ON SOME TOPOLOGICAL PROPERTIES FOR SPECIAL CLASSES OF BANACH SPACES. PART 2

P. KASYNOV, V. MEL’NIK

We consider some classes of infinite-dimensional Banach spaces with integrable derivatives. A compactness lemma for nonreflexive spaces is obtained. However some main topological properties for the given spaces are obtained.

This work is continuation of [1].

Theorem 1. \( W_0^* \subset C(S; H) \) with continuous embedding. Moreover, for every \( y, \xi \in W_0^* \) and \( s, t \in S \) the next formula of integration by parts takes place

\[
(y(t), \xi(t)) - (y(s), \xi(s)) = \int_s^t [(y'(\tau), \xi(\tau)) + (y(\tau), \xi'(\tau))]d\tau.
\]

(1)

In particular, when \( y = \xi \) we have:

\[
\frac{1}{2} (\|y(t)\|_H^2 - \|y(s)\|_H^2) = \int_s^t (y'(\tau), y(\tau))d\tau.
\]

Proof. To simplify the proof we consider \( S = [a, b] \) for some

\(-\infty < a < b < +\infty,\)

The validity of formula (1) for \( y, \xi \in C^1(S; V) \) is checked by direct calculation. Now let \( \varphi \in C^1(S) \) be such fixed that \( \varphi(a) = 0 \) and \( \varphi(b) = 1 \). Moreover, for \( y \in C^1(S; V) \) let \( \xi = \varphi y \) and \( \eta = y - \varphi y \). Then, due to (1):

\[
(\xi(t), y(t)) = \int_a^b \{\varphi'(s)(y(s), y(s)) + 2\varphi(s)(y'(s), y(s))\}ds,
\]

\[
-(\eta(t), y(t)) = \int_a^b \{-\varphi'(s)(y(s), y(s)) + 2(1 - \varphi(s))(y'(s), y(s))\}ds,
\]

from here for \( \xi_i \in L_{\eta_i}(S; V_i^*) \) and \( \eta_i \in L_{\eta_i}(S; H) \) \((i = 1, 2)\) such that \( y' = \xi_1 + \xi_2 + \eta_1 + \eta_2 \) it follows:

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\[ \|y(t)\|_H^2 = \int_0^b \left\{ \varphi'(s)(y(s), y'(s)) + 2\varphi(s)(y'(s), y(s)) \right\} ds - \frac{b}{2} \left( y'(s), y(s) \right) ds \leq \]

\[ \leq \max_{s \in S} \| \varphi'(s) \|_{C(S,Y^*)} \cdot \| y \|_{L_1(S; V)} + 2 \int_S \varphi(s) \left( y'(s), y(s) \right) ds \leq \]

\[ \leq \max_{s \in S} \| \varphi'(s) \|_{C(S,Y^*)} \| y \|_{L_1(S; V)} + \]

\[ + 2 \max_{s \in S} |\varphi(s) - 1| \left( \| \xi_1 \|_{L_{q_1}(S,V_1')} \| y \|_{L_p(S,V_1)} + \| \xi_2 \|_{L_{q_2}(S,V_2')} \| y \|_{L_p(S,V_2)} \right) \]

\[ + \| \eta_1 \|_{L_1(S; H)} \| y \|_{L_1(S; H)} + \| \eta_2 \|_{L_2(S; H)} \| y \|_{L_2(S; H)} \right) \leq \]

\[ \leq \max_{s \in S} \| \varphi'(s) \|_{C(S,Y^*)} \left( \| y \|_{L_p(S,V_1)} \max \{ \text{mes} \, (S)^{1/q_1} \} + \| y \|_{L_p(S,V_2')} \max \{ \text{mes} \, (S)^{1/q_2} \} \right) + \]

\[ + 2 \max_{s \in S} |\varphi(s) - 1| \left( \| \xi_1 \|_{L_{q_1}(S,V_1')} + \| \xi_2 \|_{L_{q_2}(S,V_2')} + \| \eta_1 \|_{L_1(S; H)} + \| \eta_2 \|_{L_2(S; H)} \right) \times \]

\[ \times \left( \| y \|_{L_p(S,V_1)} + \| y \|_{L_p(S,V_2')} + \| y \|_{C(S; H)} \max \{ \text{mes} \, (S)^{1/q_1}, \text{mes} \, (S)^{1/q_2} \} \right). \]

