УДК 62-50

©2004. **Veljko A. Vujičić**

PROGRAMMED MOTION OF MECHANICAL SYSTEMS

The main results of this contribution are new methods of solving of adjoint systems of $6n$ differential nonlinear equations and its application in classical and celestial mechanics. At first, we are observing a general dynamic system of n differential equations of the first order, which contain n independent functions $x(t)$ and n unknown composite functions $X(x(t))$. A programme of motion of such one is described by n independent finite algebraic equations $f(x) = 0$. For realization of the control motion it is necessary to define functions $X(x)$ and within them also control functions. It is shown that such dynamic systems do not correspond to mechanical systems. Defining of control motion of mechanical systems is much more complex. It is explained which of the differential equations of motion are used, and what are the consequences. It is also manifested that $3N$ Newton's differential equations of motions and $n = 3N-k, k < 3N$, Lagrange's differential equations of second kind, or $2n$ Hamilton's differential equations on manifolds, are not giving the same results at defining of forces, being of the primary importance for control motion.

П.В. Харламов: Мифы и метафизические представления неизбежны в становлении науки, но при совершенствовании ее их стремятся устранить, преодолевая установившиеся стереотипы и догмы. Однако зачастую именно их и полагают истинами и потому не пытаются анализировать.

Mathematical base. We observe the system of n differential equations (see, for example, [1], p. 247),

$$\frac{df(x)}{dx} = F(f,U) = \mathcal{F}(f(x)) + U \tag{1}$$

and n algebraic independent equations

$$\mathcal{P}(f,x) = 0, \tag{2}$$

where: x is the independent variable, $f = (f_1, \ldots, f_n)$ are functions of x, $F(f) = \big(F_1(f), \ldots, F_n(f)\big)$. With $\mathcal{F}(f)$ we will mark the known functions, and $U = (U_1, \ldots, U_n)$ are the unknown functions. In order to be able to integrate equations (1), it is necessary to determine functions F. For solving that task, one can differentiate equations (2) on x, then

$$\frac{\partial\mathcal{P}}{\partial f}\frac{df}{dx} + \frac{\partial\mathcal{P}}{\partial x} = 0.$$

Further, according to (1), a system of n linear equations on F is obtained

$$\frac{\partial\mathcal{P}}{\partial f}\big(\mathcal{F}(f) + U\big) = -\frac{\partial\mathcal{P}}{\partial x}.$$

From this equation at the well known conditions it is possible to determine the right sides of differential equations (1) as functions of $f(x)$ and independent variable x.

1. Dynamical systems. The term "dynamical system" is generally two-ways used. First, it is a mathematical term for a system of differential equations of the first order, or a smooth vector field on a differentiable manifold M

$$\dot{y} = Y(y,t), \tag{3}$$

where: t is the natural parameter, $y(t) = \big(y_1(t), \dots, y_n(t)\big)$, $\dot{y} = \dfrac{dy}{dt}$ are n dimensional vectors, while $Y(t, y)$ are the known composite real functions of $y(t)$ and t. Second, it is a general form of differential equations of motion of mechanical systems. Most frequently, these are seen as the generalized Hamilton's differential equations of the first order

$$\dot{p} = -\frac{\partial H}{\partial q}, \quad \dot{q} = \frac{\partial H}{\partial p}, \tag{4}$$

where $q = (q^1(t), \dots, q^n(t))$ and $p = (p_1, \dots, p_n)$ are Hamilton's variables on $2n$ dimensional manifolds T^*M^n. More precisely: $q(t) \in M^n$ are the generalized Lagrange's **independent** coordinates, and $p(t) = A(q)\dot{q}$ is the known linear form of the generalized velocities $\dot{q}$. In fact, equations (4) are Lagrange's differential equations of the second kind and second order on the configuration manifold M^n or tangent manifold $(q, \dot{q}) \in TM^n$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = Q^*. \tag{5}$$

Equations (4) and (5) do not contain all forces acting on material points. For their use it is not sufficient only to find a kinetic energy. It is more important to find out how the manifold M^n is defined, in order to know precisely what parts of motion are present. More general and suitable equations for determination of control motion are so-called Newton's equations of motion, however these are differential equations of the second order.

