem a sequence of approximate problems for equations with bounded lower-order terms is considered. The solvability of the approximate problem is established, for example, using the theory of pseudomonotone operators under certain assumptions on the coefficients of the initial equation. Then same appropriate a priori estimates are established for the sequence of the obtained solutions $\{u_i\}$ (uniform with respect to i), and on this basis the limit passage is made in the corresponding integral identities.

For problem (10), (11) we consider the following approximate problem:

$$\sum_{\alpha\in\Lambda}(-1)^{|\alpha|}D^\alpha A_\alpha(x,\nabla_2 u)+A_0(x,u)+B_i(x,u,\nabla_2 u)=f \quad \text{in} \quad \Omega, \tag{13}$$

$$D^\alpha u=0, \quad |\alpha|=0,1, \quad \text{on} \quad \partial\Omega, \tag{14}$$

where $i\in\mathbb{N}$, $B_i(x,s,\xi)=T_i(B(x,s,\xi))$ and $T_i(s)=\max\{\min\{s,i\},-i\}$.

Using such a scheme and approximate problem (13), (14), in [13] the existence of a bounded generalized solution of problem (10), (11) is proved in the case where $A_0\equiv 0$ and the following conditions are satisfied. The functions g_2, g_3 and f belong to $L^r(\Omega)$, $r>n/q$, for almost every $x\in\Omega$ and every $s\in\mathbb{R}$, $\xi,\xi'\in\mathbb{R}^{n,2}$, $\xi\neq\xi'$, the following inequalities hold:

$$\sum_{\alpha\in\Lambda}[A_\alpha(x,\xi)-A_\alpha(x,\xi')](\xi_\alpha-\xi'_\alpha)>0,$$

$$B(x,s,\xi)s\geqslant 0. \tag{15}$$

Without condition (15) the proof of solvability of problem (10), (11) meets the principal difficulties. First of all, we cannot use the functions $\psi(G_k(s))$, where $\psi(s)=(e^{\lambda|s|}-1)\mathrm{sign}s$, $G_k(s)=s-T_k(s)$, $\lambda,k>0$ and $s\in\mathbb{R}$, in the test functions (superpositions) to obtain the necessary estimates of solutions of the approximating problems in the same manner as for the second-order equations (see for instance [9]).

However, using in the integral identity (12) the test function $v = \psi(u)$ with a suitable λ, where u is a bounded generalized solution of problem (10), (11), we arrive at the estimate $\|u\| \leqslant C$, where $C > 0$ does not depend on u.

In this case, as in the case of the second-order equations (see for instance [9], [11]), the presence of additional absorption term A_0 (with condition (7)) in equation (10) is significant.

The following theorem is the main result of the present article.

Theorem 1. *Let the functions g_2, g_3, g_4, f belong to $L^{n/q}(\Omega)$, and let u be a generalized solution of problem* (10), (11) *such that $u \in L^\infty(\Omega)$. Then for every $\lambda > c_4/c_3$,*

$$\int_\Omega \Bigg(\sum_{|\alpha|=1} |D^\alpha u|^q + \sum_{|\alpha|=2} |D^\alpha u|^p \Bigg) \exp(\lambda|u|)dx \leqslant C_1, \tag{16}$$

where C_1 is a positive constant depending only on n, p, q, $\operatorname{meas}\Omega$, *$c$, c_0, c_2, c_3, c_4, λ, the functions g_2, g_3, g_4 and f.*

Corolarry 1. *Let the functions g_2, g_3, g_4, f belong to $L^{n/q}(\Omega)$, and let u be a generalized solution of problem* (10), (11) *such that $u \in L^\infty(\Omega)$. Then*

$$\|u\| \leqslant C_2, \tag{17}$$

where C_2 is a positive constant depending only on n, p, q, $\operatorname{meas}\Omega$, *$c$, c_0, c_2, c_3, c_4, the functions g_2, g_3, g_4 and f.*

