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Numerical Implementation of Cross-Section Method for Irregular Waveguides

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Wave scattering in irregular waveguides is investigated. The cross-section method is considered as a method for calculation of the field in a waveguide consisting of two regular waveguides with different cross-sections joined by an irregular domain. In the paper, a mathematically justified derivation of the basic equations of the method is given. An iterative procedure for their numerical solution is proposed. The algorithm is applied to the problems with the smooth and nonsmooth irregularities. In particular, numerical results for a test problem having analytical solution, are presented.

Index Terms: wave scattering, irregular waveguide, cross section method, iterative method

1. Introduction

The idea of the cross-section method (CSM) was proposed several decades ago [1, 2]. The method was developed and investigated by different authors. The most essential contribution to its foundation is given in [3-5].

CSM is suitable for investigation of the waveguides with different kinds of small and smoothly varying irregularities such as smooth and slow change of the cross-section shape, jog and shift of the axis line, alteration of the optical density of the filling etc. It is a useful technique for studying the wave scattering in closed and open irregular metallic, dielectric and impedance waveguides [6-10], field converters, cavity antennae [11-16], and other practically important problems. An attempt to justify the above method for the 3D vector problem was undertaken in [17]. However, the range of the practical applicability of the method is still not examined theoretically and numerically.

The purpose of this paper is to demonstrate the mathematical equivalence of the main equations of the method in the form presented in [3, 5] and in [1, 2], to suggest an iterative procedure for solving the main equations of CSM, and to investigate numerically the applicability of the method for the case of large enough irregularities. Application of the method is presented in the framework of the new general scheme for the investigation of the wave scattering in irregular waveguides proposed in [18]. Scattering problems in the domains with infinite boundaries were studied in [19]. A detailed analysis of the scattering by obstacles in regular waveguides is given in [20]. In [21] the waveguide theory is developed in application to optical waveguides.

Here we present the method for acoustical waveguides with soft walls; the pressure u on such walls is equal to zero. The waveguides with the continuously varying cross-section shape are considered. The two-dimensional case is equivalent to the electromagnetic problem for the E-polarization ($u = E_y$). Derivations of the main equations are made in the way similar to [1, 2]. The scattering problem is put in two forms: as a boundary value problem with inhomogeneous equation and homogeneous conditions at infinity, as well as such a problem for homogeneous equation with inhomogeneous conditions at infinity.

The idea of CSM as a method for solving the problems of wave propagation in irregular waveguides is not new, but in this paper a self-contained and rigorous derivation of its basic equations is given. The novel idea and novel result in the paper is the numerical implementation of the method based on an iterative procedure.

Numerical results are presented for two problems: for a test one having an exact solution and for a problem with the geometrical parameters varying in a wide range. The results obtained for the test problem show the character of changing the errors of the computed solution versus geometry of irregularity as well as versus number of the normal modes taken into account.

It is known that CSM is suitable for investigation of the waveguides with slowly varying irregularities (in the case considered in our test-problem the slowly varying waveguide means that the angle between the waveguide boundary and axis is not large). This limitation is necessary in order that the rate of convergence of the series representing the solution be satisfactory from the practical point of view. The numerical results obtained in the paper show that the method we use can be successfully applied for a wide range of the slopes of the waveguide irregular part (up to angles $\pi/3$ in our case). The results demonstrate also the efficiency of the proposed implementation of the method.

The time dependence of the form $\exp(-j\omega t)$ is assumed.

2. Problem Statement

Let us consider a waveguide which is a union of two regular waveguides, W_1 and W_2, with the boundaries S_1 and S_2, respectively, joined by an irregular domain W_0 with the boundary S_0 (see Fig. 1). We assume that the cross-section $D(z)$ of W_0 varies smoothly as a function of z, $0 \le z \le d$, where z is directed along the waveguide. By x we denote the transversal to the z-axis coordinate in the cross-section $D(z)$. In the 3D case the x-coordinate is two-dimensional, $x = \{x_1, x_2\}$. We also assume that the boundary of the waveguide is such that there are no trapped modes in the waveguide, that is, there are no non-trivial quadratically integrable solutions to the homogeneous boundary value problem describing the waves in the waveguide. According to Theorem 2.1 in [20], p. 92, this is the case if the boundary is described by a monotone function of z. In the 3D case the geometrical condition on the boundary in [20], p. 92, is as follows:

