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FREQUENCY EQUATIONS OF SMALL OSCILLATIONS MIXED SYSTEMS OF THE COUPLED DISCRETE AND CONTINUOUS SUBSYSTEMS

In this paper, by using examples of mixed systems of the coupled discrete subsystem of rigid bodies and continuous subsystem, the method of obtaining of small oscillations frequency equations are presented. Small oscillations frequency equations of coupled deformable body and holonomic conservative systems are obtained. By using numerical experiment connections between own small oscillations circular frequencies of the mixed system and subsystem of the rigid bodies and deformable body are studied and analyzed. By using MathCAD program graphical presentations of a set of small oscillations circular frequencies of the deformable body with "perturbations"caused on interaction of subsystem small oscillations of rigid bodies. By using examples, analogy between frequency equations of some classes of these systems is identified. Special cases of discretization and continualization of coupled subsystems with corresponding sets of proper circular frequencies and frequency equations of small oscillations are analyzed.

Introduction. In many classical textbooks on Theory of Oscillations [1] we can find many examples of classical tasks of frequency equations of discrete or continuous system oscillations, which are excited with initial perturbations of natural equilibrium state. In the long time period, as a professor of Elastodynamics and Theory of Oscillations at Faculty of Mechanical Engineering, I wrote many original examine tasks and corresponding solutions of these tasks. In the teaching process I must show to students rational explanations of some solutions, and properties of oscillatory processes of system dynamics. By introducing some of mine assistants to the teaching process I discuss possibilities for different solutions of equations of oscillatory systems dynamics and small transformations of the examined tasks definitions to compose new tasks but with same solution philosophy. Today, by using computer tools as MathCad, Mathematica, MathLab, a new powerful possibilities for visualizations of oscillatory processes in dynamical systems applied in engineering practice are very useful for university teaching of oscillations theory as an accompanying tools to the analytical method and pure mathematical explanations. By using these WEB and MathCad informational tools some presentations of examine tasks of Elastodynamics and Theory of Oscillations in the teaching process and study at Faculty of Mechanical Engineering is presented at www. masfak.ni.ac.yu/Elastodynamika [2]. Some of these original tasks with solutions were published in the three books [3–5].

Current research in theory of discrete dynamical system oscillations is directed to nonlinear phenomena [6], as well as to nonstationary processes, and also to stochasticlake and chaoticlake processes in purely deterministic dynamical systems and conditions. Nonlinear phenomena in oscillation theory of continuous systems, damage and fracture structure of dynamical systems are topics of premier journals and international scientific conferences (Issues of Journals: Applied Mechanics Reviews; Refferativniy Zhurnal Mechanika Moscow; Proceedings of conferences: ENOC (Copenhagen 1999, Moscow 2002); ICNM Shanghai 1998, 2002; Control Oscillations and Chaos COC 2000 Saint-Petersburg); IUTAM Symposium, Roma 2003. Pure elastic systems are now not in focus of researchers [7].

New materials such as new construction materials of structure in engineering systems are inspirations of many researchers for new constitutive relations for discoveries in mathematical sense and for investigations of dynamics of these constructions. In some author's papers dynamics of discrete systems of material particles which are constrained by standard hereditary, rheological, or creep light elements [8-12] are investigated. These papers inspired by papers of Goroshko O.A. and all [13].

In the monograph [14] of Goroshko O.A. and Hedrih (Stevanović) K. analytical dynamics of discrete hereditary systems and corresponding solutions are first presented as an integral theory of these kind systems.

A new material used in structure of active systems is piezoceramics. In the papers [15] and [16] piezoceramics behaviour in the vibrations regimes is presented as a results of the analytical, numerical and experimental investigations of the vibrations frequency spectra. These results are important for investigations of active structure oscillations and control of oscillations.

In papers [17] and [18] longitudinal hereditary vibrations and creep vibrations of a fractional derivative rheological rod with variable cross section are examined. Partial differential equation and particular solutions for the case of natural creep longitudinal vibrations of the rod of creep material with a fractional derivative order is accomplished. For the case of natural creep vibrations eigenfunction and time-function for different examples of boundary conditions are determined. Different boundary conditions are analysed and series of eigenvalues and natural circular frequencies of longitudinal creep vibrations as well as tables of these values are completed. By using MathCad a graphical presentation of the time-function is present.

In the papers [19–21] and [22], the problem on transversal oscillations of bar which is free or under the action of the length-wise random forces is considered.

In the paper [22], the problem on transversal oscillations of two layer straight bar, which is under the action of the length-wise random forces is considered. The excitation processes is a bounded noise excitation. It is assumed, that the layers of the bar were made of creep continuously non homogenous material and the corresponding modulus of elasticity and creep fractional derivative order constitutive relation of each layer are continuous function of length coordinate and thickness coordinates. The equation of the transversal creep vibrations of a fractional derivative order constitutive relation beam are examined. Partial fractional-differential equation and particular solutions for the case of natural creep vibrations of the beam of creep material of a fractional derivative order constitutive relation in the case of the influence of rotation inertia is derived. For the case of natural creep vibrations, eigenfunction and time-function, for different examples of boundary conditions are determined.

The paper [23] presents the discrete continuum method on examples of homogenous discrete systems with limited number of degrees of motion freedom dynamics. These systems are in the form of homogenous chains and nets in space and plain. Material points of these nets and chains are tied by elastic, standard hereditary or creep elements. By introducing the trigonometric method for studying properties and equations of dynamics of discrete homogenous continuums author sets up the discrete continuum method for the study of dynamics of chain systems with hereditary or creeping connections. These systems dynamics is described by a system of integro-differential equations or differential equations with fractional derivatives. A light standard creep element is defined by a constitutive relation of stress-strain state, for the creation of which fractional order derivatives were used.

In this paper we use keywords: Discrete continuum, discrete hereditary system, discrete homogenous chain, discrete homogenous material net, elastic element, standard hereditary light element, standard creep light element, integro-differential relation, fractional derivatives order, Jules-Lissajous figure, trigonometric method, small vibrations. We can see interaction between notions of words discrete continuum and continuous of disrete systems. It was inspiration for me to turn my attention to the mixed systems of the coupled discrete subsystem of rigid bodies and continuous subsystem and to compose characteristic – frequency equations of the small oscillations of these systems.

Papers [24] and [25] are also directed to examinations of the classical knowledge on continuous and discrete systems to make some new conclusions. Visualization of oscillatory processes and new presentation of their properties in classical oscillatory models of real systems are given in [26, 27].