Remark 1. Let $r > n/q$, let the functions g_2, g_3 and f belong to $L^r(\Omega)$, and the function g_4 belong to $L^{n/q}(\Omega)$. Let $i \in \mathbb{N}$, and let u_i be a generalized solution of problem (13), (14). By assertion (iii) of theorem 1 in [7], the function u_i belong to $L^\infty(\Omega)$. Now, using Corolarry 1 and the inequality $B_i(x,s,\xi) \leqslant B(x,s,\xi)$, $(x,s,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^{n,2}$, we establish the inequality $\|u_i\| \leqslant C_2$. Hence the sequence $\{u_i\} \subset \mathring{W}^{1,q}_{2,p}(\Omega)$ is bounded in $\mathring{W}^{1,q}_{2,p}(\Omega)$, and hence there exist an increasing sequence $\{i_j\} \subset \mathbb{N}$ and a function $u \in \mathring{W}^{1,q}_{2,p}(\Omega)$ such that $u_{i_j} \to u$ weakly in $\mathring{W}^{1,q}_{2,p}(\Omega)$.

2. Auxiliary results. By analogy with Lemma 2.2 of [14], we establish the following lemma.

Lemma 1. *Let $u \in \mathring{W}^{1,q}_{2,p}(\Omega) \cap L^\infty(\Omega)$, $h \in C^2(\mathbb{R})$ and $h(0) = 0$. Then $h(u) \in \mathring{W}^{1,q}_{2,p}(\Omega) \cap L^\infty(\Omega)$ and the following assertions hold*:

1) *for every n-dimensional multi-index α, $|\alpha| = 1$,*

$$D^\alpha h(u) = h'(u) D^\alpha u \quad \textit{a.e. in } \Omega,$$

2) *for every n-dimensional multi-index α, $|\alpha| = 2$,*

$$D^\alpha h(u) = h'(u) D^\alpha u + h''(u) D^{\alpha'} u D^{\alpha''} u \quad \textit{a.e. in } \Omega,$$

where $|\alpha'| = |\alpha''| = 1$, $\alpha' + \alpha'' = \alpha$.

Lemma 2. *Let ψ be an odd function on $\mathbb{R}$ such that $\psi \in C^1(\mathbb{R})$, $\psi \in C^2(\mathbb{R} \setminus \{0\})$ and ψ'' has a discontinuity at the origin of the first kind. Then if*

$$u \in \overset{\circ}{W}{}^{1,q}_{2,p}(\Omega) \cap L^\infty(\Omega), \tag{18}$$

then $\psi(u) \in \overset{\circ}{W}{}^{1,q}_{2,p}(\Omega) \cap L^\infty(\Omega)$ and the following assertions hold:

1) *for every n-dimensional multi-index α, $|\alpha| = 1$,*

$$D^\alpha \psi(u) = \psi'(u) D^\alpha u \quad \textit{a.e. in } \Omega,$$

2) *for every n-dimensional multi-index α, $|\alpha| = 2$,*

$$D^\alpha \psi(u) = \begin{cases} \psi'(u) D^\alpha u + \psi''(u) D^{\alpha'} u D^{\alpha''} u & \text{a.e. in } \{u \neq 0\}, \\ \psi'(0) D^\alpha u & \text{a.e. in } \{u = 0\}, \end{cases}$$

where $|\alpha'| = |\alpha''| = 1$, $\alpha' + \alpha'' = \alpha$.

Proof. Let $u \in \overset{\circ}{W}{}^{1,q}_{2,p}(\Omega) \cap L^\infty(\Omega)$. We define the function $\Psi \colon \mathbb{R} \to \mathbb{R}$ by

$$\Psi(s) = \psi(s) - \psi'(0)s, \;\; s \in \mathbb{R}. \tag{19}$$

If $|\alpha| = 1$, we put

$$w_\alpha = \Psi'(u) D^\alpha u, \tag{20}$$

and if $|\alpha| = 2$, we put

$$w_\alpha(x) = \begin{cases} \Psi'(u) D^\alpha u + \Psi''(u) D^{\alpha'} u D^{\alpha''} u & \text{if } u(x) \neq 0, \\ 0 \;\; \text{if } u(x) = 0, \end{cases} \tag{21}$$

where $|\alpha'| = |\alpha''| = 1$, $\alpha' + \alpha'' = \alpha$.