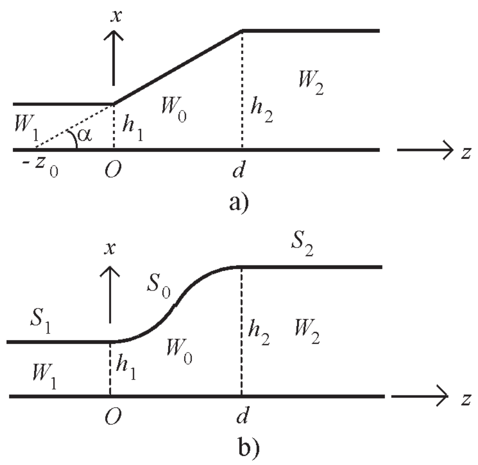

Fig. 1. *Geometry of the problems*

the exterior normal to the boundary forms an obtuse angle with the positive direction of the z-axis.

The Helmholtz equation,

$$(\Delta_x + \frac{\partial^2}{\partial z^2})u + k^2 u = f, \quad (1)$$

with a real wavenumber k holds in $W = W_1 \cup W_0 \cup W_2$, Δ_x is the Laplacian with respect to the x-variable, $f = f(x,z)$ is a compactly supported function, that is, the function vanishing outside a bounded region. We assume that the support of f is localized between some sections, $z = z_1$ and $z = z_2$ in W_1, $z_1 < z_2 < 0$: $f(x,z) = 0$ if $z \notin [z_1, z_2]$. Here z_1 and z_2 are arbitrary negative numbers, so that the support of the source function f is located in W_1. The support of f is the complement of the largest open set on which f vanishes almost everywhere. The boundary condition

$$u = 0 \qquad \text{at} \qquad S = S_1 \cup S_0 \cup S_2 \quad (2)$$

holds, and the radiation conditions at infinity are imposed:

$$u\big|_{z\to-\infty} \cong \sum_n p_{-n}^{(1)} e^{-j\beta_n^{(1)} z} v_n^{(1)}(x), \quad (3)$$

$$u\big|_{z\to\infty} \cong \sum_n p_n^{(2)} e^{j\beta_n^{(2)}(z-d)} v_n^{(2)}(x), \qquad (4)$$

where the coefficients $p_{-n}^{(1)}$ and $p_n^{(2)}$ are unknown. Here (and below) the summation is from $n = 1$ to ∞, $\beta_n^{(i)} = (k^2 - k_n^{(i)2})^{1/2}$, $i = 1, 2$; for driving modes $\beta_n^{(i)}$ are the real positive constants and for damped modes they are imaginary ones ($j\beta_n^{(i)} < 0$); $k_n^{(i)2}$, $v_n^{(i)}$ are the eigenvalues and eigenfunctions of the boundary value problem for the transversal Helmholtz equation

$$\Delta_x v_n^{(i)} + k_n^{(i)2} v_n^{(i)} = 0 \qquad (5)$$

with the boundary condition $v_n^{(i)} = 0$ at the contours ∂D_i of the cross-sections D_i. The functions $v_n^{(i)}$ are orthonormal in $L^2(D_i)$:

$$\int_{D_i} v_n^{(i)}(x)\bar{v}_m^{(i)}(x)dx = \delta_{nm},$$

where the overbar stands for complex-conjugate (we assume that in general case v_m are complex).

Let us call the problem (1)-(4) Problem A. In practice, instead of the force term $f(x)$ in (1), the excitation is often given in the form of the incident normal modes coming from $-\infty$. In this case $f = 0$, but the total field at $-\infty$ is the sum of the incident and reflected fields, while at $+\infty$ the total field is the transmitted field. Problem A can be easily reduced to this form.

