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ACCELERATED ALGORITHM OF LEAST-SQUARES APPROXIMATION OF SIGNALS BY EXPONENTIALS FOR WIDEBAND FREQUENCY-DOMAIN REFLECTOMETRY

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In the paper the possibility of acceleration of procedure of best least-squares approximation of signals by exponentials was considered. For this purpose the analytical expressions for components of gradient vector and Hessian matrix of the objective function were obtained. The algorithms of quasisolution searching were constructed. Method of conjugate gradient and modified Newton method were used. The obtained algorithms were compared with modification of Nelder-Mead method which used information about only values of objective function. The comparison of the novel method and Prony's method and matrix pencil method was held.

1. Introduction

A problem of exponential model approximation arises in different applications of radiophysics measurements: antenna measurements, determination of parameters of layered structures by results of measuring frequency characteristics of reflection, non-destructive testing, investigation of transmission lines and so on [1-3]. The additive exponential model allows ones to simulate their different properties. Exponential model is relevant to describe reflection in transmission lines and multilayered dielectric structures. Modern frequency-domain reflectometry often implement an inverse Fourier transform of the complex frequency-domain data but its ability to separate closely spaced reflections is limited by the instrument sweep range. This limitation of the Fourier transform can be avoided if one tries to fit a sum of complex exponentials to the frequency-domain data.

2. Basic Conception

The approach is based on multifrequency reflection coefficient measurement in free space or in transmission lines and additive exponential model of reflection coefficient with the finite number of addends. The model for processed frequency-domain signal can be written as:

$$R_M\left(\vec{r},\vec{t},\omega\right) = \sum_{m=1}^{M} r_m \cdot \exp(-j\omega t_m)\,, \qquad (1)$$

where parameters of model are vector of amplitudes $\vec{r} = \{r_1, r_2, ..., r_M\}$ and vector of time locations $\vec{t} = \{t_1, t_2, ..., t_M\}$. In case of representation of signal on discrete frequency grid $\omega_n = \omega_1 + (n-1)\Delta\omega$ ($n = 1,2,...,N$) model (1) can be written as

$$\vec{R}_M = \mathbf{E}\vec{r}\,, \qquad (2)$$

where $\vec{R}_M = \{R_M(\omega_1), R_M(\omega_2), ..., R_M(\omega_N)\}$ and elements of matrix $\mathbf{E}$ are given by $E_{nm} = \exp(-j\omega_n t_m)$; $n = 1,2,...,N$; $m = 1,2,...,M$. The matrix $\mathbf{E}$ is a function of vector $\vec{t} = \{t_1, t_2, ..., t_M\}$.

The determination of parameters of exponential model using results of measured frequency characteristics is inverse problem because the exponential model can be not quite adequate. Therefore, for deriving the stable solution of the indicated problem the quasisolution method based on minimization of discrepancy between the model and measured data was used. For processing measured signal $\vec{R} = \{R(\omega_1), R(\omega_2), ..., R(\omega_N)\} = \{R_1, R_2, ..., R_N\}$ the main idea of the quasisolution method is minimization of objective function

$$\rho(\vec{r},\vec{t}) = \left\|\vec{R} - \vec{R}_M\right\|^2 = \left\|\vec{R} - \mathbf{E}\,\vec{r}\right\|^2 \qquad (3)$$

with searching best values of parameter vectors $\vec{r}$ and $\vec{t}$.

The amplitude vector $\vec{r}$ may be excluding by linear problem solution for optimal amplitudes

$$\mathbf{E}^{\mathrm{H}}\mathbf{E}\,\vec{r} = \mathbf{E}^{\mathrm{H}}\vec{R} \tag{4}$$

and substitution of the solution in the objective function (3). The linear problem (4) can be rewritten in the form

$$\mathbf{H}\,\vec{r} = \vec{G} \tag{5}$$

with Gram matrix $\mathbf{H} = \mathbf{E}^{\mathrm{H}}\mathbf{E}$ and vector $\vec{G} = \mathbf{E}^{\mathrm{H}}\vec{R} = \{G(t_1),\ldots,G(t_M)\}$ which is formed by values of time domain signal corresponding to the time vector $\{t_1,t_2,\ldots,t_M\}$. Matrix $\mathbf{H}$ elements are determined in the following manner $(k,m = 1,2,\ldots,M)$:

$$H_{km} = \mathrm{sinN}\left[N, \frac{\Delta\omega}{2}(t_k - t_m)\right] \times \exp[j\omega_{\mathrm{mid}}(t_k - t_m)], \tag{6}$$