This work is one new addition to the knowledge of the mixed systems of the coupled discrete subsystem of rigid bodies and continuous subsystem to compose characteristic – frequency equations of the small oscillations.

We can conclude that new computer tools with power possibilities directed philosophy of considerations of real systems dynamics by using discretization of continuum as the way and method for solutions of problems, and by using many iterations continualizations of solutions. Discretizations and continualizations in the process of solutions and analysis of dynamical processes are opposite directions and good method for proving calculations and conclusions.

In accordance with close specializations of researchers we can find a few examples of mixed systems which consists from coupled discrete and continuous systems. And not very often there are some analytical results. In the epoque of the large numerical experiments over the dynamical systems I think that is very important to make some new classical examples of frequency equations useful for teaching process in Theory of Vibrations.

1. Model of Mixed Systems of Coupled Discrete Subsystem of Rigid Bodies and Continuous Subsystem. We consider two subsystems: one elastic and two rigid rods. The elastic rod has the straight axis and represents a continuous system – solid deformable body with the following parameters: E, ρ, ℓ, A. Two rigid rods have weights at free ends with masses m_p and m_0 (see Fig. 1). This system is constrained by spring with rigidity c_0 and coupled with discrete systems with n degree of freedom. For example, such discrete subsystem is chain system of the n material particles with masses m_i, $i = 1, 2, 3, ..., n$, translator movable along line parallel to the rod axis ; these masses are connected by springs with rigidities c_i, $i = 1, 2, 3, ..., n$. We consider connections between longitudinal vibrations of the elastic rod and free oscillations of the chain material particles system. Let us to determine frequency equations of the defined mixed system of the coupled discrete subsystem of rigid bodies and continuous subsystem.

1.1. Differential Equation of the Longitudinal Oscillations of the Elastic Rod and Boundary Conditions. In accordance with notations (see Fig. 1), we can see that $u(x,t)$ longitudinal displacement of the rod cross section at the distance x measured from left rod end in the axis direction at the time t Partial differential equation of the longitudinal oscillations is

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c_e^2 \frac{\partial^2 u(x,t)}{\partial x^2}, \tag{1}$$

where $c_e^2 = \dfrac{\mathbf{E}}{\rho}$.

Solution of equation (1) is in the following form:

$$u(x,t) = \mathbf{X}(x)\mathbf{T}(t), \tag{2}$$

where are

$$\mathbf{X}(x) = C_1 \cos\lambda x + C_2 \sin\lambda x, \qquad \mathbf{T}(t) = A\cos\omega t + B\sin\omega t.$$

By using boundary conditions of the subsystem of the longitudinal rod oscillations [1] as well as compatibility conditions of the displacements and forces as ineractions of the coupled subsystems we can write:

$$\left[m_P \frac{\partial^2 u(x,t)}{\partial t^2}\right]\Big|_{x=0} = \left[\mathbf{EA}\frac{\partial u(x,t)}{\partial x}\right]\Big|_{x=0},$$

$$\left[m_P \frac{\partial^2 u(x,t)}{\partial t^2}\right]\Big|_{x=\ell} = \left[-\mathbf{EA}\frac{\partial u(x,t)}{\partial x}\right]\Big|_{x=\ell} + \mathbf{F}(t), \qquad u(\ell,t) = x_0(t).$$

Let us to introduce the following notations:

$$\mu_P = \frac{m_P}{\rho\mathbf{A}\ell}, \qquad \mu_0 = \frac{m_0}{\rho\mathbf{A}\ell}, \qquad \xi = \lambda\ell, \qquad \omega^2 = \lambda^2\frac{\mathbf{E}}{\rho} = \frac{\xi^2}{\ell^2}\frac{\mathbf{E}}{\rho} = \xi^2\omega_0^2, \qquad \omega_0^2 = \frac{\mathbf{E}}{\rho\ell^2},$$

$$\frac{d\mathbf{X}(x)}{dx} = \lambda\frac{d\mathbf{X}(\xi)}{d\xi} = \frac{\xi}{\ell}\frac{d\mathbf{X}(\xi)}{d\xi}, \qquad c_e = \frac{\mathbf{EA}}{\ell}, \qquad \kappa = \frac{c_0}{c_e}, \qquad u_0 = \frac{m_0\omega_0^2}{c_0}.$$

By introducing the proposed solutions (2) into boundary conditions and conditions of the compatibility displacement and forces we can write

$$\mu_P\xi^2\mathbf{X}(0) + \xi\mathbf{X}'(0) = 0, \qquad (\mu_0\xi^2 - \kappa)X(\xi) - \xi\mathbf{X}'(\xi) + \kappa A_1 = 0.$$

By using the relation: $\mathbf{X}(\xi) = C_1\cos\xi + C_2\sin\xi$ and its derivative with respect to the argument $\xi : \mathbf{X}'(\xi) = -C_1\sin\xi + C_2\cos\xi$, from previous equations we can obtain

$$\mu_P\xi^2 C_1 + \xi C_2 = 0,$$

$$C_1[(\mu_0\xi^2 - \kappa)\cos\xi + \xi\sin\xi] + C_2[(\mu_0\xi^2 - \kappa)\sin\xi - \xi\cos\xi] = -\kappa A_1.$$

Determinant of this previous systems of algebraic equations with respect to the C_1, C_2 is

$$\Delta(\xi) = \begin{vmatrix} \mu_P\xi^2 & \xi \\ [(\mu_0\xi^2 - \kappa)\cos\xi + \xi\sin\xi] & [(\mu_0\xi^2 - \kappa)\sin\xi - \xi\cos\xi] \end{vmatrix} =$$

$$= \xi\{\mu_P\xi[(\mu_0\xi^2 - \kappa)\sin\xi - \xi\cos\xi] - [(\mu_0\xi^2 - \kappa)\cos\xi + \xi\sin\xi]\} \tag{3}$$

and these coefficients we can express by following expressions:

$$C_1 = \frac{\kappa A_1\xi}{\xi\{\mu_P\xi[(\mu_0\xi^2 - \kappa)\sin\xi - \xi\cos\xi] - [(\mu_0\xi^2 - \kappa)\cos\xi + \xi\sin\xi]\}},$$

$$C_2 = -\frac{\kappa\mu_P\xi^2 A_1}{\xi\{\mu_P\xi[(\mu_0\xi^2 - \kappa)\sin\xi - \xi\cos\xi] - [(\mu_0\xi^2 - \kappa)\cos\xi + \xi\sin\xi]\}}.$$