Clearly,

$$w_\alpha \in L^q(\Omega), \;\; |\alpha| = 1; \quad w_\alpha \in L^p(\Omega), \;\; |\alpha| = 2. \tag{22}$$

We fix $\varepsilon > 0$. Let Ψ_ε be the function on $\mathbb{R}$ such that

$$\Psi_\varepsilon(s) = \begin{cases} \Psi(s) + (\frac{1}{2}\varepsilon\Psi''(\varepsilon) - \Psi'(\varepsilon))(s - \varepsilon) + \frac{1}{6}\varepsilon^2\Psi''(\varepsilon) - \Psi(\varepsilon) \text{ if } s > \varepsilon, \\ \Psi''(\varepsilon)s^3/(6\varepsilon) \;\; \text{if } |s| \leqslant \varepsilon, \\ \Psi(s) + (\frac{1}{2}\varepsilon\Psi''(\varepsilon) - \Psi'(\varepsilon))(s + \varepsilon) - \frac{1}{6}\varepsilon^2\Psi''(\varepsilon) + \Psi(\varepsilon) \text{ if } s < -\varepsilon. \end{cases}$$

We have

$$\Psi_\varepsilon \in C^2(\mathbb{R}), \tag{23}$$

$$\Psi'_\varepsilon(s) = \begin{cases} \Psi'(s) + \varepsilon\Psi''(\varepsilon)/2 - \Psi'(\varepsilon) & \text{if } |s| > \varepsilon, \\ \Psi''(\varepsilon)s^2/(2\varepsilon) & \text{if } |s| \leqslant \varepsilon, \end{cases}$$

$$\Psi''_\varepsilon(s) = \begin{cases} \Psi''(s) & \text{if } |s| > \varepsilon, \\ \Psi''(\varepsilon)s/\varepsilon & \text{if } |s| \leqslant \varepsilon. \end{cases}$$

The following limit relations hold:

$$\lim_{\varepsilon\to 0}\Psi_\varepsilon(s)=\Psi(s)\quad\text{for every } s\in\mathbb{R}, \tag{24}$$

$$\lim_{\varepsilon\to 0}\Psi'_\varepsilon(s)=\Psi'(s)\quad\text{for every } s\in\mathbb{R}, \tag{25}$$

$$\lim_{\varepsilon\to 0}\Psi''_\varepsilon(s)=\begin{cases}\Psi''(s) & \text{if } s\in\mathbb{R}\setminus\{0\},\\ 0 & \text{if } s=0.\end{cases} \tag{26}$$

Using the inclusions (18) and (23), the equality $\Psi_\varepsilon(0)=0$ and Lemma 1, we establish that

$$\Psi_\varepsilon(u)\in \overset{\circ}{W}{}^{1,q}_{2,p}(\Omega)\cap L^\infty(\Omega),$$

for $|\alpha|=1$

$$D^\alpha\Psi_\varepsilon(u)=\Psi'_\varepsilon(u)D^\alpha u,$$

and for $|\alpha|=2$

$$D^\alpha\Psi_\varepsilon(u)=\Psi'_\varepsilon(u)D^\alpha u+\Psi''_\varepsilon(u)D^{\alpha'}uD^{\alpha''}u,$$

where $|\alpha'|=|\alpha''|=1$, $\alpha'+\alpha''=\alpha$. Hence, using (18), (24)–(26) and Lebesgue's dominated convergence theorem, we deduce

$$\lim_{\varepsilon\to 0}\|\Psi_\varepsilon(u)-\Psi(u)\|_{L^q(\Omega)}=0, \tag{27}$$

$$\lim_{\varepsilon\to 0}\sum_{|\alpha|=1}\|D^\alpha\Psi_\varepsilon(u)-w_\alpha\|_{L^q(\Omega)}=0,\quad \lim_{\varepsilon\to 0}\sum_{|\alpha|=2}\|D^\alpha\Psi_\varepsilon(u)-w_\alpha\|_{L^p(\Omega)}=0. \tag{28}$$