One can present the solution of (1) in W_1 as $u = U^0 + U^s$, where U^0 is any partial solution of the problem (1)-(3) in W_1 and U^s satisfies the homogeneous Helmholtz equation in W_1 with the conditions (2), (3). The function U^0 may be found as the solution of the inhomogeneous problem in the regular waveguide W_1 extended to ∞ with the condition (3) at $-\infty$ and the condition of type of (4) at ∞. This problem can be easily solved through the separation of variables. Here z_2 has the same meaning as on the line above formula (2). Since $f = 0$ at $z_2 < z < 0$, the function U^0 satisfies the homogeneous Helmholtz equation in this region and has the form

$$U^0 = \sum_n p_n^{(1)} e^{j\beta_n^{(1)} z} v_n^{(1)}(x) \qquad (6)$$

with the known coefficients $p_n^{(1)}$. This function describes the incident field, it represents the waves propagating in the positive direction of the z-axis.

The general solution of homogeneous Eq. (1) in W_1 satisfying (2), (3) in W_1 is as follows:

$$U^s = \sum_n p_{-n}^{(1)} e^{-j\beta_n^{(1)} z} v_n^{(1)}(x). \qquad (7)$$

So the solution of (1)-(3) in W_1 is

$$u\big|_{z_2<z\le 0} = \sum_n [p_n^{(1)} e^{j\beta_n^{(1)} z} + p_{-n}^{(1)} e^{-j\beta_n^{(1)} z}] v_n^{(1)}(x). \qquad (8)$$

Since k = const, the general solution of the problem in W_2 follows from (4):

$$u\big|_{z\ge d} = \sum_n p_n^{(2)} e^{j\beta_n^{(2)}(z-d)} v_n^{(2)}(x). \qquad (9)$$

Putting $z = 0$ in (8) and $z = d$ in (9) yields the conditions for $u(x,z)$ at the vertical sides of W_0 (that is, on the sections $z = 0$ and $z = d$):

$$u\big|_{z=0} = \sum_n [p_n^{(1)} + p_{-n}^{(1)}] v_n^{(1)}(x), \qquad (10)$$

$$u\big|_{z=d} = \sum_n p_n^{(2)} v_n^{(2)}(x). \qquad (11)$$

In a similar way, differentiating (8), (9) with respect to z and putting $z = 0$ and $z = d$, respectively, yields two more conditions for $\partial u/\partial z$ at these boundaries:

$$\left.\frac{\partial u}{\partial z}\right|_{z=0} = j\sum_n \beta_n^{(1)} [p_n^{(1)} - p_{-n}^{(1)}] v_n^{(1)}(x), \qquad (12)$$

$$\left.\frac{\partial u}{\partial z}\right|_{z=d} = j\sum_n \beta_n^{(2)} p_n^{(2)} v_n^{(2)}(x). \qquad (13)$$

Thus, the problem A is reduced to the non-standard interior boundary problem (1), (2), (10)-(13) for the irregular domain W_0.

Let us call this problem Problem B. Here $p_n^{(1)}$ are the given magnitudes of the excitation modes calculated in (6) and $p_{-n}^{(1)}$, $p_n^{(2)}$ are the reflection and transmission factors, which are to be found.

Often a statement of the problem, alternative to (1)-(4) is used based on the concept of the scattering matrix. Namely, the source in the Eq. (1) can be taken in the form of one normal mode of the left waveguide W_1 coming from $-\infty$. Then the problem lies in finding the set of functions u_m, m = 1, 2, ... satisfying the homogeneous Eq. (1) in W with the condition (2) and the following conditions at infinity:

$$\left.u_m\right|_{z\to-\infty} \cong e^{j\beta_m^{(1)}z} v_m^{(1)}(x) + \sum_n r_{mn} e^{-j\beta_n^{(1)}z} v_n^{(1)}(x) \qquad (14)$$

$$\left.u_m\right|_{z\to\infty} \cong \sum_n t_{mn} e^{j\beta_n^{(2)}(z-d)} v_n^{(2)}(x) \qquad (15)$$

Here $\{r_{mn}\}$, $\{t_{mn}\}$ are the unknown reflection and transmission matrices, respectively.

Let us call this problem Problem C. If this problem is solved for all m = 1, 2, ..., then the solution of Problem A is given as

$$u = \sum_m p_m^{(1)} u_m, \qquad (16)$$

where $p_m^{(1)}$ are the same as in (6).