Dynamical system (3) for control motion can be, according to systems (1) and (2), presented in the form

$$\dot{y} = Y(y, t) + U; \tag{6}$$
$$\mathcal{P}(y, t) = 0, \quad \mathcal{P} \in C^1,$$

where $Y(y, t)$ are functions known in advance, in contrast to functions U, which are unknown. Determination of the right side of equations (6) and among them, of the unknown factors of control U, is deduced from system (3)

$$\dot{\mathcal{P}} = \frac{\partial \mathcal{P}}{\partial y}\dot{y} + \frac{\partial \mathcal{P}}{\partial t} = 0,$$

or, according to equation (6),

$$\frac{\partial \mathcal{P}}{\partial y}(Y + U) = -\frac{\partial \mathcal{P}}{\partial t}.$$

From here, as it is obvious, it is easy to define functions $U(y, t)$ needed for realization of control motion of dynamical system (6).

2. Forces of programmed motion of mechanical system. Dynamical systems (3) in mechanics have more composite and denominated structure. The very word "dynamic" originating from a Greek word means science on forces, as $\delta\upsilon\nu\alpha\mu\iota\varsigma$ is signifying a force producing a motion. Mechanical motion of body means moving a body from one place to another with a velocity during some time. Let us assume that there are bodies characterized by a mass m_ν, dim $m = M$; distance ρ, dimension L, and time t, dim $t = T$. Here we base the mathematical theory on Newton's axioms. We consider motion of a system of N bodies as the material points with masses $m_\nu, (\nu = 1, \dots, N)$, and the position vectors $\mathbf{r}_\nu = y^1_\nu \mathbf{e}_1 + y^2_\nu \mathbf{e}_2 + y^3_\nu \mathbf{e}_3 = = y^i_\nu \mathbf{e}_i$, where $\mathbf{e}_i$ are the orthonormal basic vectors.

Velocities are determined by vectors $\mathbf{v}_\nu = \dot{\mathbf{r}}_\nu = \dot{y}^i_\nu \mathbf{e}_i$. According to Newton's second axiom, there are N vectors equations

$$m_\nu \frac{d\mathbf{v}_\nu}{dt} = \mathbf{F}_\nu, \quad \mathbf{v}_\nu \in \mathbf{R}_3,$$

or $3N$ related equations in the coordinate form

$$\frac{d\dot{y}_\nu(t)}{dt} = \frac{1}{m_\nu} Y_\nu, \tag{6}$$

where $Y_\nu = (Y^1_\nu, Y^2_\nu, Y^3_\nu)$ are now $3N$ unknown coordinates of the vectors forces $\mathbf{F}_\nu$.

In general, a programme of motion is made by a system of mutually independent algebraic and kinematics equations:

$$\mathcal{P}_r\big(y(t)\big) = 0, \quad r = 1, \ldots, k_1, \tag{8}$$

$$\mathcal{P}_v\big(y(t), \dot{y}(t)\big) = 0, \quad v = 1, \ldots, k_2, \tag{9}$$

$$\mathcal{P}_\sigma(Y) = 0, \quad \sigma = 1, \ldots, k_3; \quad k_1 + k_2 + k_3 = 3N.$$

Such programme of $3N$ equations and together with $3N$ differential equation of motion (7) are providing a determination of $3N$ forces, producing a motion and $3N$ function of positions $Y^i_\nu(t)$, which seems to be pretty simple. Let us differentiate equations (8) and (9) so that

$$\ddot{\mathcal{P}}_r = \sum_{\mu=1}^{3N} \sum_{\nu=1}^{3N} \frac{\partial^2 \mathcal{P}_r}{\partial y_\mu \partial y_n u} \dot{y}_\mu \dot{y}_\nu + \sum_{\nu=1}^{3N} \frac{\partial \mathcal{P}_r}{\partial y_\nu} \ddot{y}_\nu = 0, \tag{10}$$

$$\dot{\mathcal{P}}_v = \sum_{\nu=1}^{3N} \frac{\partial \mathcal{P}_v}{\partial y_\nu} \dot{y}_\nu + \sum_{\nu=1}^{3N} \frac{\partial \mathcal{P}_v}{\partial \dot{y}_\nu} \ddot{y}_\nu = 0. \tag{11}$$

By the substitution of a second derivative $\ddot{y}_\nu$ from equations (7) in equations (10) and (11), the system of $3N$ linear equations on Y_ν is obtained, based on which it is possible to determine $3N$ unknown Y_ν as functions of $y_\nu, \dot{y}_\nu$. As resultant forces $\mathbf{F}_\nu$ represent a total of all forces acting on ν material point, it is understood that among them forces of control motion are present.