Using these limit relations, we establish in the usual way that the generalized derivative $D^\alpha\Psi(u)$ exists for every $\alpha\in\Lambda$, and $D^\alpha\Psi(u)=w_\alpha$ a.e. on Ω. Then, by (22), (27) and (28), the function $\Psi(u)$ belong to $\overset{\circ}{W}{}^{1,q}_{2,p}(\Omega)\cap L^\infty(\Omega)$, and (19)–(21) imply that assertions 1) and 2) hold. The lemma is proved. □

3. Proof of Theorem 1. Let the functions g_2, g_3, g_4 and f belong to $L^{n/q}(\Omega)$, and let u be a bounded generalized solution of problem (10), (11). We fix an arbitrary positive number λ such that

$$\lambda>c_4/c_3. \tag{29}$$

By c_i, $i=6,7,...$, we shall denote positive constants depending only on n, p, q, $\operatorname{meas}\Omega$, c, c_0, c_2, c_3, c_4, λ, the functions g_2, g_3, g_4 and f.

We define the function $\psi:\mathbb{R}\to\mathbb{R}$ by

$$\psi(s)=(e^{\lambda|s|}-1)\operatorname{sign}s,\quad s\in\mathbb{R}.$$

We set $c_6=c_3\lambda-c_4$. By (29), we have $c_6>0$. Elementary calculations show that

$$c_3\psi'-c_4|\psi|>c_6\psi'\quad\text{on }\ \mathbb{R}. \tag{30}$$

We set

$$I' = \int\limits_{\{u\neq 0\}} \Big\{ \sum_{|\alpha|=2} |A_\alpha(x, \nabla_2 u)| \Big\} \Big\{ \sum_{|\beta|=1} |D^\beta u|^2 \Big\} |\psi''(u)| dx,$$

$$\Phi = \sum_{|\alpha|=1} |D^\alpha u|^q + \sum_{|\alpha|=2} |D^\alpha u|^p.$$

By virtue of Lemma 2, $\psi(u) \in \overset{\circ}{W}{}^{1,q}_{2,p}(\Omega) \cap L^\infty(\Omega)$. Thus, by (12), we have

$$\int_\Omega \Big\{ \sum_{\alpha\in\Lambda} A_\alpha(x, \nabla_2 u) D^\alpha \psi(u) + A_0(x,u)\psi(u) + B(x,u,\nabla_2 u)\psi(u) \Big\} dx = \int_\Omega f\psi(u) dx.$$

From this equality and assertions 1) and 2) of Lemma 2 we deduce that

$$\int_\Omega \Big\{ \sum_{\alpha\in\Lambda} A_\alpha(x, \nabla_2 u) D^\alpha u \Big\} \psi'(u) dx + \int_\Omega A_0(x,u)\psi(u) dx$$

$$\leqslant \int_\Omega |B(x,u,\nabla_2 u)||\psi(u)| dx + \int_\Omega |f||\psi(u)| dx + I'.$$

Hence, using (5)-(7) and the fact that $0 < \psi' = \lambda|\psi| + \lambda$ and $\operatorname{sign}\psi(s) = \operatorname{sign} s$ on $\mathbb{R}$, we obtain

$$\int_\Omega \Phi(c_3\psi'(u) - c_4|\psi(u)|) dx + c_0 \int_\Omega |u|^{q-1}|\psi(u)| dx$$

$$\leqslant \int_\Omega (\lambda g_3 + g_4 + |f|)|\psi(u)| dx + \lambda \int_\Omega g_3 dx + I'.$$