1.2. Differential Equations of the Material Particles Discrete System with Boundary Condition. Now, we consider subsystem of discrete material particles with n degree of freedom and we choose n generalized coordinates x_i, $i = 1,2,3,...,n$, and corresponding matrix $\mathbf{A}$ of inertia coefficients and matrix $\mathbf{C}$ quazielastic coefficients

$$\mathbf{A} = (a_{ij}), \quad i,j = 1,2,3,...,n; \qquad C = (c_{ij}), \quad i,j = 1,2,3,...,n.$$

System of the differential equations of the discrete subsystem with boundary condition is

$$\mathbf{A}\{\ddot{x}\} + \mathbf{C}\{x\} = -c_0(x_1 - x_0)\mathbf{I}_0\{I\}, \tag{4}$$

where

$$\{x\} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{Bmatrix}, \qquad \mathbf{I}_0 = \begin{pmatrix} 1 & & & & & \\ & 0 & & & & \\ & & 0 & & & \\ & & & 0 & & \\ & & & & 0 & \\ & & & & & 0 \end{pmatrix}, \qquad \{I\} = \begin{Bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{Bmatrix}. \tag{5}$$

Solution of the previous system (5) is assumed in the following form:

$$\{x\} = \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \\ \vdots \\ A_{n-1} \\ A_n \end{Bmatrix} \mathbf{T}(t) = \{A\}\mathbf{T}(t), \qquad \{\ddot{x}\} = -\omega^2\{A\}\mathbf{T}(t) = -\xi^2\omega_0^2\{A\}\mathbf{T}(t).$$

1.3. Frequency Equation of the Coupled Longitudinal Oscillations of the Elastic Rod and Discrete System of the Material Particles. Taking into consideration the relation

$$u(\ell,t) = x_0(t) = \mathbf{X}(\ell)\mathbf{T}(t) = (C_1\cos\xi + C_2\sin\xi)\mathbf{T}(t) = \frac{\kappa A_1\xi}{\Delta(\xi)}(\cos\xi - \mu_P\xi\sin\xi)\mathbf{T}(t)$$

and desighations $\frac{1}{c_0}\mathbf{C} = \overline{\mathbf{C}}$, $\frac{1}{m_0}\mathbf{A} = \overline{\mathbf{A}}$ from the system of differential equations in matrix form (4) we can obtain following matrix equation:

$$\left(\overline{\mathbf{C}} - \xi^2 u_0\overline{\mathbf{A}} + \left[1 - \frac{\kappa\xi}{\Delta(\xi)}(\cos\xi - \mu_P\xi\sin\xi)\right]\mathbf{I_0}\right)\{A\} = \{0\}.$$

The previous matrix equations are algebraic homogeneous equations and for its nontrivial solutions it is necessary that determinant of this system is equal to zero. From this condition we can obtain the following characteristic frequency transcendent equation:

$$\left|\overline{\mathbf{C}} - \xi^2 u_0\overline{\mathbf{A}} + \left[1 - \frac{\kappa\xi}{\Delta(\xi)}(\cos\xi - \mu_P\xi\sin\xi)\right]\mathbf{I_0}\right| = 0. \tag{6}$$

This is main result of this consideration of the mixed system of coupled subsystems with free oscillations. We can see that this equation consists of two parts: one part is expression of the frequency equations of the discrete system oscillations, and second part of the deformable body frequency equation is connected by one member with previous.

1.4. Special Cases of the Frequency Equation of the Coupled Longitudinal Oscillations of the Elastic Rod and Discrete System of the Material Particles. The frequency equation (6) can be presented in the following form

$$\begin{vmatrix} \begin{matrix} k_{11}-\xi^2u_0\mu_{11}+\Big[1- \\ -\frac{\kappa\xi}{\Delta(\xi)}(\cos\xi-\mu_P\xi\sin\xi)\Big] \end{matrix} & k_{12}-\xi^2u_0\mu_{12} & k_{13}-\xi^2u_0\mu_{13} & \dots & k_{1n}-\xi^2u_0\mu_{1n} \\ k_{21}-\xi^2u_0\mu_{21} & k_{22}-\xi^2u_0\mu_{22} & k_{23}-\xi^2u_0\mu_{23} & \dots & k_{2n}-\xi^2u_0\mu_{2n} \\ k_{31}-\xi^2u_0\mu_{31} & k_{32}-\xi^2u_0\mu_{32} & k_{33}-\xi^2u_0\mu_{33} & \dots & k_{3n}-\xi^2u_0\mu_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ k_{n1}-\xi^2u_0\mu_{n1} & k_{2n}-\xi^2u_0\mu_{2n} & k_{3n}-\xi^2u_0\mu_{3n} & \dots & k_{nn}-\xi^2u_0\mu_{nn1} \end{vmatrix}=0.$$

For the case of the coupled elastic rod longitudinal oscillations and chain discrete material particles system oscillations previous frequency equations take the following form

$$\begin{vmatrix} \begin{matrix} k_1-\xi^2u_0\mu_1+\Big[1- \\ -\frac{\kappa\xi}{\Delta(\xi)}(\cos\xi-\mu_P\xi\sin\xi)\Big] \end{matrix} & -k_1 & & \dots & \\ -k_1 & k_1+k_2-\xi^2u_0\mu_2 & -k_2 & \dots & \\ & -k_2 & k_2+k_3-\xi^2u_0\mu_3 & \dots & \\ & & \dots & \dots & \\ & & & \dots & -k_{n-1} \\ & & & \dots & k_{n-1}+k_n-\xi^2u_0\mu_n \end{vmatrix}=0.$$

For the case that elastic rod is connected with one material particle with two springs we obtain

$$k_1-\xi^2u_0\mu_1+\Big[1-\frac{\kappa\xi}{\Delta(\xi)}(\cos\xi-\mu_P\xi\sin\xi)\Big]=0$$

and taking into account expression (3) we can write the following :

$$\xi(k_1+1-\xi^2u_0\mu_1)\{\mu_P\xi[(\mu_0\xi^2-\kappa)\sin\xi-\xi\cos\xi]-$$

$$-[(\mu_0\xi^2-\kappa)\cos\xi+\xi\sin\xi]\}-\kappa\xi(\cos\xi-\mu_P\xi\sin\xi)=0.$$

For the case that one end of the rod is fixed – case of the cantilever rod, in the previous frequency equation we can introduce $\mu_P\to\infty$, and than we obtain

$$\mathrm{tg}\xi=\frac{\xi(k_1+1-\xi^2u_0\mu_1)}{\{\xi^2[\mu_0(k_1+1)+\kappa u_0\mu_1-\mu_0\xi^2u_0\mu_1]-k_1\kappa\}}.$$

For the case of free material particle and connected by one spring we can write:

$$\xi\mathrm{tg}\xi=\frac{1-\xi^2u_0\mu_1}{(\mu_0+\kappa u_0\mu_1-\mu_0\xi^2u_0\mu_1)}.$$

For two material particles connected to a rod as a chain, we obtain:

$$\left\{k_1 - \xi^2 u_0 \mu_1 + \left[1 - \frac{\kappa\xi}{\Delta(\xi)}(\cos\xi - \mu_P \xi \sin\xi)\right]\right\}(k_1 + k_2 - \xi^2 u_0 \mu_2) - k_1^2 = 0$$

where $\Delta(\xi)$ from (3).