Hence, due to [1, theorem 3], definition of \( \| \cdot \|_X \), if we take in last inequality \( \varphi(t) = \frac{t-a}{b-a} \) for all \( t \in S \) we obtain

\[ \| y \|_{C(S; H)}^2 \leq C_2 \| y \|_{W_0}^2 + C_3 \| y \|_{w_0}^2 \| y \|_{C(S; H)}, \]

where \( C_1 \) is the constant from inequality \( \| y \|_{C(S,Y^*)} \leq C_1 \| y \|_{W_0} \) for every \( y \in W_0^* \),

\[ C_2 = 2 + \frac{C_1}{\min \{ \text{mes} \, (S)^{1/p_1}, \text{mes} \, (S)^{1/p_2} \}}, \quad C_3 = 2 \max \left\{ \max \{ \text{mes} \, (S)^{1/min \{q_1,p_2\}}, \text{mes} \, (S)^{1/q_2} \} \right\} \]

Remark that \( \frac{1}{+\infty} = 0 \), \( C_2, C_3 > 0 \). From (2) it obviously follows that

\[ \| y \|_{C(S; H)} \leq C_4 \| y \|_{W_0^*} \] for all \( y \in C^1(S; V) \),

where \( C_4 = \frac{C_3 + \sqrt{C_2^2 + 4C_2}}{2} \) does not depend on \( y \).

Now let us apply [1, theorem 4]. For arbitrary \( y \in W_0^* \) let \( \{ y_n \}_{n \geq 1} \) be a sequence of elements from \( C^1(S; V) \) converging to \( y \) in \( W_0^* \). Then in virtue of relation (3) we have

\[ \| y_n - y_k \|_{C(S; H)} \leq C_4 \| y_n - y_k \|_{W_0^*} \rightarrow 0, \]
therefore, the sequence \( \{ y_n \}_{n \geq 1} \) converges in \( C(S;H) \) and it has only limit \( \chi \in C(S;H) \) such that for a.e. \( t \in S \) \( \chi(t) = y(t) \). So, we have \( y \in C(S;H) \) and now the embedding \( W_0^* \subset C(S;H) \) is proved. If we pass to limit in (3) with \( y = y_n \) as \( n \to \infty \) we obtain the validity of the given estimation \( \forall y \in W_0^* \). It proves the continuity of the embedding \( W^* \) into \( C(S;H) \).

Now let us prove formula (1). For every \( y, \xi \in W_0^* \) and for corresponding approximating sequences \( \{ y_n, \xi_n \}_{n \geq 1} \subset C^1(S;V) \) we pass to the limit in (1) with \( y = y_n, \xi = \xi_n \) as \( n \to \infty \). In virtue of Lebesgue's theorem and \( W_0^* \subset C(S;V^*) \) with continuous embedding formula (1) is true for every \( y \in W_0^* \).

The theorem is proved.

In virtue of \( W^* \subset W_0^* \) with continuous embedding and due to the latter theorem the next statement is true.

**Corollary 1.** \( W^* \subset C(S;H) \) with continuous embedding. Moreover, for every \( y, \xi \in W^* \) and \( s,t \in S \) formula (1) takes place.

For every \( n \geq 1 \) let us define the Banach space \( W_n^* = \{ y \in X_n^* \mid y' \in X_n \} \) with the norm

\[
\| y \|_{W_n^*} = \| y \|_{X_n^*} + \| y' \|_{X_n},
\]

where the derivative \( y' \) is considered in sense of scalar distributions space \( \mathcal{D}^*(S;H_n) \). As far as

\[
\mathcal{D}^*(S;H_n) = \mathcal{L}(\mathcal{D}(S);H_n) \subset \mathcal{L}(\mathcal{D}(S);V_{\omega n}^*) = \mathcal{D}^*(S;V^*)
\]

it is possible to consider the derivative of an element \( y \in X_n^* \) in the sense of \( \mathcal{D}^*(S;V^*) \). Remark that for every \( n \geq 1 \) \( W_n^* \subset W_{n+1}^* \subset W^* \).

**Proposition 1.** For every \( y \in X^* \) and \( n \geq 1 \) \( P_n y' = (P_n y)' \), where derivative of element \( x \in X^* \) is in the sense of the scalar distributions space \( \mathcal{D}^*(S;V^*) \).

**Remark 1.** We pay our attention that in virtue of the previous assumptions the derivatives of an element \( x \in X_n^* \) in the sense of \( \mathcal{D}(S;V^*) \) and in the sense of \( \mathcal{D}(S;H_n) \) coincide.