From this and (30) it follows that

$$\begin{aligned} c_6 \int_\Omega \psi'(u)\Phi dx + c_0 \int_\Omega |u|^{q-1}|\psi(u)| dx \\ \leqslant \int_\Omega (\lambda g_3 + g_4 + |f|)|\psi(u)| dx + \lambda \int_\Omega g_3 dx + I'. \end{aligned} \tag{31}$$

Let us estimate the integral I'. We fix an arbitrary $\varepsilon > 0$. It is obvious that

$$\frac{p-1}{p} + \frac{2}{q} + \frac{q-2p}{qp} = 1.$$

Using this equality and Young's inequality, we establish that if $\alpha \in \Lambda$, $|\alpha| = 2$, and $\beta \in \Lambda$, $|\beta| = 1$, then

$$\begin{aligned} |A_\alpha(x, \nabla_2 u)|\, |D^\beta u|^2 \leqslant \varepsilon^2 |A_\alpha(x,\nabla_2 u)|^{p/(p-1)} \\ + \varepsilon^2 |D^\beta u|^q + \varepsilon^{2-2qp/(q-2p)} \quad \text{on} \;\; \Omega. \end{aligned}$$

From this and (4) we deduce that

$$I' \leqslant n(c_2+n)\varepsilon^2 \int\limits_{\{u\neq 0\}} \Phi|\psi''(u)|dx + n\varepsilon^2 \int\limits_{\{u\neq 0\}} g_2|\psi''(u)|dx$$

$$+ n^3\varepsilon^{2-2qp/(q-2p)} \int\limits_{\{u\neq 0\}} |\psi''(u)|dx.$$

Putting in the last inequality $\varepsilon = (\frac{c_6}{2\lambda n(c_2+n)})^{1/2}$, and noting that $|\psi''| = \lambda\psi'$ and $|\psi''| = \lambda^2|\psi| + \lambda^2$ on $\mathbb{R}\setminus\{0\}$, we obtain

$$I' \leqslant \frac{c_6}{2}\int_\Omega \psi'(u)\Phi dx + c_7\int_\Omega (g_2+1)|\psi(u)|dx + c_8.$$

From this and (31) it follows that

$$\frac{c_6}{2}\int_\Omega \psi'(u)\Phi dx + c_0\int_\Omega |u|^{q-1}|\psi(u)|dx \leqslant c_9\int_\Omega F|\psi(u)|dx + c_{10}, \tag{32}$$

where $F = g_2 + \lambda g_3 + g_4 + |f| + 1$.

We now estimate the integral $\int_\Omega F|\psi(u)|dx$. We fix an arbitrary $H > 0$. It is clear that

$$\int_\Omega F|\psi(u)|dx = \int\limits_{\{F>H,|u|\geqslant 1\}} F|\psi(u)|dx + \int\limits_{\{F<H\}} F|\psi(u)|dx + \int\limits_{\{F>H,|u|<1\}} F|\psi(u)|dx, \tag{33}$$

$$\int\limits_{\{F<H\}} F|\psi(u)|dx < H\int_\Omega |\psi(u)|dx, \qquad \int\limits_{\{F>H,|u|<1\}} F|\psi(u)|dx < (e^\lambda - 1)\int_\Omega F dx. \tag{34}$$

Before estimating the first integral in the right-hand side of equality (33), we remark that there exists a positive constant $c_{q,\lambda}$ depending only on q and λ such that

$$|\psi(s)| \leqslant c_{q,\lambda}|\psi(s/q)|^q \quad \text{for every} \quad s \geqslant 1. \tag{35}$$

Note also that, by (2), assertion 1) of Lemma 1 and equality $(\psi'(s/q))^q = \lambda^{q-1}\psi'(s)$, $s \in \mathbb{R}$, we have

$$\left(\int_\Omega |\psi(u/q)|^{q^*} dx\right)^{q/q^*} \leqslant (c\lambda^{q-1}/q^q)\int_\Omega \psi'(u)\Phi dx. \tag{36}$$