If $f(x,z) = 0$, then the right-hand sides of (14), (15) give the solution of Problem C in W_i, and the scattering problem in W is reduced to the boundary value problem (1), (2), (10)-(13) in W_0 with $u = u_m$, $p_n^{(1)} = \delta_{mn}$, $p_{-n}^{(1)} = r_{mn}$, $p_n^{(2)} = t_{mn}$. Thus, Problem C is reduced to Problem B.

Hereafter we study Problem B. From the above arguments it follows that in general the scattering problems in irregular waveguides can be reduced to Problem B.

3. Mathematical Description of the Cross-Section Method

The solution of Problem B can be found in the form

$$u(x,z) = \sum_n c_n(z) v_n(z,x), \qquad (17)$$

where $c_n = (u, v_n)$, $(u, v_n) = \int\limits_{D(z)} u(x,z)\bar{v}_n(x,z)\mathrm{d}x$ is the inner product in $L^2(D(z))$, $v_n(x,z)$ are the eigenfunctions of the equation of type of (5) in the cross-section $D(z)$ with the eigenvalues $k_n^2(z)$ and boundary condition $v_n = 0$ at $\partial D(z)$. The functions $v_n(x,z)$ are orthonormal in $L^2(D(z))$:

$$(v_m, v_n) = \delta_{nm}. \qquad (18)$$

The above formulation is valid in 2D and 3D cases. We assume that the eigenfunctions $v_n(x,z)$ can be easily calculated for each cross-section $D(z)$. Otherwise the practical application of the cross-section method is difficult.

To obtain a set of ordinary differential equations for c_n let us multiply (1) by v_n and integrate over $D(z)$ to get

$$\beta_n^2(z)c_n + (u'', v_n) = 0, \qquad (19)$$

where $\beta_n^2(z) = k^2 - k_n^2(z)$, $u'' = \partial^2 u/\partial z^2$.

Differentiating $c_n(z)$ with respect to z yields:

$$c'' = (u'', v_n) + 2(u', v') + (u, v_n''). \qquad (20)$$

From (19), (20) one can obtain:

$$\beta_n^2 c_n + c_n'' - 2(u', v_n') - (u, v_n'') = 0. \qquad (21)$$

According to (17), the last term in (21) takes the form:

$$(u, v_n'') = \sum_m b_{nm} c_m, \tag{22}$$

where

$$b_{nm} = (v_m, v_n''). \tag{23}$$

If the boundary $\partial D(z)$ is varied smoothly with respect to z, then the series (17) can be once differentiated termwise:

$$u' = \sum_m [v_m' c_m + c_m' v_m]. \tag{24}$$

Using (24) one writes the term (u', v_n') as

$$(u', v_n') = \sum_m [d_{nm} c_m + c_m' a_{nm}], \tag{25}$$

where

$$d_{nm} = (v_m', v_n'), \qquad a_{nm} = (v_m, v_n'). \tag{26}$$

From (21)-(25) one gets

$$c_n'' + \beta_n^2 c_n - 2\sum_m a_{nm} c_m' - \sum_m (b_{nm} + 2d_{nm}) c_m = 0. \tag{27}$$

Note that

$$a_{nm} = -\bar{a}_{mn}, \tag{28}$$

as follows from differentiating the identity (18) with respect to z. Similarly, differentiating (18) twice, one gets:

$$2d_{nm} = -(b_{nm} + \bar{b}_{mn}) \tag{29}$$

and (27) takes the form

$$c_n'' + \beta_n^2 c_n - 2\sum_m a_{nm} c_m' + \sum_m \bar{b}_{mn} c_m = 0. \tag{30}$$

Eq. (30) must be satisfied for n = 1, 2, ...

It is useful to rewrite (30) in the matrix form:

$$C'' + K^2 C - 2AC' + B^* C = 0, \tag{31}$$

where $C = \{c_n\}$, $K = \mathrm{diag}(\beta_n)$, $A = \{a_{nm}\}$, $B^* = \{\bar{b}_{mn}\}$ is the matrix adjoint to B.