**Proof.** It is sufficient to show that for every \( \varphi \in \mathcal{D}(S) \) \( P_n y' \varphi = (P_n y)' \varphi \). In virtue of definition of derivative in sense of \( \mathcal{D}^*(S;V^*) \) we have

\[
\forall \varphi \in \mathcal{D}(S) \quad P_n y' \varphi = -P_n y \varphi' = -P_n \int_S y(\tau) \varphi'(\tau) d\tau = \int_S P_n y(\tau) \varphi'(\tau) d\tau = -(P_n y)(\varphi) = (P_n y)' \varphi.
\]

The proposition is proved.
Due to [1, propositions 3, 4] it follows the next

**Proposition 2.** For every \( n \geq 1 \) \( W^*_n = P_n W^* \), i.e.
\[
W^*_n = \{ P_n y(\cdot) \mid y(\cdot) \in W^* \}.
\]

Moreover, if the triple \( (H_i)_{i \in \mathbb{Z}}; V_j; H \), \( j = 1, 2 \) satisfies condition (\( \gamma \)) with \( C = C_j \). Then for every \( y \in W^* \) and \( n \geq 1 \)
\[
\| P_n y(\cdot) \|_{W^*} \leq \max \{ C_1, C_2 \} \| y(\cdot) \|_{W^*}.
\]

**Theorem 2.** Let the triple \( (H_i)_{i \in \mathbb{Z}}; V_j; H \), \( j = 1, 2 \) satisfy condition (\( \gamma \)) with \( C = C_j \). We consider bounded in \( X^* \) set \( D \subset X^* \) and \( E \subset X \) that is bounded in \( X \). For every \( n \geq 1 \) let us consider
\[
D_n := \{ y_n \in X^* \mid y_n \in D \text{ and } y'_n \in P_n E \} \subset W^*_n.
\]
Then
\[
\| y_n \|_{W^*} \leq \| D \|_+ + C \| E \|_+ \quad \text{for all } n \geq 1 \text{ and } y_n \in D_n, \tag{4}
\]
where \( C = \max \{ C_1, C_2 \} \), \( \| D \|_+ = \sup_{y \in D} \| y \|_{X^*} \) and \( \| E \|_+ = \sup_{f \in E} \| f \|_X \).

**Remark 2.** Due to proposition 2 \( D_n \) is well-defined and \( D_n \subset W^*_n \) is true.

**Remark 3.** A priori estimates (like (4)) appear at studying of solvability of differential–operator equations, inclusions and evolutional variational inequalities in Banach spaces with maps of \( w_2 \)-pseudomonotone type by using Faedo–Galerkin method (see [2, 3]) at boundary transition, when it is necessary obtain a priori estimates of approximate solutions \( y_n \) in \( X^* \) and of its derivatives \( y'_n \) in \( X \).

**Proof.** Due to proposition 2 for every \( n \geq 1 \) and \( y_n \in D_n \)
\[
\| y_n \|_{W^*} = \| y_n \|_{X^*} + \| y'_n \|_X \leq \| D \|_+ + \| P_n E \|_+ \leq \| D \|_+ + \max \{ C_1, C_2 \} \| E \|_+.
\]
The theorem is proved.

Further, let \( B_0, B_1, B_2 \) be some Banach spaces such that
\[
B_0, B_2 \text{ are reflexive } B_0 \subset B_1 \text{ with compacting embedding} \tag{5}
\]
\[
B_0 \subset B_1 \subset B_2 \text{ with compacting embedding}. \tag{6}
\]

**Lemma 1.** ([4] lemma 1.5.1, p.71) Under the assumptions (5), (6) for an arbitrary \( \eta > 0 \) there exists \( C_\eta > 0 \) such that
\[
\| x \|_{B_1} \leq \eta \| x \|_{B_0} + C_\eta \| x \|_{B_2} \quad \forall x \in B_0.
\]

**Corollary 2.** Let the assumptions (5), (6) for the Banach spaces \( B_0, B_1 \) and \( B_2 \) are verified, \( p_1 \in [1; +\infty] \), \( S = [0, T] \) and the set \( K \subset L^{p_1}(S; B_0) \) such that

a) \( K \) is precompact set in \( L^{p_1}(S; B_2) \);
b) $K$ is bounded set in $L_{p_1}(S; B_0)$.

Then $K$ is precompact set in $L_{p_1}(S; B_1)$.

**Proof.** Due to lemma 1 and to the norm definition in $L_{p_i}(S; B_i)$, $i = 0, 2$ it follows that for an arbitrary $\eta > 0$ there exists such $C_\eta > 0$ that

$$
\|y\|_{L_{p_1}(S; B_1)} \leq 2\eta\|y\|_{L_{p_1}(S; B_0)} + 2C_\eta\|y\|_{L_{p_1}(S; B_2)} \quad \forall y \in L_{p_1}(S; B_0)
$$

Let us check inequality (7), when $p_1 \in [0, +\infty)$ (the case $p_1 = +\infty$ is direct corollary of lemma 1):

$$
\|y\|^2_{L_{p_1}(S; B_1)} = \int_S \|y(t)\|^2_{B_1} \, dt \leq \int_S \|y(t)\|_{B_0} + C_\eta\|y(t)\|_{B_2} \, dt \leq \eta\|y\|^2_{L_{p_1}(S; B_0)} + C_\eta\|y\|^2_{L_{p_1}(S; B_2)}
$$