where $\omega_{\mathrm{mid}} = \omega_1 + (N-1)\Delta\omega/2$ is the middle frequency of the range and function $\mathrm{sinN}(N,x) = \dfrac{\sin(N\cdot x)}{N\cdot\sin(x)}$ is discrete analogy of function $\mathrm{sinc}(x)$. Elements of the vector $\vec{G}$ are calculated according to the following expression $(k = 1,2,\ldots,M)$:

$$G_k = G(t_k) = \frac{1}{N}\sum_{n=1}^{N} R_n \exp(j\omega_n t_k). \tag{7}$$

Substituting in the objective function (3) the optimal values of amplitudes

$$\vec{r}_{\mathrm{opt}} = \mathbf{H}^{-1}\vec{G} \tag{8}$$

provides new objective function of smaller dimension which depends on only vector $\vec{t}$:

$$\rho(\vec{t}) = \|\vec{R}\|^2 - \vec{G}^{\mathrm{H}}\mathbf{H}^{-1}\vec{G}. \tag{9}$$

Minimization of the objective function (9) for searching the quasisolution is complicated nonlinear problem with heavy computational resources. Using of methods based on first and second derivatives should provide acceleration of the algorithm. For this purpose the analytical expressions for components of gradient vector and Hessian matrix of the objective function were obtained.

3. The First and Second Derivatives of the Objective Function

As for the given signal its norm is a constant, minimization of the objective function (9) is completely equivalent to maximization of the objective function

$$\delta(\vec{t}) = \vec{G}^{\mathrm{H}}\mathbf{H}^{-1}\vec{G}. \tag{10}$$

The derivative of the objective function with respect to location t_m is calculated as

$$\frac{\partial\delta}{\partial t_m} = 2\,\mathrm{Re}\left\{\vec{G}^{\mathrm{H}}\mathbf{H}^{-1}\frac{\partial\vec{G}}{\partial t_m}\right\} - \vec{G}^{\mathrm{H}}\mathbf{H}^{-1}\frac{\partial\mathbf{H}}{\partial t_m}\mathbf{H}^{-1}\vec{G}. \tag{11}$$

Choosing the amplitude vector $\vec{r}$ in optimal form $\vec{r} = \vec{r}_{\mathrm{opt}} = \mathbf{H}^{-1}\vec{G}$ for the objective function (10) calculation, the derivate of the objective function (11) can be rewritten in the form

$$\frac{\partial\delta}{\partial t_m} = 2\,\mathrm{Re}\left\{\vec{r}^{\mathrm{H}}\frac{\partial\vec{G}}{\partial t_m}\right\} - \vec{r}^{\mathrm{H}}\frac{\partial\mathbf{H}}{\partial t_m}\vec{r} = 2\,\mathrm{Re}\left\{r_m^* G'_m - r_m^*\sum_{\substack{k=1\\k\neq m}}^{M} r_k \frac{\partial S(t_m - t_k)}{\partial t_m}\right\}, \tag{12}$$

where $G'_m = \dfrac{\partial G(t_m)}{\partial t_m} = \dfrac{j}{N}\sum_{n=1}^{N} R_n\omega_n \exp(j\omega_n t_m)$.

The derivative of the function $S(t)$ is calculated in the form

$$\frac{\partial S(t)}{\partial t} = S(t)\times \left\{j\omega_{\mathrm{mid}} + \frac{\Delta\omega}{2}\left[N\,\mathrm{ctg}\left(N\frac{\Delta\omega}{2}t\right) - \mathrm{ctg}\left(\frac{\Delta\omega}{2}t\right)\right]\right\}.$$

The second derivative of the objective function is equal to

$$\frac{\partial^2\delta}{\partial t_k\partial t_m} = 2\,\mathrm{Re}\left\{\left(\frac{\partial\vec{G}^{\mathrm{H}}}{\partial t_k} - \vec{r}^{\mathrm{H}}\frac{\partial\mathbf{H}}{\partial t_k}\right)\mathbf{H}^{-1}\left(\frac{\partial\vec{G}}{\partial t_m} - \frac{\partial\mathbf{H}}{\partial t_m}\vec{r}\right) + \vec{r}^{\mathrm{H}}\frac{\partial^2\vec{G}}{\partial t_k\partial t_m}\right\} - \vec{r}^{\mathrm{H}}\frac{\partial^2\mathbf{H}}{\partial t_k\partial t_m}\vec{r}.$$