Now, using Holder's inequality, (35) and (36), we obtain

$$\int\limits_{\{F>H,|u|\geqslant 1\}} F|\psi(u)|dx \leqslant \left(\int\limits_{\{F>H\}} F^{n/q}dx\right)^{q/n}\left(\int\limits_{\{|u|\geqslant 1\}} |\psi(u)|^{n/(n-q)}dx\right)^{(n-q)/n}$$

$$\leqslant c_{q,\lambda}\|F\|_{L^{n/q}(\{F>H\})}\bigg(\int_\Omega |\psi(u/q)|^{q^*}dx\bigg)^{q/q^*} \leqslant c_{q,\lambda}c\lambda^{q-1}q^{-q}\|F\|_{L^{n/q}(\{F>H\})}\int_\Omega \psi'(u)\Phi dx.$$

From this and (33) and (34) it follows that

$$\int_\Omega F|\psi(u)|dx \leqslant c_{11}\|F\|_{L^{n/q}(\{F>H\})}\int_\Omega \psi'(u)\Phi dx + \int_\Omega H|\psi(u)|dx + c_{12}. \tag{37}$$

Now, choosing $H>0$ such that $c_9c_{11}\|F\|_{L^{n/q}(\{F>H\})} < c_6/4$, from (32) and (37) we obtain

$$\frac{c_6}{4}\int_\Omega \psi'(u)\Phi dx + \int_\Omega c_0|u|^{q-1}|\psi(u)|dx \leqslant \int_\Omega c_9H|\psi(u)|dx + c_{13}. \tag{38}$$

It is clear that

$$\int_\Omega c_0|u|^{q-1}|\psi(u)|dx \geqslant \int\limits_{\{c_0|u|^{q-1}>c_9H\}} c_0|u|^{q-1}|\psi(u)|dx > \int\limits_{\{c_0|u|^{q-1}>c_9H\}} c_9H|\psi(u)|dx, \tag{39}$$

$$\int_\Omega c_9H|\psi(u)|dx = \int\limits_{\{c_0|u|^{q-1}>c_9H\}} c_9H|\psi(u)|dx + \int\limits_{\{c_0|u|^{q-1}\leqslant c_9H\}} c_9H|\psi(u)|dx$$
$$\leqslant \int\limits_{\{c_0|u|^{q-1}>c_9H\}} c_9H|\psi(u)|dx + c_9H(e^{\lambda(c_9H/c_0)^{1/(q-1)}}-1)\,\mathrm{meas}\,\Omega\,. \tag{40}$$

From (38)–(40) it follows that

$$\frac{c_6}{4}\int_\Omega \psi'(u)\Phi dx \leqslant c_{14}.$$

Hence, using the fact that for every $s\in\mathbb{R}$, $\psi'(s)=\lambda\exp(\lambda|s|)$, we deduce (16). Theorem 1 is proved.