Then

$$
= 2^{p_1-1}\left[\eta\|y\|^2_{L_{p_1}(S; B_0)} + C_\eta\|y\|^2_{L_{p_1}(S; B_2)}\right]
$$

$$
\leq 2^{p_1}\left[\eta\|y\|^2_{L_{p_1}(S; B_0)} + C_\eta\|y\|^2_{L_{p_1}(S; B_2)}\right] \quad \forall y \in L_{p_1}(S; B_0).
$$

The last inequality follows from

$$
\frac{a^{p_1} + b^{p_1}}{2} \leq (a + b)^{p_1} \leq 2^{p_1-1}(a^{p_1} + b^{p_1}) \quad \forall a, b \geq 0.
$$

Now let $\{y_n\}_{n \geq 1}$ be an arbitrary sequence from $K$. Then by the conditions of the given statement there exists $\{y_{n_k}\}_{k \geq 1} \subset \{y_n\}_{n \geq 1}$ that is a Cauchy subsequence in the space $L_{p_1}(S; B_2)$. So, thanks to inequality (7) for every $k, m \geq 1$

$$
\|y_{n_k} - y_{n_m}\|_{L_{p_1}(S; B_1)} \leq 2\eta\|y_{n_k} - y_{n_m}\|_{L_{p_1}(S; B_0)} + 2C_\eta\|y_{n_k} - y_{n_m}\|_{L_{p_1}(S; B_2)} \leq \eta C + 2C_\eta\|y_{n_k} - y_{n_m}\|_{L_{p_1}(S; B_2)}
$$

where $C > 0$ is a constant that does not depend on $m, k, \eta$. Therefore, for every $\varepsilon > 0$ we can choose $\eta > 0$ and $N \geq 1$ such that

$$
\eta C < \varepsilon / 2 \quad \text{and} \quad 2C_\eta\|y_{n_k} - y_{n_m}\|_{L_{p_1}(S; B_2)} < \varepsilon / 2 \quad \forall m, k \geq N
$$

Thus,

$$
\forall \varepsilon > 0 \quad \exists N \geq 1: \quad \|y_{n_k} - y_{n_m}\|_{L_{p_1}(S; B_1)} < \varepsilon \quad \forall m, k \geq N.
$$

This fact means, that $\{y_{n_k}\}_{k \geq 1}$ converges in $L_{p_1}(S; B_1)$. The corollary is proved.
Theorem 3. Let conditions (5), (6) for $B_0, B_1, B_2$ are satisfied, $p_0, p_1 \in [1;+\infty)$, $S$ be a finite time interval and $K \subset L_{p_1}(S;B_0)$ be such that

a) $K$ is bounded in $L_{p_1}(S;B_0)$;

b) for every $\varepsilon > 0$ there exists such $\delta > 0$ that from $0 < h < \delta$ it results in

$$
\int_S \|u(\tau) - u(\tau + h)\|_{B_2}^{P_0} d\tau < \varepsilon \quad \forall u \in K.
$$

Then $K$ is precompact in $L_{\min\{p_0,p_1\}}(S;B_1)$.

Furthermore, if for some $q > 1$ $K$ is bounded in $L_q(S;B_1)$, then $K$ is precompact in $L_p(S;B_1)$ for every $p \in [1,q)$.

Remark 4. Further we consider that every element $x \in (S \to B_1)$ is equal to $0$ out of the interval $S$.

Proof. At the beginning we consider the first case. For our goal it is enough to show, that it is possible to choose a Cauchy subsequence from every sequence $(y_n)_{n \geq 1} \subset K$ in $L_{\min\{p_0,p_1\}}(S;B_1)$. Due to corollary 2 it is sufficient to prove this statement for $L_{\min\{p_0,p_1\}}(S;B_2)$.

For every $x \in K \quad \forall h > 0 \quad \forall t \in S$ we put

$$
\int_{\tau}^{\tau+h} x(t) \, dt,
$$

where the integral is regarded in the sense of Bochner integral. We point out that $\forall h > 0 \quad x_h \in C(S;B_0) \subset C(S;B_2)$.