For three material particles the chain frequency equation is

$$\begin{vmatrix} \begin{matrix} k_1 - \xi^2 u_0 \mu_1 + \\ + \left[1 - \frac{\kappa\xi}{\Delta(\xi)}(\cos\xi - \mu_P \xi \sin\xi)\right] \end{matrix} & -k_1 & \\ -k_1 & k_1 + k_2 - \xi^2 u_0 \mu_2 & -k_2 \\ & -k_2 & k_2 + k_3 - \xi^2 u_0 \mu_3 \end{vmatrix} = 0.$$

For the case of discrete material particles homogeneous chain, frequency equation takes the following form:

$$\begin{vmatrix} \begin{matrix} 1 - \xi^2 \tilde{u}_0 + \\ + \frac{1}{k_1}\left[1 - \frac{\kappa\xi}{\Delta(\xi)}(\cos\xi - \mu_P \xi \sin\xi)\right] \end{matrix} & -1 & & & & \\ -1 & 2 - \xi^2 \tilde{u}_0 & -1 & & & \\ & -1 & 2 - \xi^2 \tilde{u}_0 & \dots & & \\ & & \dots & \dots & \dots & \\ & & & \dots & 2 - \xi^2 \tilde{u}_0 & -1 \\ & & & & -1 & 2 - \xi^2 \tilde{u}_0 \end{vmatrix} = 0.$$

For special case of three material particles homogeneous chain, frequency equation is

$$[(2 - \xi^2 \tilde{u}_0)^2 - 1]\left\{1 - \xi^2 \tilde{u}_0 + \frac{1}{k_1}\left[1 - \frac{\kappa\xi}{\Delta(\xi)}(\cos\xi - \mu_P \xi \sin\xi)\right]\right\} - (2 - \xi^2 \tilde{u}_0) = 0.$$

2. Analogy between the Frequency Equation of the Coupled Longitudinal Oscillations of the Elastic Rod and Discrete System of the Material Particles and Coupled Torsion Oscillations of the Elastic Rod and Corresponding Discrete System of the Material Particles. By using analogy [28, 29] between two systems, and especially between longitudinal and torsional oscillations of elastic rod with circle cross section we can use previous analytical results for determining frequency equation of coupled small oscillations of mixed systems presented in Fig. 2 and 3.

In general case the mixed system consists of two subsystems: one elastic rod-shaft, whose axis is straight with parameters: $\mathbf{G}, \rho, \ell, \mathbf{A}, \mathbf{I}_0$, and with two rigid discs at the free ends with mass inertia moments with respect to the shaft axis: $\mathbf{J}_P$ and $\mathbf{J}_0$. This rod-shaft is constrained by torsion spring with rigidity c_0 and coupled with discrete systems with n degrees of freedom. For example, this discrete subsystem is mechanism in the form of chain system of n material particles (or rigid bodies) with generalized masses $m_i,\ i = 1, 2, 3, ..., n$ torsion (rotation) movable along circle line coaxial to the rod-shaft's axis; these masses are connected by torsion springs with rigidities $c_i,\ i = 1, 2, 3, ..., n$. We consider connections between torsional vibrations of elastic rod-shaft and free oscillations of generalized chain of material particles of system-mechanism (examples in Fig. 2 and 3). Let us to determine frequency equations of the defined mixed system of the coupled discrete subsystem of rigid bodies and continuous subsystem.

By using analogy $\theta(x,t) \Rightarrow u(x,t)$, $\mathbf{J}_P \Rightarrow m_P$, $\mathbf{J}_0 \Rightarrow m_0$, $x_i \Rightarrow \theta_i$, and taking into account that

$$\mu_P = \frac{\mathbf{J}_P}{\rho \mathbf{I}_0 \ell}, \qquad \mu_0 = \frac{\mathbf{J}_0}{\rho \mathbf{I}_0 \ell}, \qquad \xi = \lambda \ell, \qquad \omega^2 = \lambda^2 \frac{\mathbf{G}}{\rho} = \frac{\xi^2}{\ell^2}\frac{\mathbf{G}}{\rho} = \xi^2 \omega_0^2,$$

$$\omega_0^2 = \frac{\mathbf{E}}{\rho \ell^2}, \qquad c_t = \mathbf{G}\mathbf{I}_0/\ell, \qquad \kappa = c_0/c_t, \qquad u_0 = \mathbf{J}_0 \omega_0^2/c_0,$$

we can write the following frequency equation of the considered mixed system with torsional oscilations of the shaft and coupled mechanisms (Fig. 2):

$$\left| \overline{\mathbf{C}} - \xi^2 u_0 \overline{\mathbf{A}} + \Big[1 - \frac{\kappa \xi}{\Delta(\xi)}(\cos\xi - \mu_P \xi \sin\xi)\Big]\mathbf{I}_0 \right| = 0$$

in same form as expression (6).