Let us now eliminate the matrix B^* from (31). For this purpose we differentiate the second equation in (26) and get $a_{nm}' = b_{nm} + d_{nm}$ or

$$B = A' - D. \tag{32}$$

To eliminate the matrix D note that

$$v_n' = \sum_m \bar{a}_{nm} v_m. \tag{33}$$

Then $d_{nm} = (v_m', v_n') = \sum_p a_{np} (v_m', v_p) = \sum_p a_{np} \bar{a}_{mp}$,

so that

$$D = AA^*. \tag{34}$$

From (28) one gets $A = -A^*$. Therefore

$$B^* = -A' + A^2 \tag{35}$$

and Eq. (31) can be written as

$$C'' - 2AC' + (K^2 - A' + A^2)C = 0. \tag{36}$$

Let us introduce a new pair of unknown vectors, C and G, in place of C; $c_n(z)$ and $g_n(z)$ are the components of the vectors C and G, respectively. Namely, let

$$G = C' - AC. \tag{37}$$

Then

$$C'' = G' + A'C + AC' = G' + A'C + A(G + AC) =$$
$$= G' + AG + (A' + A^2)C$$

and (36) yields:

$$G' - AG + (A' + K^2 - A^2 + B^*)C =$$
$$= G' - AG + K^2 C = 0.$$

Thus we have the following set of equations for C, G:

$$C' = AC + G, \qquad G' = AG - K^2 C. \tag{38}$$

The boundary conditions for Eqs. (38) can be obtained from (10)-(13) with account that $v_n(x,0) = v_n^{(1)}$, $v_n(x,d) = v_n^{(2)}$.

From (10), (12) one gets

$$c_n(0) = p_n^{(1)} + p_{-n}^{(1)}, \tag{39}$$

$$c_n'(0) - \sum_m a_{nm} c_m(0) = g_n(0) = j\beta_n^{(1)}[p_n^{(1)} - p_{-n}^{(1)}], \tag{40}$$

where $g_n(0)$ is the component of the vector $G(0)$, G is defined in (37). One can eliminate the unknown p_{-n} from (39), (40) to obtain the following condition:

$$C(0) - jK^{-1}(0)G(0) = 2P_+^{(1)}, \tag{41}$$

where $P_+^{(1)} = \{p_n^{(1)}\}$.

Similarly, from (11), (13) we have

$$C(d) + jK^{-1}(d)G(d) = 0. \tag{42}$$

Eqs. (38) together with (41), (42) state the interior boundary value problem for the functions C, G in W_0. We assume that $\beta_n^{(i)} \neq 0$ for any n.

The set of Eqs. (38) is stiff [22], because the functions c_n, g_n contain both exponentially increasing and exponentially decreasing components (if $\mathrm{Im}(\beta_j) \neq 0$). The computational methods developed for solving these equations are rather expensive.

To eliminate this difficulty, let us introduce the new unknown functions $P_+ = \{p_n(z)\}$ and $P_- = \{p_{-n}(z)\}$ describing the magnitudes of the forward and backward normal modes in the irregular domain:

$$C = P_+ + P_-, \qquad G = jK(P_+ - P_-). \tag{43}$$

Then one has the new set of equations from (38):

$$P_+' = (Z_1 + jK)P_+ + Z_2 P_-, \tag{44}$$

$$P_-' = (Z_1 - jK)P_- + Z_2 P_+, \tag{45}$$

where

$$Z_1 = (A - K^{-1}K' + K^{-1}AK)/2, \tag{46}$$

$$Z_2 = (A + K^{-1}K' - K^{-1}AK)/2. \tag{47}$$

The boundary conditions for P_+ and P_- can be easily obtained from (41), (42):

$$P_+(0) = P_+^{(1)}, \tag{48}$$

$$P_-(d) = 0. \tag{49}$$

Eqs. (44), (45) are equivalent to Eqs. (2.46) from [5].