Fixing a positive number $\varepsilon$, we construct for a set $K \subset L_{p_0}(S;B_0) \subset L_{p_0}(S;B_2)$ a final $\varepsilon$-web in $L_{p_0}(S;B_2)$. For $\varepsilon > 0$ we choose $\delta > 0$ from (8). Then for every fixed $h \ (0 < h < \delta)$ we have:

$$
\|x_h(t + u) - x_h(t)\|_{B_2} \leq \frac{1}{h} \int_{t}^{t+h} x(\tau) \, d\tau - \int_{t}^{t+h} x(\tau) \, d\tau \leq \frac{1}{h} \int_{t}^{t+h} x(\tau) \, d\tau.
$$

Moreover, from the Hölder inequality we obtain

$$
\int_{t}^{t+h} \|x(\tau + u) - x(\tau)\|_{B_2} \, d\tau \leq \left( \frac{1}{h} \right)^{p_0} \left( \int_{t}^{t+h} \|x(\tau + u) - x(\tau)\|_{B_2}^{p_0} \, d\tau \right)^{\frac{1}{p_0}} \leq
$$
\[
\left( \frac{1}{T} \right) \int_0^T \| x(\tau) - x(\tau) \|_{B_2}^p \, d\tau \leq \left( \frac{1}{h} \right)^{\frac{1}{p_0}} \frac{1}{p_0} \left( \frac{1}{h} \int_0^T \| x(\tau) \|_{B_2}^p \, d\tau \right)^{\frac{1}{p_1}} \frac{1}{p_1} \leq \left( \frac{C}{h} \right)^{\frac{1}{p_1}} \frac{1}{p_1},
\]

the family of functions \( \{x_h\}_{x \in K} \) is uniformly bounded, because of the constant \( C \geq 0 \) does not depend on \( x \in K \). Hence, \( \forall h: 0 < h < \delta \) the family of functions \( \{x_h\}_{x \in K} \) is precompact in \( C(S; B_2) \), so in \( L_{\min\{p_0, p_1\}}(S; B_2) \) too.

On the other hand, \( \forall 0 < h < \delta, \forall x \in K, \forall t \in S \)
\[
\| x(t) - x_h(t) \|_{B_2} \leq \frac{1}{h} \int_t^{t+h} \| x(\tau) - x(t) \|_{B_2} \, d\tau \leq
\]
\[
\leq \left( \frac{1}{h} \right)^{\frac{1}{p_1}} \frac{1}{p_1} \left( \int_0^T \| x(\tau) \|_{B_2}^p \, d\tau \right)^{\frac{1}{p_1}} \frac{1}{p_1} \leq \left( \frac{C}{h} \right)^{\frac{1}{p_1}} \frac{1}{p_1},
\]

From here, taking into account inequality (8) we receive:
\[
\left( \frac{1}{h} \int_0^T \| x(t) - x_h(t) \|_{B_2}^p \, d\tau \right)^{\frac{1}{p_0}} \leq \frac{1}{h} \int_0^T \| x(t) - x(t + \tau) \|_{B_2}^p \, d\tau \leq \left( \frac{1}{h} \right)^{\frac{1}{p_0}} \frac{1}{p_0} \left( \int_0^T \| x(t) - x(t + \tau) \|_{B_2}^p \, d\tau \right)^{\frac{1}{p_0}} \frac{1}{p_0} = e^\frac{1}{p_0}.
\]

Hence, by virtue of the precompactness of system \( \{x_h\}_{x \in K} \) in \( L_{\min\{p_0, p_1\}}(S; B_2) \) \( \forall 0 < h < \delta \) we have that \( K \) is a precompact set in \( L_{\min\{p_0, p_1\}}(S; B_2) \).

Let us consider the second case. Assume that for some \( q > 1 \) the set \( K \) is bounded in \( L_q(S; B_1) \). Similarly to the previous case, it is enough to show that for every \( p \in [1; q) \) and \( \{y_n\}_{n \geq 1} \subset K \) there exists a subsequence \( \{y_{n_k}\}_{k \geq 1} \subset \{y_n\}_{n \geq 1} \) and \( y \in L_p(S; B_1) \) so that
\[
y_{n_k} \to y \text{ in } L_p(S; B_1) \text{ as } k \to \infty.
\]
Because of \( y_n \to y \) in \( L_{\min\{p_0, p_1\}}(S; B_1) \), up to a subsequence, as \( n \to \infty \), we have \( \exists \{y_{n_k}\}_{k \geq 1} \subset \{y_n\}_{n \geq 1} \) such that \( \lambda(B_{n_k}) \to 0 \) as \( k \to \infty \), where \( B_n := \{ t \in S : \|y_n(t) - y(t)\|_{B_1} \geq 1 \} \) for every \( n \geq 1 \), \( \lambda \) is the Lebesgue measure on \( S \).

Then for every \( k \geq 1 \)

\[
\int_S \|y_{n_k}(s) - y(s)\|_{B_1}^p \, ds = \int_{A_{n_k}} \|y_{n_k}(s) - y(s)\|_{B_1}^p \, ds + \int_{B_{n_k}} \|y_{n_k}(s) - y(s)\|_{B_1}^p \, ds \\
+ \left( \int_S \|y_{n_k}(s) - y(s)\|_{B_1}^q \, ds \right)^{\frac{p}{q}} \left( \lambda(B_{n_k}) \right)^{\frac{q-p}{q}} =: I_{n_k} + J_{n_k},
\]

where \( A_n = S \setminus B_n \) for every \( n \geq 1 \).