For the special case of the mixed system presented in Fig. 3 with cantilever shaft, and corresponding mechanisms in accordance with presentation in Fig. $3, b^*$, or $3, c^*$ or $3, d^*$ we can write the following frequency equation:

$$\mu \xi \mathrm{tg} \xi = \frac{9\xi^2 - k^2}{56\xi^2 - 9k^2}. \tag{7}$$

When ξ is small we can take that is $\mathrm{tg}\xi \approx \xi$ and for approximation of previous frequency equation we can write

$$\mu \xi^2 (56\xi^2 - 9k^2) - 9\xi^2 + k^2 \approx 0 \Rightarrow 56\xi^4 - 9\xi^2\big(k^2 + \frac{1}{\mu}\big) + \frac{k^2}{\mu} = 0$$

and minimal value of the eigenvalues of small oscillations of the mixed system is

$$\xi_{1,2}^2 = \frac{9\big(k^2 + \frac{1}{\mu}\big) \mp \sqrt{81\big(k^2 + \dfrac{1}{\mu}\big)^2 - 224\dfrac{k^2}{\mu}}}{112}.$$

3. Numerical Experiment and Visualization of the Frequency Equation of the Coupled Torsional Oscillations of the Elastic Rod-shaft and Discrete System of the Mechanisms. For numerical experiment we take into account the special case of the mixed system presented in Fig. 3 with cantilever shaft and corresponding mechanisms in accordance with presentation in Fig. $3, b^*$, or $3, c^*$ or $3, d^*$. For that case frequency equation is in the form (7). By changing parameters k and μ of mixed system the Fig. 4 – 11 are composed for the frequency functions $f1(x) = -\mu x \mathtt{tg} x + \dfrac{9x^2 - k^2}{56x^2 - k^2}$ and corresponding parts to the influence of discrete $f3(x) = \dfrac{9x^2 - k^2}{56x^2 - k^2}$ or continuous $f2(x) = -\mu x \mathtt{tg} x$ subsystem.

4. Concluding Remarks. *From the obtained analytical and numerical results for natural longitudinal vibrations of the elastic rod coupled with material particle discrete system, it can be seen that connections are convenient for changing characteristic function depending on discrete system material parameters, and that fundamental eigen function depending on* *space coordinate is dependent on boundary conditions and geometrical properties of coupled discrete system.*

In this paper we returned to classical problems of the theory of oscillations, coupled elastic bodies and systems of discrete material points on selected examples and at the same time we determined the corresponding frequency equations. Results of numerical experiment are shown on frequency function graphs that consist out of members expressing the influence of discrete systems on frequency equations over potential functions and members expressing the influence of deformable bodies and which contain transcedent functions withing themselves. We can see from the graphs the visualizations of perturbations of frequency spectre of own circular frequencies, deformable bodies oscillations or vice versa. Similar disturbances can be seen on the frequency spectre of a discrete system but with opposite effects. We can see "the continualization of frequency spectres of discrete system"on the graph of frequency functions. At the same time we can interpret these results as discretization of the part of frequency spectre of continual system as a result of coupling with a discrete system. It should also be stated here the analogy used between these mixed systems with coupled subsystems, continuous and discrete when it is possible to establish a direct analogy between longitudinal and torsional oscillations of deformable body with annular cross-sections. That enabled an analythical analysis to be conducted for one type of system and results to be used on another type. And at the end, it should be stressed again that the goal of this paper was the solution of a classical but very concurrent task since the literature contains a very small number of examples of such a task. Methodology of continuum discretization and of continualization of discrete system which meet at border cases of study of properties of real systems.

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Appendix A. Figures

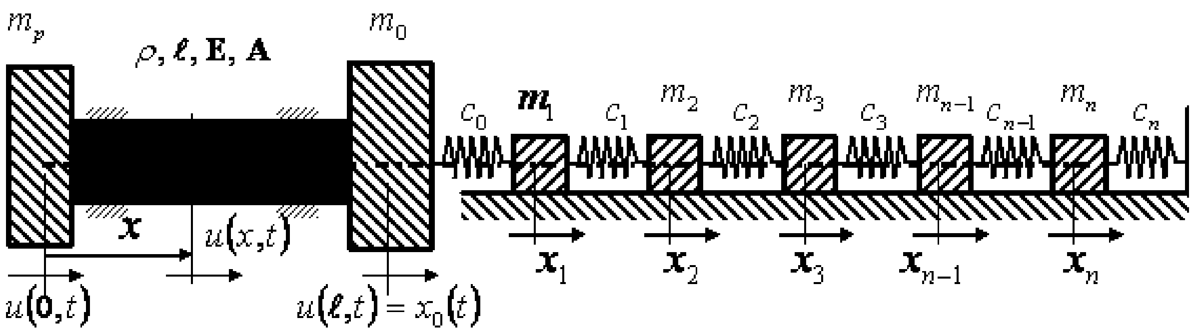

Fig.1. Small oscillations of the mixed system of coupled discrete and continuous subsystems Longitudinal oscillations of the beam with multibody chain with changeable numbers of material particles.

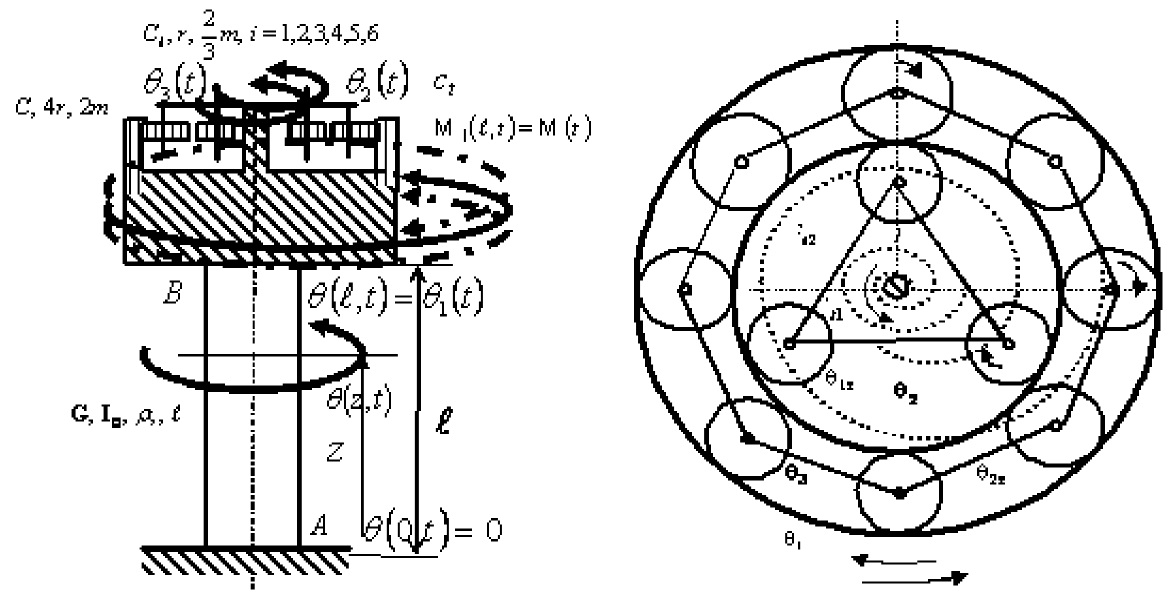

Fig.2. Small oscillations of the mixed system of the coupled discrete and continuous subsystems. Torsion oscillations of the cantilever shaft with multibody mechanisms with two chain of the changeable numbers of discs.