Matrix of the second derivatives of the objective function (10) can be written in following manner:

$$\mathbf{Q} = 2\,\mathrm{Re}\{\mathbf{V}^{\mathrm{H}}\mathbf{H}^{-1}\mathbf{V} + \mathbf{U}\}. \tag{13}$$

In (13) $\mathbf{V} = [\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_M]$ is the matrix of size $M \times M$, formed by vector-column $\vec{v}_m$, which is equal to ($m = 1, \ldots, M$)

$$\vec{v}_m = \begin{pmatrix} r_m \dfrac{\partial S(t_1 - t_m)}{\partial t_m} \\ r_m \dfrac{\partial S(t_2 - t_m)}{\partial t_m} \\ \vdots \\ \dfrac{\partial G(t_m)}{\partial t_m} - \sum\limits_{\substack{k=1 \\ k \neq m}}^{M} r_k \dfrac{\partial S(t_m - t_k)}{\partial t_m} \\ \vdots \\ r_m \dfrac{\partial S(t_M - t_m)}{\partial t_m} \end{pmatrix},$$

$\mathbf{U}$ is the matrix of size $M \times M$ with (km)-th element in form

$$U_{km} = \vec{r}^{\mathrm{H}} \frac{\partial^2 \vec{G}}{\partial t_k \partial t_m} - \frac{1}{2} \vec{r}^{\mathrm{H}} \frac{\partial^2 \mathbf{H}}{\partial t_k \partial t_m} \vec{r} = \begin{cases} r_k r_m^* \dfrac{\partial^2 S(t_m - t_k)}{\partial t_k \partial t_m}, \; k \neq m \\ r_m^* \dfrac{\partial^2 G(t_m)}{\partial t_m^2} - \sum\limits_{\substack{i=1 \\ i \neq m}}^{M} r_i r_m^* \dfrac{\partial^2 S(t_m - t_i)}{\partial t_m^2}, \; k = m, \end{cases}$$

where

$$\frac{\partial^2 G(t_m)}{\partial t_m^2} = -\frac{1}{N} \sum_{n=1}^{N} R_n \omega_n^2 \exp(j \omega_n t_m).$$

The second derivative of the function $S(t)$ is calculated as

$$\frac{\partial^2 S(t)}{\partial t^2} = S(t) \left\langle \left\{ j\omega_{\text{mid}} + \frac{\Delta\omega}{2} \left[N \operatorname{ctg}\left(N \frac{\Delta\omega}{2} t \right) - \operatorname{ctg}\left(\frac{\Delta\omega}{2} t \right) \right] \right\}^2 - \left(\frac{\Delta\omega}{2} \right)^2 \left[\frac{N^2}{\sin^2\left(N \frac{\Delta\omega}{2} t \right)} - \frac{1}{\sin^2\left(\frac{\Delta\omega}{2} t \right)} \right] \right\rangle.$$

It is possible to simplify the calculation of the first derivative of the objective function (11) using the columns of the matrix $\mathbf{V}$:

$$\frac{\partial \delta}{\partial t_m} = \operatorname{Re}\left\{ \vec{r}^{\mathrm{H}} \left[\frac{\partial \vec{G}}{\partial t_m} + \vec{v}_m \right] \right\}. \qquad (14)$$

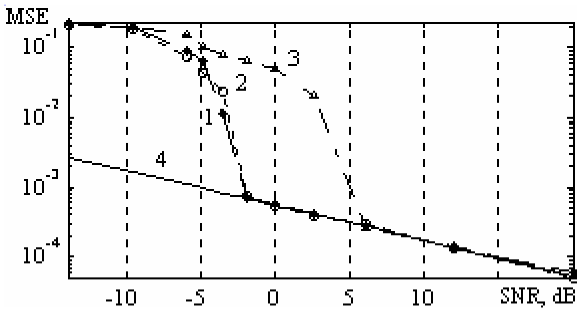

Fig. 1. *MSE as function SNR for the different realizations of the algorithms:*
1 – with Nelder-Mead method;
2 – with conjugate gradient method;
3 – with modified Newton method;
4 – Cramer-Rao bound

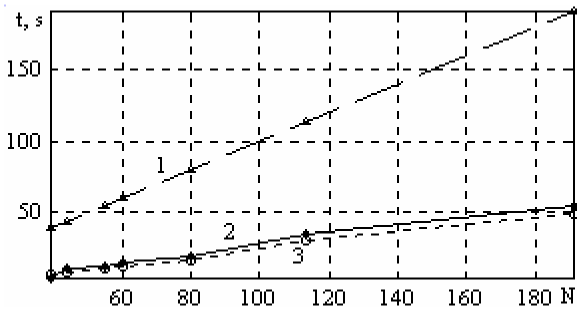

Fig. 2. *Computational time for 2500 trials for the different realizations of the algorithm:*
1 – with Nelder-Mead method;
2 – with conjugate gradient method;
3 – with modified Newton method

4. Realisation of Accelerated Algorithm and Results of Testing

We have compared accelerated algorithm of least-squares approximation of signals by exponentials based on the information about the first and second derivatives and the algorithm of order zero, which operation do not need information about derivatives. The general scheme of algorithm implementation has been presented in [3].