It is clear that \( J_{n_k} \to 0 \) as \( k \to \infty \). Let us consider \( I_{n_k} \). Since \( \{y_{n_k}\}_{k \geq 1} \) is precompact in \( L_{\min\{p_0, p_1\}}(S; B_1) \), there exists such \( \{y_{m_k}\}_{k \geq 1} \subset \{y_{n_k}\}_{k \geq 1} \) that \( y_{m_k}(t) \to y(t) \) in \( B_1 \) as \( k \to \infty \) almost everywhere in \( S \). Setting

\[
\forall k \geq 1, \quad \forall t \in S \quad \varphi_{m_k}(t) := \begin{cases} \|y_{m_k}(t) - y(t)\|_{B_1}^p, & t \in A_n, \\ 0, & \text{otherwise}, \end{cases}
\]

using definition of \( A_{m_k} \), sequence \( \{\varphi_{m_k}\}_{k \geq 1} \) satisfies the conditions of the Lebesgue theorem with the integrable majorant \( \phi \equiv 1 \). So \( \varphi_{m_k} \to 0 \) in \( L_1(S) \) as \( k \to \infty \). Thus, within to a subsequence, \( y_n \to y \) in \( L_q(S; B_1) \).

The theorem is proved.

Let Banach spaces \( B_0, B_1, B_2 \) satisfy all assumptions (5), (6), \( p_0, p_1 \in [1; +\infty) \) be arbitrary numbers. We consider the set with the natural operations

\[
W = \{ v \in L_{p_0}(S; B_0) | v' \in L_{p_1}(S; B_2) \},
\]

where the derivative \( v' \) of an element \( v \in L_{p_0}(S; B_0) \) is considered in the sense of the scalar distribution space \( D(S; B_2) \). It is clear, that

\[
W \subset L_{p_0}(S; B_0).
\]

**Theorem 4.** The set \( W \) with the natural operations and the graph norm

\[
\|v\|_W = \|v\|_{L_{p_0}(S; B_0)} + \|v'\|_{L_{p_1}(S; B_2)}
\]

is a Banach space.
Proof. The executing of the norm properties for $\| \cdot \|_{W}$ immediately follows from its definition. Now we consider the completeness of $W$ referring to just defined norm. Let $\{y_n\}_{n \geq 1}$ be a Cauchy sequence in $W$. Hence, due to the completeness of $L_{p_0}(S;B_0)$ and $L_{p_1}(S;B_2)$ it follows that for some $y \in L_{p_0}(S;B_0)$ and $v \in L_{p_1}(S;B_2)$

$$y_n \rightarrow y \text{ in } L_{p_0}(S;B_0) \text{ and } y'_n \rightarrow v \text{ in } L_{p_1}(S;B_2) \text{ as } n \rightarrow +\infty.$$ 

Due to [5, lemma IV.1.10] and in virtue of continuous dependence of the derivative by the distribution in $\mathcal{D}'(S;B_2)$ (see [5, p. 169]) it follows, that $y'=v \in L_{p_1}(S;B_2)$.

The theorem is proved.

Theorem 5. Under conditions (5), (6) $W \subset C(S;B_2)$ with the continuous embedding.

Proof. For a fixed $y \in W$ let us show that $y \in C(S;B_2)$ . Let us put

$$\xi(t) = \int_{t_0}^{t} y'(\tau) d\tau \quad \forall t_0, t \in S.$$ 

The integral is well-defined because $y' \in L_1(S;B_2)$. On the other hand, from the inequality [5, p. 153]

$$\|\xi(t) - \xi(s)\|_{B_2} \leq \int_{t}^{s} |y'(\tau)| d\tau \quad \forall s \geq t, \ s \in S$$

it follows that $\xi \in C(S;B_2)$. Due to [5] (lemma IV.1.8) $\xi'=y'$, so from [5] (lemma IV.1.9) it follows that

$$y(t) = \xi(t) + z \quad \text{for a.e. } t \in S.$$ 

for some fixed $z \in B_2$.

Thus the function $y$ also lies in $C(S;B_2)$.