Theoretical problems concerning the numerical solution of the problem (44)-(49) need further investigation. Note that the functions P_+ do not contain exponentially increasing component, whereas P_- do not contain exponentially decreasing component. This fact allows one to apply an iterative method for solving this problem: at each iteration a Cauchy problem is solved for one subset of Eqs. (44), (45) in the forward or backward directions, respectively. Namely, the equation

$$\left(P_+^{[2q+1]}\right)' = (Z_1 + jK)P_+^{[2q+1]} + Z_2 P_-^{[2q]} \tag{50}$$

(the value in the square brackets denotes the serial number of iteration) with the initial condition of the type of (48) is solved with respect to $P_+^{[2q+1]}$ for $0 \le z \le d$ in the forward direction at each odd $(2q + 1)$-th iteration ($q = 0, 1, 2, ...$) with $P_-^{[2q]}$ taken from the previous iteration. At the first iteration one takes $P_-^{[0]} \equiv 0$. Similarly, the equation

$$\left(P_-^{[2q+2]}\right)' = (Z_1 - jK)P_-^{[2q+2]} + Z_2 P_+^{[2q+1]} \tag{51}$$

with the initial condition of the type of (49) is solved with respect to $P_1^{[2q+2]}$ for $0 \le z \le d$ in the backward direction at each even $(2q+2)$-th iteration with $P_+^{[2q+1]}$ taken from the previous iteration.

Such a technique can be interpreted as taking into account successive transformations of the normal waveguide modes at the irregularities.

The definition of the functions $p_n(z)$, $p_{-z}(z)$ (43) is unique everywhere except the "critical sections" $z = z_n$ where $\beta_n(z) = 0$. At these points the components of K^{-1} in Eqs. (44), (45) are not defined. In this paper we do not investigate the properties of the solution in the neighborhoods of "critical sections". Let us note that the functions $p_n(z)$, $p_{-z}(z)$ are introduced by (43) only for $z \notin (z_n - \delta, z_n + \delta)$ with some small δ. In the intervals $(z_n - \delta, z_n + \delta)$ Eqs. (38) should be used. The matching conditions at $z = z_n \pm \delta$ for these equations follow from Eqs. (43) used at these points.

Then the n-th Eq. (50) or (51) is solved at each iteration only for $|z - z_n| > h_n$, where h_z is a step size of the variable z in the numerical method for the Cauchy problem, and the equation is substituted by the n-th pair of Eqs. (38) at the last discretization point preceding z_n. At the first discretization point after z_n the above pair of Eqs. (38) is substituted by the n-th Eq. of (50) or (51). At the points of substitution the functions c_n, g_n and p_n, p_{-n} are matched by formula (43). This technique for dealing with critical cross-sections was used in [3].

4. Numerical Results

The applicability of the CSM is defined by the rate of convergence of the series (17). There are no theoretical estimates of this rate. Numerical results suggest that this rate decreases as the slope of the boundary of the waveguide irregular part increases. The numerical experiments were carried out to find out the above dependence and the practical limitation of the method.

The numerical results presented refer to the $2D$ problems with the same regular waveguides W_1, W_2 but different shapes of the irregular domain W_0 (Fig. 1 (a), (b)). To demonstrate the dependence of the errors on the number of the terms kept in series (17) we show the numerical results for the test problem concerning the waveguide shown in Fig. 1 (a). In this case such errors are expected to be greater than in the second problem because of the nonsmoothness of the waveguide upper boundary. Next, the results of the numerical solution of both problems are presented for the case when the incident field is the first normal mode of the left waveguide.

The first problem is a problem for the waveguide with W_i of the height h_i, i=1,2, and the height of W_0 given by the formula:

$$h(z) = h_1 + z(h_2 - h_1)/d$$

(see Fig. 1 (a)). As a test problem for the method, we choose the above one with the initial data allowing an exact analytical solution. These data are taken in the following way. First, an exact solution of the homogeneous Eq. (1) with conditions (2) in W_0 is analytically constructed. Then the magnitudes $p_n^{(1)}$, $p_{-n}^{(2)}$ of incident waves in W_1, W_2, respectively, are calculated, which excite jointly the field in W_0 described by this exact solution. Next, these magnitudes are used as initial data in the Problem B, which is solved numerically. Finally, the obtained solution is compared with the exact one.