Notes: In the literature some modern and respectable authors are of opinion that the functions of forces are known in advance, for example, these functions are determined experimentally. This approach does correspond to neither our previous stipulation on determination of force, nor Newton's theory [2, p.27]. We consider first the Newton's conditions and other existing conditions and the needed programmes. In order to better understand the proposed methods, let us discuss the famous problem of two bodies. It is well-known and accepted that two bodies of masses m_1 and m_2 are reacting on each other according to the so-called universal Newton's law of gravitation

$$F = \kappa \frac{m_1 m_2}{\rho^2}; \quad \kappa = 6,67 \cdot 10^{-11} m^3 kg^{-1} s^{-2}. \tag{12}$$

Starting from the Newton's axioms of mechanics and conditions, in which the distance $\rho = \rho(t)$ is defined according to some law, the more general force can be deducted

$$\mathbf{F} = \frac{m_1 m_2}{m_1 + m_2} \ddot{\boldsymbol{\rho}},$$

or

$$\mathcal{F}_\rho = \frac{m_1 m_1}{m_1 + m_2}(\ddot{\rho} - \rho\dot{\theta}^2),$$

or

$$F = \chi\frac{m_1 m_2}{\rho},$$

where

$$\chi = \frac{\dot{\rho}^2 + \rho\ddot{\rho} - v_{or}^2}{m_1 + m_2}.$$

Really, there is a show of reason in it.

Example. Two bodies problem. The motion of a system of two bodies, observed as the the material points, is known in celestial mechanics as "two bodies problem". Kepler's laws as well as Newton's gravitational force, are the ones that relate to the motion of two bodies mutually attracting each other. This is a simple mechanical system of two material points, but its reduction to the Newton's theorems of gravity is a significant problem.

The main goal is to determine the formula for the force of mutual attraction of bodies. Thus we considered two material points whose masses are m_1 and m_2, which move towards each other so that the distance between their inertia centers is a time function $\rho(t) = \|\mathbf{r}_2 - \mathbf{r}_1\|$. If $\mathbf{r}_1$ and $\mathbf{r}_2$ are the position vectors of mass points m_1 and m_2, the distance is

$$\rho_0 = \|\mathbf{r}_2 - \mathbf{r}_1\|,$$

because

$$\boldsymbol{\rho}(t) = \mathbf{r}_2 - \mathbf{r}_1 = \rho\boldsymbol{\rho}_0, \tag{13}$$

where $\boldsymbol{\rho}_0 = \dfrac{\boldsymbol{\rho}}{\rho}$. The main goal is to determine the formula for magnitude F of the mutual attraction force of bodies $\mathbf{F}_1$ or $\mathbf{F}_2$.

Solution. Differential equations of motion of two bodies are:

$$m_i\frac{d^2\mathbf{r}_i}{dt^2} = \mathbf{F}_i, \quad i = 1, 2, \tag{14}$$

and, according to Newton's third axiom

$$\mathbf{F}_1 = -\mathbf{F}_2. \tag{15}$$

The task consists of determination of forces $\mathbf{F}_i$, as functions of distance $\rho(t)$. The derivative of second order of the vector function (13) is

$$\ddot{\boldsymbol{\rho}} = \ddot{\mathbf{r}}_2 - \ddot{\mathbf{r}}_1 = (\ddot{\rho} - \rho\dot{\theta}^2)\boldsymbol{\rho}_0 + (\rho\ddot{\theta} + 2\dot{\rho}\dot{\theta})\mathbf{n}_0, \tag{16}$$

where θ is the angle between the vector $\boldsymbol{\rho}$ and some fixed direction; $\mathbf{n}_0 \perp \boldsymbol{\rho}_0$. Substituting the derivatives $\ddot{\mathbf{r}}_1$ and $\ddot{\mathbf{r}}_2$ from equation (14) into (16), according to (15), we obtain

$$\frac{\mathbf{F}_2}{m_2} + \frac{\mathbf{F}_2}{m_1} = \ddot{\boldsymbol{\rho}},$$

or

$$\mathbf{F}_2 = -\mathbf{F}_1 = \frac{m_1 m_2}{m_1 + m_2}\ddot{\boldsymbol{\rho}}, \tag{17}$$

and further

$$\frac{m_1+m_2}{m_1m_2}\mathbf{F}_2 = (\ddot{\rho}-\rho\dot{\theta}^2)\boldsymbol{\rho}_0 + (\rho\ddot{\theta}+2\dot{\rho}\dot{\theta})\mathbf{n}_0. \tag{18}$$