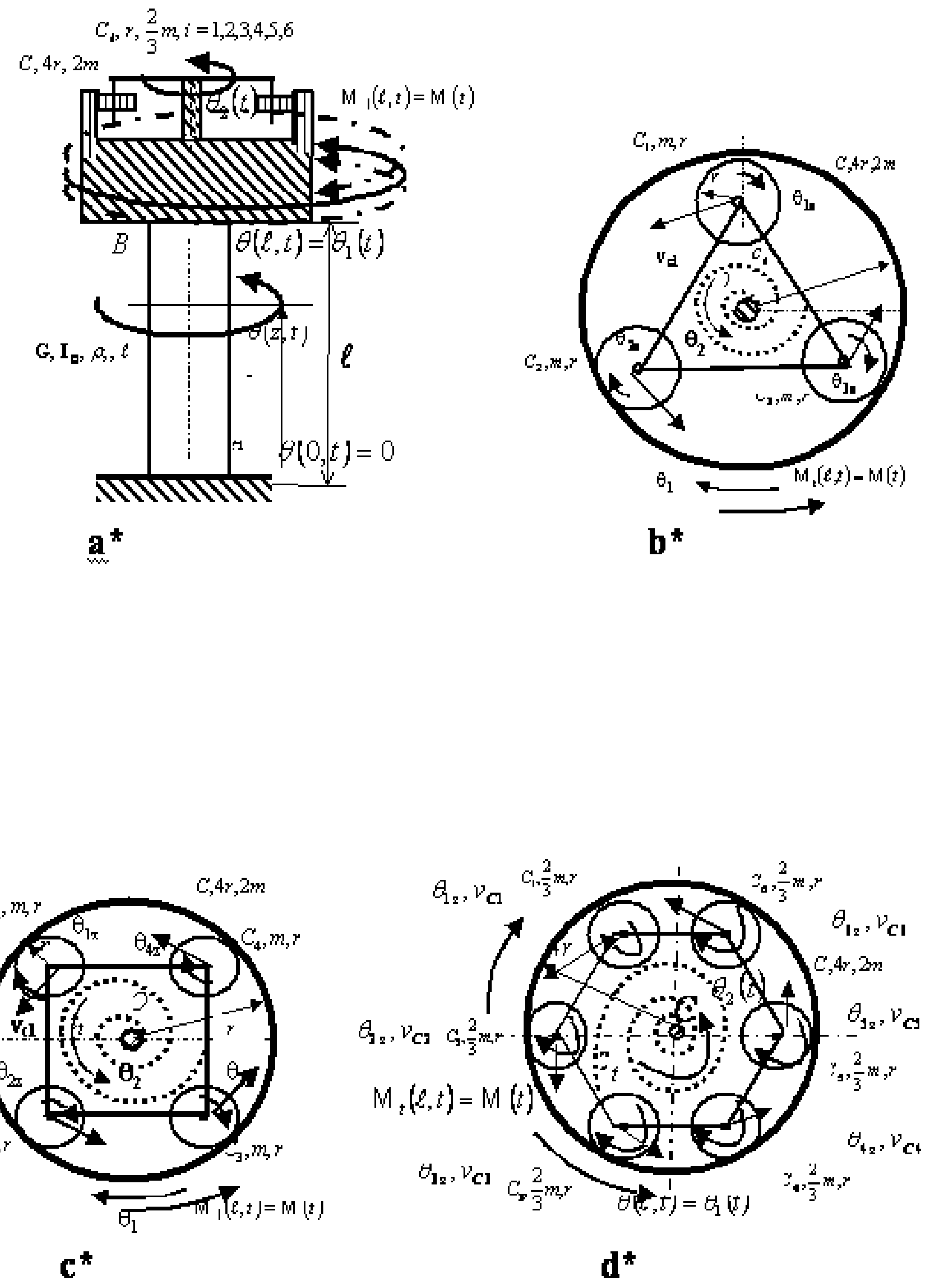

Fig.3. Small oscillations of the mixed system of the coupled discrete and continuous subsystems. Torsion oscillations of the cantilever shaft with multi body mechanisms with changeable numbers of discs.

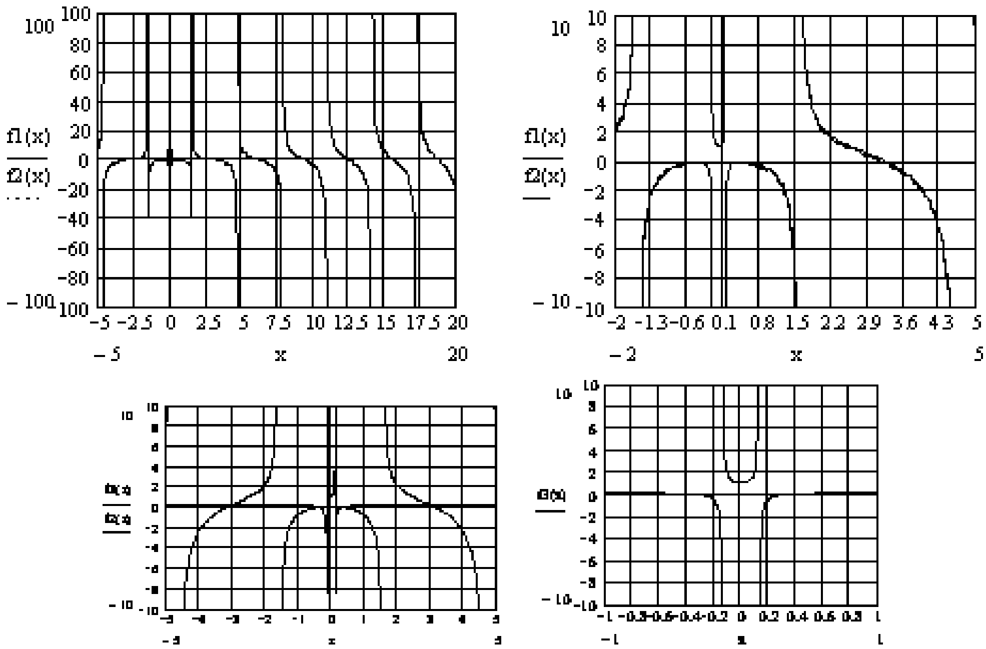

Fig. 4. Numerical simulations and graphical presentations. The figure shows frequency functions and frequency function members that introduce discretization in continual system of transcedent frequency function for parameters: $k = 1, \mu = 0.5$.