In virtue of the continuous embedding of $L_{p_1}(S;B_2)$ in $L_1(S;B_2)$ we have that for some constant $k > 0$, which does not depend on $y$,

$$\|\xi(t)\|_{B_2} \leq \int_{S} |y'(\tau)| d\tau \leq k\|y'\|_{L_{p_1}(S;B_2)} \quad \forall t \in S.$$ 

From here, due to the continuous embedding $B_0 \subset B_2$, we have

$$\|z\|_{B_2} = (\text{mes}(S))^{1/p_1} \left( \int_{S} |y|_{B_2}^{p_1} ds \right)^{1/p_1} = \|y - \xi\|_{L_{p_1}(S;B_2)} \leq$$

$$\leq k_1 \left( \|y\|_{L_{p_1}(S;B_2)} + \|\xi\|_{C(S;B_2)} \right) \leq k_2 \left( \|y\|_{L_{p_0}(S;B_0)} + \|y'\|_{L_{p_1}(S;B_2)} \right),$$
where \( \text{mes}(S) \) is the “length” (the measure) of \( S \), \( k_3 > 0 \) is a constant that does not depend on \( y \in W \). Therefore, from the last two relations there exists \( k_3 \geq 0 \) such that

\[
\|y\|_{C(S,B_2)} \leq k_3 \|y\|_W \quad \forall y \in W.
\]

The theorem is proved.

The next result represents a generalization of the compactness lemma [4, theorem 1.5.1, p. 70] into the case \( p_0, p_1 \in [1;+\infty) \).

**Theorem 6.** Under conditions (5), (6), for all \( p_0, p_1 \in [1;+\infty) \) the Banach space \( W \) is compactly embedded in \( L_{p_0}(S;B_1) \).

**Proof.** At the beginning we prove the compact embedding of \( W \) in \( L_1(S;B_2) \).

For every \( y \in W \) and \( h \in \mathbb{R} \) let us take

\[
y_h(t) = \begin{cases} y(t + h), & \text{if } t + h \in S, \\ 0, & \text{otherwise.} \end{cases}
\]

In virtue of theorem 5 the given definition is correct.

**Lemma 2.** For every \( y \in W \) and \( h \in \mathbb{R} \)

\[
\|y - y_h\|_{L_1(S;B_2)} \leq h\|y\|_{L_1(S;B_2)}.
\]

**Proof.** Let \( y \in W \) be fixed. Then

\[
\|y - y_h\|_{L_1(S;B_2)} = \int_S \|y(t + h) - y(t)\|_{B_2} dt = \int_t^{t+hb} \|y'(\tau)\|_{B_2} d\tau dt.
\]

Let us put \( g_y(t) = \int_t^{t+hb} y'(\tau) d\tau = y(t + h) - y(t) \quad \forall t \in S, \ i=1,2 \). Due to theorem 5 the element \( g_y \in C(S;B_2) \). So, as \( S \) is a compact set, we have that \( g_y \in L_1(S;B_2) \). Therefore, due to proposition [6, p.191] with \( X = L_1(S;B_2) \) and to [1, theorem 2] it follows the existence of \( h_y \in L_\infty(S;B_2^*) \equiv X^* \) such that

\[
\int_S \|g_y(t)\|_{B_2} dt = \int_S \langle h_y(t), g_y(t) \rangle_{B_2} dt \quad \text{and} \quad \|h_y\|_{L_\infty(S;B_2^*)} = 1
\]

Hence,

\[
\int_t^{t+hb} \|y'(\tau)\|_{B_2} d\tau = \int_S \|g_y(t)\|_{B_2} dt = \int_S \langle h_y(t), g_y(t) \rangle_{B_2} dt = \int_t^{t+hb} \langle h_y(t), y'(\tau) \rangle_{B_2} d\tau dt = \int_t^{t+hb} \langle h_y(t), y'(\tau) \rangle_{B_2} d\tau dt
\]
\[\int_{S\gamma-h}^{\tau} \left\{ h_{\gamma}(t), y'(\tau) \right\} dtd\tau = \int_{S\gamma-h}^{\tau} \left\{ \int_{S} h_{\gamma}(t)dt, y'(\tau) \right\} d\tau \leq \\text{esssup}_{t\in S} \left\{ \|h_{\gamma}(t)\| + h \int_{S} \|y'(\tau)\| d\tau \right\} \leq h\|y'\|_{L_{1}(S;B_{2})}^{2} \cdot \]

So, we have obtained necessary estimation (9).

The lemma is proved.

Let us continue the proof of the given theorem. Let \( K \subset W \) be an arbitrary bounded set. Then for some \( C > 0 \)

\[\|y\|_{L_{p_{0}}(S;B_{0})} \leq C, \quad \|y'\|_{L_{p_{1}}(S;B_{2})} \leq C \quad \forall y \in K. \quad (10)\]

In order to prove the precompactness of \( K \) in \( L_{1}(S;B_{1}) \) let us apply theorem 4 with \( B_{0} = B_{0}, \; B_{1} = B_{1}, \; B_{2} = B_{2}, \; p_{0} = 1, \; p_{1} = p_{1} \). Due to estimates (9) and (10) the all conditions of the given theorem hold. So, the set \( K \) is precompact in \( L_{1}(S;B_{1}) \) and hence in \( L_{1}(S;B_{2}) \). In virtue of theorem 5 and the Lebesgue theorem it follows that the set \( K \) is precompact in \( L_{p_{0}}(S;B_{0}) \). Hence, due to corollary 2 we obtain the necessary statement.