By the scalar multiplication of relation (17) with the vector $\boldsymbol{\rho}_0$, we get

$$F_\rho = \mathcal{M}(\ddot{\rho}-\rho\dot{\theta}^2), \tag{19}$$

where

$$\mathcal{M} = \frac{m_1m_2}{m_1+m_2}.$$

If we take into consideration that $v_{or}^2 = \dot{\rho}^2+\rho^2\dot{\theta}^2$, the formula (19) can be written in the form

$$F = \chi\frac{m_1m_2}{\rho}, \tag{20}$$

where

$$\chi = \frac{\dot{\rho}^2+\rho\ddot{\rho}-v_{or}^2}{m_1+m_2}.$$

In order to make the generalization of these formulas more clear we present next examples.

1. Two bodies (as material points) are moving at the line $z=0, y=0$, i.e. at the axe x. The distance between the bodies changes according to

$$\rho = x_2-x_1 = l+c\sin\omega t.$$

For this example we have $\dot{\theta}=0$ in formula (19), and

$$\ddot{\rho} = -\omega^2c\sin\omega t = -\omega^2(\rho-l) = -\omega^2(x_2-x_1-l),$$

hence

$$F = -\frac{m_1m_2}{m_1-m_2}\omega^2(\rho-l).$$

2. The bodies of the masses m_1 and m_2 move with respect to each other at the constant distance $\rho=R=\text{const}$. Based on formulas (19) and (20), it follows that

$$F = -\mathcal{M}\frac{v_{0r}^2}{R} = -\mathcal{M}\frac{R^2\dot{\theta}^2}{R} = -\mathcal{M}R\dot{\theta}^2 = -\mathcal{M}\frac{4\pi^2}{T}R = -\mathcal{M}\frac{4\pi^2R^3}{T^2R^2} = -f\frac{m_1m_2}{R^2},$$

where

$$f = \frac{4\pi^2R^3}{(m_1+m_2)T^2}.$$

3. Let us consider

$$\text{if } \rho = ct+R,\ \theta=\text{const},\ \text{then } F=0; \tag{21}$$

$$\text{if } \rho = -gt^2,\ \dot{\theta}=t^{-1},\ \text{then } F=\mathcal{M}g; \tag{22}$$

$$\text{if } \rho^3 = c_1t^2,\ \dot{\theta}=c_2t^{-1},\ \text{then } c_1,c_2=\text{const}. \tag{23}$$

From the condition (23) it follows that the size of the requested force is inversely proportional to the squared distance ρ^2, namely

$$F = \mathcal{M}\frac{C}{\rho^2}. \tag{24}$$

4. Let $\rho(t)$ changes according to the law

$$\rho = a - c\cos\omega t, \quad \omega = \dot{\theta} = \text{const}$$

In this case

$$\ddot{\rho} = c\omega^2 \cos\omega t,$$

then

$$F = -\mathcal{M}(2\rho - a_{)}. \tag{25}$$

As much as there is a resemblance of examples (21) and (22), that much examples (24) and (25) are alike Kepler's motion and Newton's theorems of gravitation. Let's dedicate more attention to this question. Majority of scientists agree upon that Isaac Newton derived "law of general gravitation", based on Kepler's laws. At the first instance our formula (20) or (19) differs much from formula (18). However, for the various conditions of change of distance from formula (19) or (20), the different formulas of forces (21)-(25) follow. Formula (12) is obtained as the consequence of formulas (19) only from the conditions of Kepler's laws. In order to prove it, let's write Kepler's laws on motion of planets around the Sun, using the mathematical relations:

$$\rho(t) = \frac{p}{1 + e\cos\theta(t)}, \quad p = \frac{b^2}{a}, \quad a, b, e = \text{const};$$

$$\rho^2\dot{\theta} = C, \quad C = \frac{2\pi ab}{T} = \text{const}; \tag{26}$$

$$a^3 = kT^2, k = \text{const};$$

where p is the parameter of the elliptic trajectory, e is the excentricity, $e \leq 1$, a is the large semi-axis of the ellipse, T is time of rotation of planet around the Sun. According to the law (26), it is obtained

$$\ddot{\rho} = \frac{C}{p}\dot{\theta}e\cos\theta = \frac{C^2(p-\rho)}{p\rho^3}.$$

The substitution of these derivatives $\ddot{\rho}$ and $\dot{\theta}$ in the formula (18) gives the "Newton law of gravitation"in the classical form

$$F = f\frac{m_1 m_2}{\rho^2}, \tag{27}$$

where, as it is known,

$$f = \frac{4\pi^2 a^3}{(m_1 + m_2)T^2}.$$

So, in the fact, formula (27) appears as the consequence of formula (18), with the precision of Kepler's laws for the Sun planetary system.