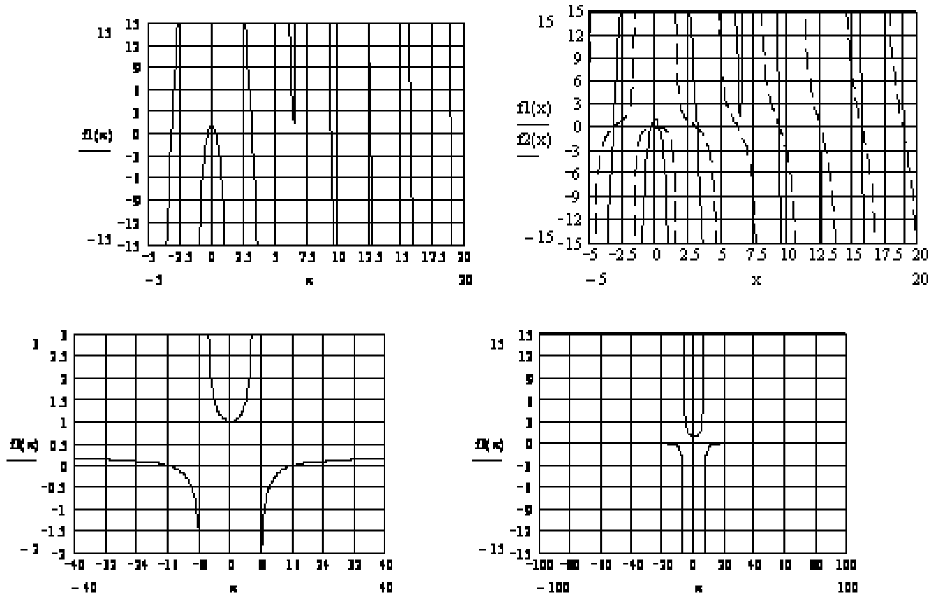

Fig.5. Numerical simulations and graphical presentations. The figure shows frequency functions and frequency function members that introduce discretization in continual system of transcedent frequency function for parameters: $k = 50, \mu = 10$.

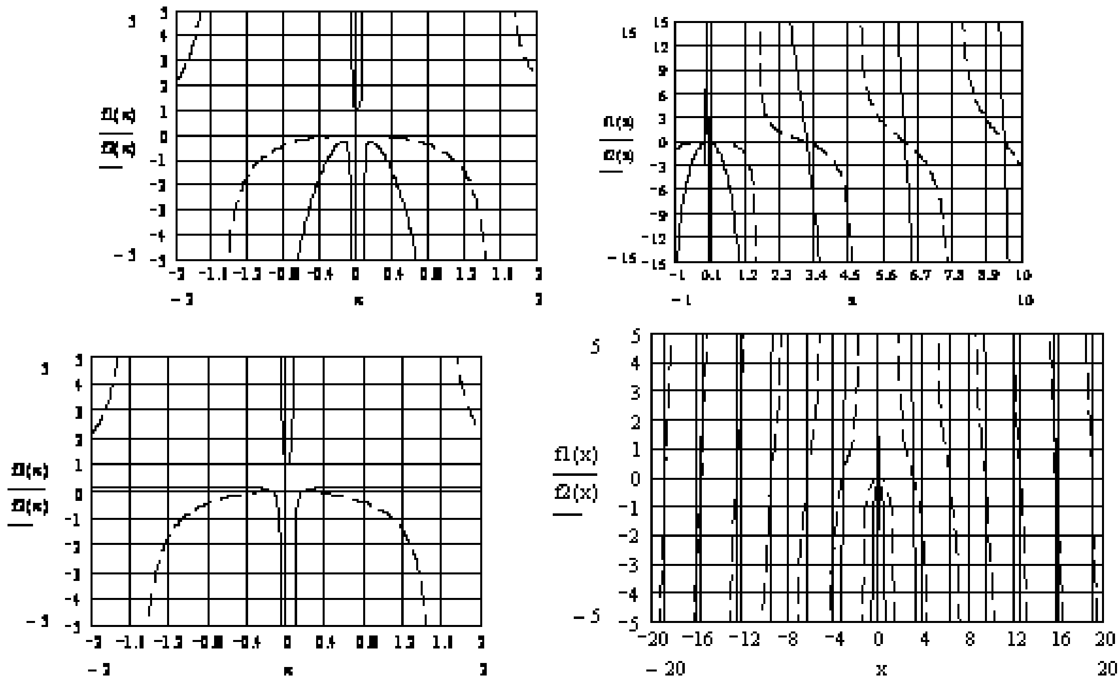

Fig.6. Numerical simulations and graphical presentations. The figure shows frequency functions and frequency function members that introduce discretization in continual system of transcedent frequency function for parameters: $k = 0.5, \mu = 10$.

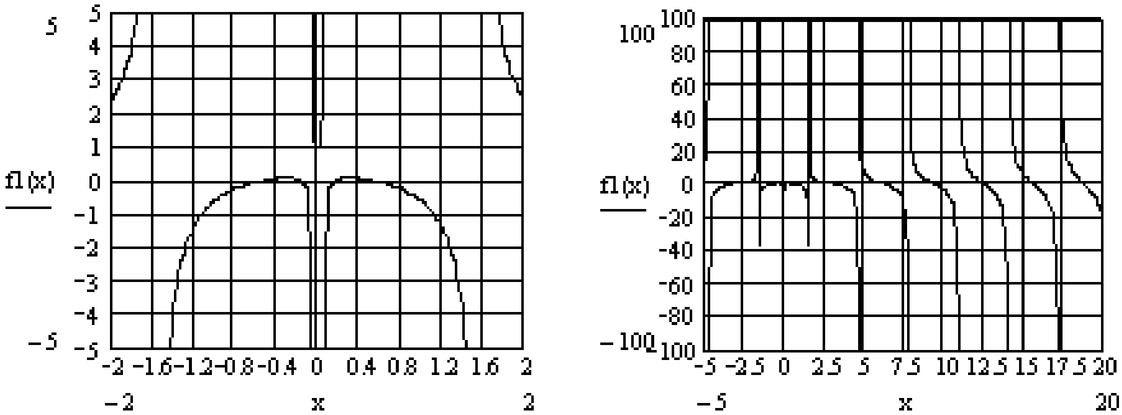

Fig.7. Numerical simulations and graphical presentations. The figure shows frequency functions and frequency function members that introduce discretization in continual system of transcedent frequency function for parameters: $k = 0.5, \mu = 0.5$.