1. *Gilbarg D., Trudinger N.S.* Elliptic partial differential equations of second order. – Berlin: Springer-Verlag, 1983. – 513 p.
2. *Skrypnik I.V.* Higher order quasilinear elliptic equations with continouos generalized solutions, (Russian) // Differentsialnye Uravneniya. – 1978. – **14**, № 6. – P. 1104-1118.
3. *Moser J.* A new proof of De Giorgi's theorem concerning the regularity problem for elliptic differential equations // Comm. Pure Appl. Math. – 1960. – **13**. – P. 457-468.
4. *Stampacchia G.* Régularisation des solutions de problèmes aux limites elliptiques à données discontinues // Proc. Internat. Sympos. Linear Spaces (Jerusalem, 1960), Pergamon, Oxford. – 1961. – P. 399-408.
5. *Stampacchia G.* Èquations elliptiques du second ordre à coefficients discontinus, Séminaire de Mathématiques Supérieures, No. 16 (Été, 1965), Les Press. Univ. Montreal, Montreal. – 1966.
6. *Kinderlehrer D. and Stampacchia G.* An Introduction to Variational Inequalities and Their Applications, Academic Press, New York-London. – 1980.
7. *Kovalevskii A.A. and Voitovich M.V.* On increasing the summability of generalized solutions of the Dirichlet problem for fourth-order nonlinear equations with strengthened ellipticity // Ukrainian Math. J. – 2006. – **58**, № 11. – P. 1717-1733.
8. *Voitovich M.V.* Integrability properties of generalized solutions of the Dirichlet problem for higher-order nonlinear equations with strengthened ellipticity, (Russian) // Tr. Inst. Prikl. Mat. Mekh. – 2007. – **15**. – P. 3-14.

9. *Boccardo L., Murat F. and Puel J.-P.* L^∞-estimate for some nonlinear elliptic partial differential equations and application to an existence result // SIAM J. Math. Anal. – 1992. – **23**, № 2. – P. 326-333.
10. *Drábek P. and Nicolosi F.* Existence of bounded solutions for some degenerated quasilinear elliptic equations // Ann. Mat. Pura Appl. (4). – 1993. – **165**. – P. 217-238.
11. *Dall'aglio A., Giachetti D. and Puel J.-P.* Nonlinear elliptic equations with natural growth in general domains // Ann. Mat. Pura Appl. – 2002. – **181**. – P. 407-426.
12. *Grenon N., Murat F. and Porretta A.* Existence and a priori estimate for elliptic problems with subquadratic gradient dependent terms // C. R. Acad. Sci. Paris, Ser I. – 2006. – **342**. – P. 23–28.
13. *Voitovich M.V.* Existence of bounded solutions for a class of nonlinear fourth-order equations // Differential Equations and Applications. – 2011. – **3**, № 2. – P. 247-266.
14. *Kovalevskii A.A.* Entropy solutions of the Dirichlet problem for a class of fourth-order nonlinear elliptic equations with L^1-right-hand sides // Izv. Math. – 2001. – **65**, № 2. – P. 231-283.

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Энергетические оценки ограниченных решений задачи Дирихле для одного класса нелинейных эллиптических уравнений четвертого порядка.

Рассматривается класс нелинейных эллиптических уравнений четвертого порядка со старшими коэффициентами, удовлетворяющими условию усиленной коэрцитивности, абсорбцией и младшим коэффициентом, имеющим рост порядков, совпадающих с показателями соответствующего уравнениям энергетического пространства, относительно производных неизвестной функции. Не делается никаких предположений о выполнении определенных знаковых условий относительно младшего коэффициента. Устанавливаются энергетические оценки ограниченных обобщенных решений задачи Дирихле для уравнений рассматриваемого класса.

Ключевые слова: *нелинейные эллиптические уравнения четвертого порядка, усиленная коэрцитивность, задача Дирихле, ограниченные решения, энергетические оценки.*

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Енергетичні оцінки обмежених розв'язків задачі Діріхле для одного класу нелінійних еліптичних рівнянь четвертого порядку.

Розглядається клас нелінійних еліптичних рівнянь зі старшими коефіцієнтами, які задовольняють умову підсиленої коерцитивності, абсорбцією і молодшим коефіцієнтом, що має зростання порядків, які співпадають з показниками відповідного рівнянням енергетичного простору, відносно похідних невідомої функції. Не робиться жодних припущень щодо виконання певних знакових умов відносно молодшого коефіцієнта. Встановлено енергетичні оцінки обмежених узагальнених розв'язків задачі Діріхле для рівнянь, що розглядаються.

Ключові слова: *нелінійні еліптичні рівняння четвертого порядку, підсилена коерцитивність, задача Діріхле, обмежені розв'язки, енергетичні оцінки.*

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