Our approach to this problem, as it is shown, is not only formal, but it has significant consequences. It is known that, as per standard formula: the force that the Sun is attracting the Moon is bigger than the force that the Earth is attracting the Moon. According to our formula (18) or (20), that paradox is removed, the Earth attracting force is multi times bigger than the force the Sun is attracting the Moon.

3. Forces on configurations manifolds. Let us observe N material points of masses m_ν $(\nu = 1, \ldots, N)$. With respect to arbitrarily chosen pole O and orthonormal coordinate system $(y, \mathbf{e})$, the position of the ν-th point will be determined by the vector $\mathbf{r}_\nu = y^i_\nu \mathbf{e}_i$. Let the motion of the point be limited by $k \leq 3N$ of the bilateral constraints, or the programme, which can be represented by means of the independent equations:

$$f_\mu(\mathbf{r}_1, \dots, \mathbf{r}_N, \tau(t)) = 0, \quad \mu = 1, \dots, k,$$

or

$$f_\mu(y^1_1, y^2_1, y^3_1; \dots; y^1_N, y^2_N, y^3_N, \tau(t)) = 0,$$

that is,

$$f_\mu(y^1, \dots, y^{3N}, y^0) = 0, \quad y^0 = \tau(t). \tag{28}$$

Functions f_μ are ideally smooth and regular in the area constraining the material points. The condition for the constraints' independence is, in the simplest way, reflected in the velocity conditions on the constraints:

$$\dot{f}_\mu = \frac{\partial f_\mu}{\partial y^i}\dot{y}^i + \frac{\partial f_\mu}{\partial y^0}\dot{y}^0 = 0. \tag{29}$$

These equations will be written in the following form:

$$\frac{\partial f_\mu}{\partial y^1}\dot{y}^1 + \dots + \frac{\partial f_\mu}{\partial y^k}\dot{y}^k = -\big(\frac{\partial f_\mu}{\partial y^{k+1}}\dot{y}^{k+1} + \dots + \frac{\partial f_\mu}{\partial y^{3N}}\dot{y}^{3N} + \frac{\partial f_\mu}{\partial y^0}\dot{y}^0\big).$$

From this system, linear with respect to velocities $\dot{y}$, it is possible to determine k velocities $\dot{y}^1, \dots, \dot{y}^k$ by means of remaining $3N - k + 1$ velocities $\dot{y}^{k+1}, \dots, \dot{y}^{3N}$ under the condition that the determinant is

$$\left\| \frac{\partial f_\mu}{\partial y^m} \right\|^k_k \neq 0 \quad (\mu, m = 1, \dots, k).$$

A multitude of ways, or, briefly, a manifold choice of sets of coordinates q^α, by means of which the position or configuration of the system's points in a moment of time is determined, suggests that a set of independent coordinates $q = (q^0, q^1, \dots, q^n) \in M^{n+1}$ should be called configurational manifolds. Accordingly, a set of coordinates q and velocity $\dot{q} = (\dot{q}^0, \dot{q}^1, \dots, \dot{q}^n)^T$ should be called tangential manifolds TM^{n+1}. The pencil of all the velocities vectors at the point q will consequently be denoted as $\mathbf{T}_q M^{n+1}$ which implies $n+1$ base vectors $\dfrac{\partial \mathbf{r}}{\partial q^\alpha}$ at each point upon manifolds M^{n+1}. Hence, we will further consider two sets, namely M^{n+1} and TM^{n+1}, as well as the pencil $\mathbf{T}_q M^{n+1}$ of the linear vectors. For the sake of brevity, the following notations are introduced [5]:

$$\mathcal{N} = M^{n+1}, \ M = M^n,$$

$$T\mathcal{N} = TM^{n+1}, \ TM = TM^n.$$

Considering this condition as well as the above-stated properties of functions f_μ, it is possible, according to the implicit functions theory, to determine from equations (4.1) k dependent coordinates $y^1, \dots, y^k$ by means of remaining $3N - k + 1$ coordinates $y^{k+1}, \dots, y^{3N}, y^0$. The choice of dependent and independent coordinates is arbitrary, along with a special choice of q^0 coordinates, so that each of coordinates $y^1, \dots, y^{3N}$ can itself be expressed as function $3N - k + 1$ of coordinates y. Since, as needed, constraints (28) can be expressed in curvilinear coordinate systems, the possibility of selecting independent coordinates is enlarged. If the independent generalized coordinates are denoted by letters $q^0, q^1, \ldots, q^n$, it follows that constraints (28) can be written down in the parametric form:

$$y^i = y^i(q^0, q^1, \ldots, q^n), \quad q^0 = \tau(t),$$

and also as

$$\mathbf{r}_\nu = \mathbf{r}_\nu(q^0, q^1, \ldots, q^n).$$

Velocity conditions (29) are substituted by relations

$$\mathbf{v}_\nu = \frac{\partial \mathbf{r}_\nu}{\partial q^0}\dot{q}^0 + \frac{\partial \mathbf{r}_\nu}{\partial q^1}\dot{q}^1 + \cdots + \frac{\partial \mathbf{r}_\nu}{\partial q^n}\dot{q}^n = \frac{\partial \mathbf{r}_\nu}{\partial q^\alpha}\dot{q}^\alpha,$$

These differential equations (see [4] and [5]) amount to $n+1$, that is

$$\frac{d}{dt}\frac{\partial E_k}{\partial \dot{q}^i} - \frac{\partial E_k}{\partial q^i} = Q_i, \quad i = 1, \ldots, n,$$

$$\frac{d}{dt}\frac{\partial E_k}{\partial \dot{q}^0} - \frac{\partial E_k}{\partial q^0} = Q_0 = Q_0^* + R_0, \tag{30}$$

where

$$Q_\alpha = Y_i \frac{\partial y^i}{\partial q^\alpha}, \quad i = 0, 1, \ldots, 3N, \quad \alpha = 0, 1, \ldots, n.$$

are *generalized forces*, are reduced to the differential equations in the extended form since it is clear that the kinetic energy is easy to set up with $a_{\alpha\beta}$ known inertia tensor.

For the systems with invariable constraints and with such potential energy E_p that the active forces are

$$Q_i = -\frac{\partial E_p}{\partial q^i}, \quad i = 1, \ldots, n,$$

equation (30) does not exist.

Motion on $T^*\mathcal{N}$. The notation $T^*\mathcal{N}$ here implies $2n+2$ dimensional manifolds which form $n+1$ generalized coordinates $q = (q^0, q^1, \ldots, q^n)$ and $n+1$ generalized impulses $p = (p_0, p_1, \ldots, p_n)$. Regarding the fact that $q \& \dot{q} \in T\mathcal{N}$ denotes the tangent manifolds, then the symbol $T^*\mathcal{N}$ is called the cotangent manifolds. In the literature other terms can be sometimes found such as "phase space", "state space", "Hamilton's variables", or "cotangential spaces". If the starting point is the fact that the motion state is characterized by the position coordinates of point q as well as the coordinates of impulse p, then it could be said that $T^*\mathcal{N}$ is the state of the system's motion or state manifolds. Since $\mathcal{N} = M^{n+1}$, $T^*\mathcal{N}$ can also be called the extended manifolds if it is necessary to stress its difference from configurational manifolds M^n and its respective cotangent manifolds T^*M [4].

What is even more important than the term itself is the understanding and acceptance that $p_0, p_1, \ldots, p_n$ are the impulses whose essence is determined by a mutually linear combination between generalized impulses p_α and generalized velocities $\dot{q}^\alpha$:

$$p_\alpha = a_{\alpha\beta}\dot{q}^\beta \quad \Leftrightarrow \quad \dot{q}^\alpha = a^{\alpha\beta}p_\beta.$$

The next step in considering the action principle upon $T^*\mathcal{N}$ implies the substitution of velocities $\dot{q}^\alpha$ in the above-discussed relations by means of generalized impulses p_β, because the kinetic energy has the following forms:

$$E_k = \frac{1}{2}a_{\alpha\beta}\dot{q}^\alpha\dot{q}^\beta = \frac{1}{2}p_\beta\dot{q}^\beta = \frac{1}{2}a^{\beta\gamma}p_\beta p_\gamma,$$

where

$$H = E_k + E_p = \frac{1}{2}a^{\beta\gamma}p_\beta p_\gamma + E_p(q). \tag{31}$$