Let us choose the solution of (1), (2) in W_0 in the form of standing field:

$$u(x, z) = J_{\pi/\alpha}(kr)\sin(\pi\varphi/\alpha), \tag{52}$$

where $J_{\pi/\alpha}$ is the Bessel function of the first kind,

$\alpha = \arctan((h_2 - h_1)/d)$, $\varphi = \arctan(x/(z+z_0))$, $r = (x^2 + (z+z_0)^2)^{1/2}$. The function (52) satisfies homogeneous Eq. (1) in W_0 as any function of the form $J_\nu(kr)\sin(\nu\varphi)$ does, and conditions (2) at $\varphi = 0$, $\varphi = \alpha$ in W_0. Denote $u^{(i)} = u|_{z=z^{(i)}}$, $i = 1, 2$, $z^{(1)} = 0$, $z^{(2)} = d$. Expand these functions and their derivatives with respect to z as the Fourier series with respect to the basis functions $v_n^{(i)}(x) = (2/h_i)^{1/2}\sin(n\pi x/h_i)$:

$$u^{(i)}(x) = \sum_n c_n^{(i)} v_n^{(i)}(x), \tag{53}$$

$$\left.\frac{\partial u(x,z)}{\partial z}\right|_{z=z^{(i)}} = \sum_n g_n^{(i)} v_n^{(i)}(x), \tag{54}$$

and calculate $p_n(0)$ and $p_{-n}(d)$ by the formulas:

$$p_n(0) = (c_n^{(1)} - jg_n/\beta_n^{(1)})/2, \tag{55}$$

$$p_{-n}(d) = (c_n^{(2)} + jg_n/\beta_n^{(2)})/2, \tag{56}$$

where $\beta_n^{(i)} = (k^2 - (n\pi/h_i)^2)^{1/2}$. Eqs. (55), (56) are the boundary conditions for Eqs. (44), (45). To reduce them to the form (48), (49) one should consider two problems with the incident wave coming from $-\infty$ with the magnitudes of incident modes (55) and from ∞ with the magnitudes of incident modes (56), respectively, and add the solutions of these problems. But the numerical implementation of the above iterative procedure shows that it converges not only for the problem with the boundary conditions (48), (49), but also for the more complete conditions (55), (56). This fact allows one to apply the above procedure for solving the problem (44), (45), (55), (56).

To investigate the dependence of the calculating errors on the wall inclination in W_0 and the number of normal modes taken into account, the problem has been numerically solved at the different values of these parameters. In Fig. 2 the values

$$\varepsilon_N^{(i)} = \left\| (u_N^{(i)} - u^{(i)})/u^{(i)} \right\| \tag{57}$$

are given as functions of d and N for the waveguide with $kd_1 = 1.5\pi$, $kd_2 = 4.5\pi$, where $u_N^{(i)}$ are the approximate values of $u^{(i)}$ calculated using the series (17) in which N first terms are kept. There are one driving mode (with $\mathrm{Im}\beta_n^{(1)} = 0$) in the left section and four such modes (with $\mathrm{Im}\beta_n^{(2)} = 0$) in the right one.

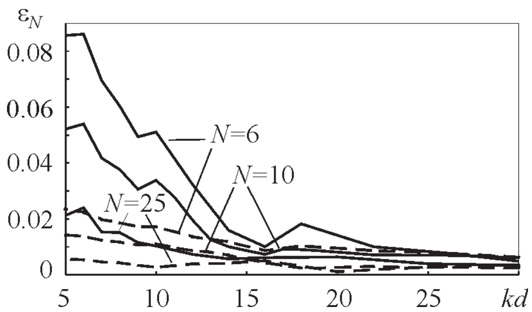

Fig. 2. *Relative computation accuracy of the field at z = 0 (solid lines) and z = d (dashed lines) keeping N terms in (17) for the waveguide shown in Fig. 1 (a)*

One can see that a high accuracy (the error is less than 1 per cent) is achieved for $\tan\alpha = (h_2 - h_1)/d < 0.5$ ($\alpha < 25°$) with $N = 6$, that is by taking into account only two decreasing modes (with $\mathrm{Im}\beta_n^{(2)} > 0$) in W_2. With $N = 25$ this accuracy is achieved for the values of α up to $\tan\alpha = 2$ ($\alpha < 60°$).

In all the variants of the input data the iterative procedure yielded the solution with the accuracy 0.01 per cent in 20÷30 iterations.