Algorithms were differed by a objective function minimization method and a volume of using information. Algorithm on base of the objective function values uses the Nelder-Mead method of the minimization. Algorithm using additional information about values of the first derivative includes a conjugate gradient method. Algorithm using additional information about values of the first and second derivatives includes a modified Newton method.

The proposed algorithms have been tested using numerical simulated data. The data used are described by the formula (2) with adding white complex Gaussian noise sequence with variance $2\sigma^2$.

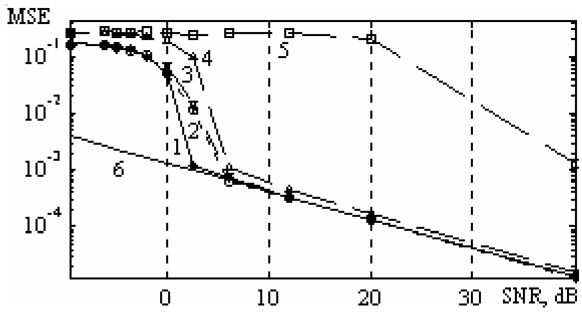

Fig. 3. *MSE as function SNR for the different algorithms*
1 – with Nelder-Mead method;
2 – with conjugate gradient method;
3 – with modified Newton method;
4 – matrix pencil method;
5 – Prony's method;
6 – Cramer-Rao bound.

The mean and variance of the locations have been estimated over 2500 trials were calculated. The model parameters were $\omega_1 = -2\pi \cdot 50$, $\Delta\omega = 2\pi$, $N = 101$, $r_1 = 1$, $r_2 = 1$, $t_1 = 0.3$, $t_2 = 0.8$.

Obtained values of mean square error (MSE) of location estimations versus SNR are presented in Fig. 1. Time of calculation for processing by means of different realization of algorithm as function of N is shown in Fig. 2. Comparison of the algorithm and traditional algorithm (matrix pencil method and Prony's method) has been carried out for the same model with other parameter t_2 equals 0.31. The results of comparison are presented in Fig. 3.

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УСКОРЕННЫЙ АЛГОРИТМ АППРОКСИМАЦИИ СИГНАЛОВ ЭКСПОНЕНТАМИ МЕТОДОМ НАИМЕНЬШИХ КВАДРАТОВ ДЛЯ ШИРОКОПОЛОСНОЙ РЕФЛЕКТОМЕТРИИ В ЧАСТОТНОЙ ОБЛАСТИ

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В статье рассмотрена возможность ускорения аппроксимации сигналов экспонентами методом наименьших квадратов. Для этого были получены аналитические выражения для компонент вектора градиента и матрицы Гессе оптимизируемой функции. Был сконструирован алгоритм поиска квазирешения. Использовались метод сопряженных градиентов и модифицированный метод Ньютона. Полученные алгоритмы были сравнены с модификацией метода Нелдера-Мида, который использует информацию только о значениях оптимизируемой функции. Было проведено сравнение нового метода с методом Прони и методом пучка матриц.

ПРИСКОРЕНИЙ АЛГОРИТМ АПРОКСИМАЦІЇ СИГНАЛІВ ЕКСПОНЕНТАМИ ЗА МЕТОДОМ НАЙМЕНШИХ КВАДРАТІВ ДЛЯ ШИРОКОСМУГОВОЇ РЕФЛЕКТОМЕТРІЇ В ЧАСТОТНІЙ ОБЛАСТІ

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У статті розглянуто можливість прискорення апроксимації сигналів експонентами методом найменших квадратів. Для цього було отримано аналітичні вирази для компонент вектора градієнта та матриці Гессе функції, що оптимізується. Було сконструйовано алгоритм пошуку квазірозв'язку. Використано метод спряжених градієнтів та модифікований метод Ньютона. Отримані алгоритми були порівняні з модифікацією методу Нелдера-Міда, який використовує інформацію тільки про значення функції, що оптимізується. Було проведене порівняння нового методу з методом Проні та методом пучка матриць.