If the generalized forces Q_α are separated into potential and non-potential P_α, then

$$Q_\alpha = -\frac{\partial E_p}{\partial q^\alpha} + P_\alpha.$$

It can be seen from formula $H = E_k + Ep$, that it is

$$\frac{\partial H}{\partial p_\alpha} = a^{\alpha\beta}p_\alpha,$$

so that, due to linear combinations (31),

$$\dot{q}^\alpha = \frac{\partial H}{\partial p_a}.$$

From here it follows

$$\dot{p}_\alpha = -\frac{\partial H}{\partial q^\alpha} + P_\alpha, \quad (\alpha = 0, 1, \ldots, n),$$

these are differential equations of the system's motion, and, along with transformations (30), form the system of $2n + 2$ differential equations:

$$\dot{p}_i = -\frac{\partial H}{\partial q^i} + P_i, \quad \dot{q}^i = \frac{\partial H}{\partial p_i}, \tag{32}$$

$$\dot{p}_0 = -\frac{\partial H}{\partial q^0} + P_0, \quad \dot{q}^0 = \frac{\partial H}{\partial p_0}. \tag{33}$$

In the case, [4] or [5], that $P_i = 0$ and $P_0^* = -\dfrac{\partial \mathcal{P}}{\partial q^0} = 0$, the function H can be extended to the total mechanical energy

$$E = H + \mathcal{P},$$

so that the system of differential equations of motion can be written in the canonical form:

$$\dot{p}_\alpha = -\frac{\partial E}{\partial q^\alpha}, \quad \dot{q}^\alpha = \frac{\partial E}{\partial p_\alpha}, \quad \alpha = 0, 1, \ldots, n.$$

In the case of the system's invariable constraints, when there is no rheonomic coordinate q^0, equations (33) vanish, while in equations (32) indices range from 1 to n.

The equations above are usually used for describing motion and motion control. But, as we said in the comment of (4) and (5), the generalized forces are essential for motion control. See, for example, the motion of a material point of mass m along cyclic trajectory $x^2 + y^2 = r^2, z = 0$ with velocity $\dot{x}^2 + \dot{y}^2 = v^2 = const.$

In this example, the system of differential equations of motion in E^3 is

$$m\ddot{x} = X, \qquad m\ddot{y} = Y, \qquad 0 = Z.$$

It follows

$$F = \left(Y^2 + Y^2\right)^{1/2} = -m\frac{v^2}{r} = -m\frac{4\pi^2}{T^2}r = -\frac{4\pi^2 r^3}{T^2}\frac{m}{r^2} = -\frac{4\pi^2 r^3}{MT^2}\frac{mM}{r^2}. \tag{34}$$

With respect to the curvilinear cylindric coordinate system r, θ for previous condition would be obtained $r = \text{const}, \quad z = 0; \quad v = r\dot{\theta} = \text{const}$ and manifold $\theta \in M^1$. On this manifold M^1 only one differential equation exists

$$\frac{d}{dt}\frac{\partial E_k}{\partial \dot{\theta}} - \frac{\partial E_k}{\partial \theta} = Q_\theta,$$

where is $2E_k = m(r\dot{\theta})^2$. Hence $Q_\theta = 0$. So, if one does not know the manifolds (in question) one could conclude that there are no forces, contradicting to (34).

1. *Харламов П.В.* Очерки об основаниях механики. – Киев: Наукова думка, 1995. – 406 с.
2. *Ньютон Ис.* Математические начала натуральной философии. – М.: Наука, 1989. – 631 с.
3. *Ševarlić B. and Brkić* Opšta astronomija. – Beograd: Naučna knjiga, 1981. – 598 p.
4. *Мартынюк А.А., Вуйичич В.А.* Некоторые задачи механики неавтономных систем. – Белград-Киев, 1991. – 108 с.
5. *Vujičić V.A.* Preprinciples of mechanics. Matematički institut SANU, Beograd, 1999. – 225 p.

Matematički institut SANU, Beograd, Yugoslavia
vvujicic@turing.mi.sanu.ac.yu

Получено 25.06.04