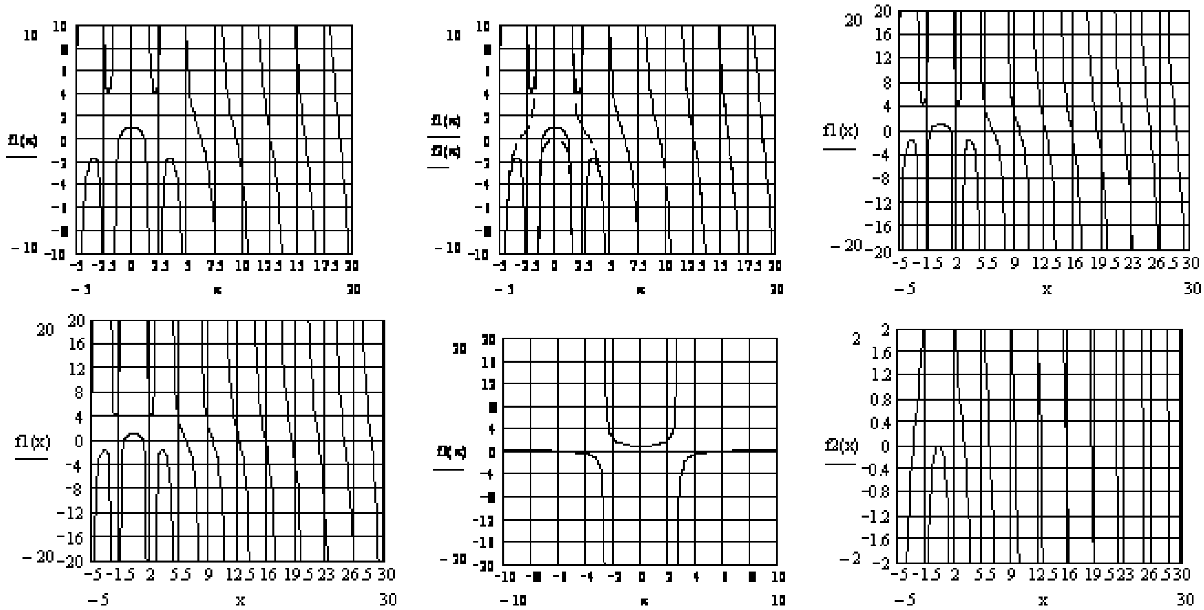

Fig.8. Numerical simulations and graphical presentations. The figure shows frequency functions and frequency function members that introduce discretization in continual system of transcedent frequency function for parameters: $k = 20, \mu = 0.5$.

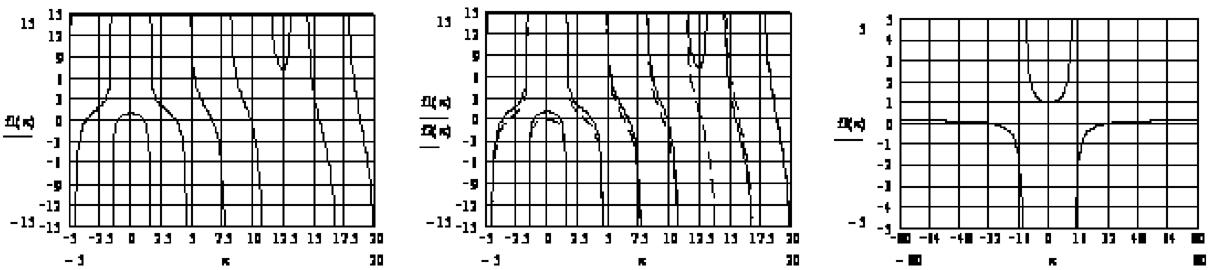

Fig.9. Numerical simulations and graphical presentations. The figure shows frequency functions and frequency function members that introduce discretization in continual system of transcedent frequency function for parameters: $k = 100, \mu = 0.5$.

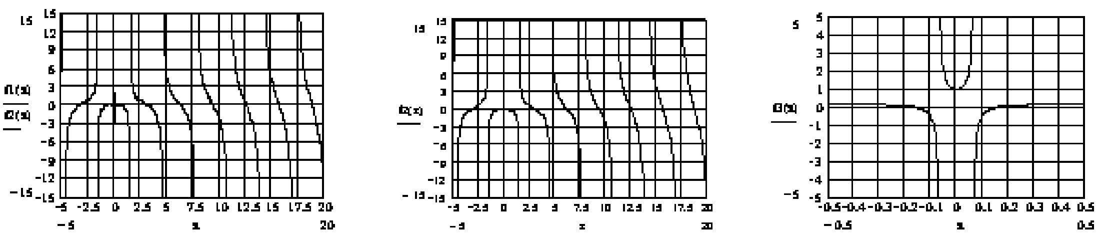

Fig.10. Numerical simulations and graphical presentations. The figure shows frequency functions and frequency function members that introduce discretization in continual system of transcedent frequency function for parameters: $k = 0.5, \mu = 1/3$.

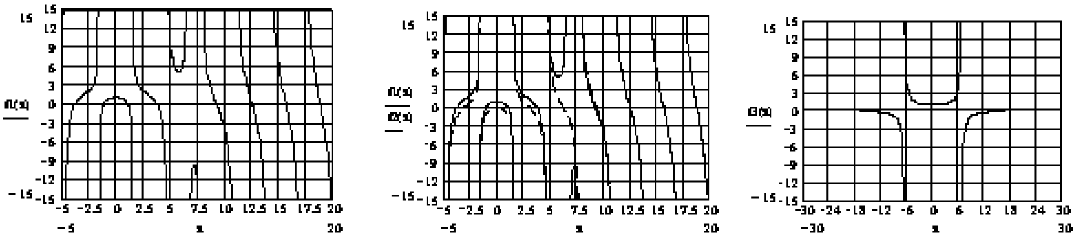

Fig.11. Numerical simulations and graphical presentations. The figure shows frequency functions and frequency function members that introduce discretization in continual system of transcedent frequency function for parameters: $k = 50$, $\mu = 0.5$.

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