In the second problem the height of W_0 is taken in the form of a cubic spline:

$$h(z) = h_1 + (z/d)^2(3 - 2z/d)(h_2 - h_1)$$

(see Fig. 1 (b)). Both problems are solved numerically by the above method with the same geometry of the regular waveguides: $kh_1 = 1.5\pi$, $kh_2 = 4.5\pi$. In this case only one driving mode exists in the left waveguide, and four ones exist in the right waveguide. The length of irregular domain W_0 varies in the range $kd = 5 \div 30$. The number of retained normal modes in series (17) is taken in the inverse dependence of kd in the range

$N = 25 \div 6.$ The excitation is assumed to be of the form of the driving mode of W_1 coming from $-\infty$.

The magnitudes of the reflection and transmission factors of driving modes in W_1 and W_2 for the both problems are shown in Fig. 3. As it was expected, the reflection factor $\left|p_{-1}^{(1)}\right|$ is negligibly small in the second problem where the shape of the regular domain is smooth. But this factor is also not large in the first problem, even in the case of a large angle of the wall break. The difference between transmission factors $\left|p_n^{(2)}\right|$, $n = 1, ..., 4$, in the problems is visible and it varies not too strongly with the length of the irregular domain.

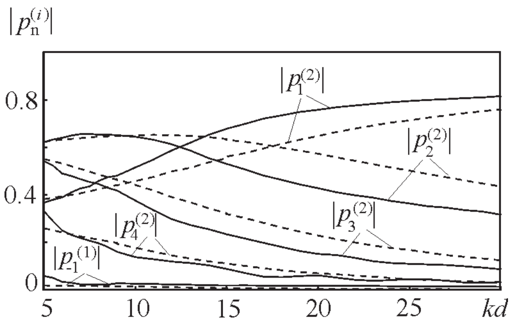

Fig. 3. *Magnitudes of the reflection and transmission factors in the waveguides shown in Fig. 1 (a) (solid lines) and Fig. 1 (b) (dashed lines) with $kh_1 = 1.5\pi$, $kh_2 = 4.5\pi$ excited by the dominant mode of the left waveguide with $p_1^{(1)} = 1$*

5. Conclusion

The problem on wave scattering in irregular waveguide with different asymptotics of the boundary at $-\infty$ and ∞ and the irregular domain with the continuously varied cross-section has been investigated by the cross-section method. Derivation of the main equations of the method has been applied, which does not use the differentiation of the non-uniformly converging series. The problem is reduced to a boundary value problem for a countable set of ordinary differential equations in the irregular part of the waveguide.

An iterative procedure has been proposed for solving these equations. It allows one to avoid the exponentially increasing errors in the stiff set of the differential equations to which the problem is reduced originally.

Numerical results have been obtained for two 2D problems with smoothly and nonsmoothly varied cross-section of the irregular domain. In particular, a test problem with the two-side excitation forming a standing field in the irregular part of the waveguide has been considered. The numerical results demonstrate high efficiency and stability of the method. For both problems the dependences of the reflection and transmission factors on the geometry are calculated and compared with each other.

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Численная реализация метода поперечных сечений для нерегулярных волноводов

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Исследуется рассеяние волн в нерегулярных волноводах. Рассматривается метод поперечных сечений для вычисления поля в волноводной системе, состоящей из двух регулярных волноводов, соединенных нерегулярной областью. В статье дается математически строгий вывод основных уравнений метода и предлагается итерационная процедура их решения. Алгоритм применяется к задачам с гладкими и негладкими неоднородностями. На примере модельной задачи, имеющей аналитическое решение, устанавливаются границы применимости метода.

Числова реалізація методу поперечних перерізів для нерегулярних хвилеводів

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Досліджується розсіювання хвиль в нерегулярних хвилеводах. Розглядається метод поперечних перерізів для обчислення поля в хвилеводній системі, що складається із двох регулярних хвилеводів, з'єднаних нерегулярною областю. В статті дається математично строге виведення основних рівнянь методу і пропонується ітераційна процедура для їх розв'язування. Алгоритм застосовується до задач з гладкими та негладкими нерегулярностями. На прикладі модельної задачі, що має аналітичний розв'язок, встановлюються границі застосовності методу.