The theorem is proved.

**Proposition 3.** Let Banach spaces \( B_{0}, B_{1}, B_{2} \) satisfy conditions (5), (6), \( p_{0}, p_{1} \in [1;+\infty) \), \( \{u_{h}\}_{h \in \mathbb{A}} \subset L_{p_{1}}(S;B_{0}) \), where \( I = (0, \delta) \subset \mathbb{R}_{+}, \; S=[a,b] \) such that

a) \( \{u_{h}\}_{h \in \mathbb{A}} \) is bounded in \( L_{p_{1}}(S;B_{0}) \);

b) there exists such \( c : I \rightarrow \mathbb{R}_{+} \) that \( \lim_{n \rightarrow \infty} c \left( \frac{b-a}{2^{n}} \right) = 0 \) and

\[\forall h \in I \quad \int_{S} |u_{h}(t) - u_{h}(t + h)|^{p_{0}}_{B_{2}} dt \leq c(h)h^{p_{0}}. \]

Then there exists \( \{h_{n}\}_{n \geq 1} \subset I \) ( \( h_{n} \searrow 0^{+} \) as \( n \rightarrow \infty \)) so that \( \{u_{h_{n}}\}_{n \geq 1} \) converges in \( L_{p_{0}}(S;B_{1}) \).

**Remark 5.** We assume \( u_{h}(t) = 0 \) when \( t > b \).

**Remark 6.** Without loss of generality let us consider \( S = [0,1] \).

**Proof.** At first we prove this statement for \( L_{p_{0}}(S;B_{2}) \). In virtue of Minkowski inequality for every \( h = \frac{1}{2^{N}} \in I \) and \( k \geq 1 \)

\[\left( \int_{0}^{1} |u_{h}(t) - u_{h_{k}}(t)|^{p_{0}}_{B_{2}} dt \right)^{\frac{1}{p_{0}}} \leq \left( \int_{0}^{1} |u_{h}(t) - u_{h}(t + h)|^{p_{0}}_{B_{2}} dt \right)^{\frac{1}{p_{0}}} + \]

\[\left( \int_{0}^{1} |u_{h_{k}}(t) - u_{h}(t + h)|^{p_{0}}_{B_{2}} dt \right)^{\frac{1}{p_{0}}} \]

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\[ + \left[ \int_0^1 \|u_h(t + h) - u_h(t + h)\|_{B_2}^{p_0} dt \right]^{1/p_0} + \left[ \int_0^1 \|u_h(t) - u_h(t)\|_{B_2}^{p_0} dt \right]^{1/p_0} \leq \]

\[ \leq \frac{1}{c^{p_0}} (h)h + \frac{1}{c^{p_0}} \left( \sum_{i=0}^{2^{k-1}} \int_0^{2^{k-1}} \|u_h(t) - u_h(t)\|_{B_2}^{p_0} dt \right) \leq \frac{1}{c^{p_0}} (h)h + \frac{1}{c^{p_0}} (h/2^k) + \]

\[ - \frac{1}{2^k} \int_0^{2^{k-1}} \|u_h(t) - u_h(t)\|_{B_2}^{p_0} dt \leq \frac{1}{c^{p_0}} (h)h + \frac{1}{c^{p_0}} (h/2^k) + \]

\[ + \left[ \int_0^1 \|u_h(t) - u_h(t + h)\|_{B_2}^{p_0} dt \right]^{1/p_0} + \left[ \int_0^1 \|u_h(t + h) - u_h(t + h)\|_{B_2}^{p_0} dt \right]^{1/p_0} \leq \]

\[ + \frac{1}{c^{p_0}} (h)h + \frac{1}{c^{p_0}} (h/2^k) + \]

\[ + \frac{1}{2^k} \int_0^{2^{k-1}} \|u_h(t) - u_h(t + h)\|_{B_2}^{p_0} dt \leq \frac{1}{c^{p_0}} (h)h + \frac{1}{c^{p_0}} (h/2^k) + \]

\[ + \int_0^1 \frac{1}{2^{N+k}} (t) - u_{1/2^{N+k}} (t)\|_{B_2}^{p_0} dt \right]^{1/p_0} \leq \frac{1}{c^{p_0}} \left( \frac{1}{2^N} \right) + c^{p_0} \left( \frac{1}{2^{N+k}} \right). \]

So, for every \( N \geq 1 \) and \( k \geq 1 \) it results in

\[ \int_0^1 \frac{1}{2^{N+k}} (t) - u_{1/2^{N+k}} (t)\|_{B_2}^{p_0} dt \right]^{1/p_0} \leq \frac{1}{c^{p_0}} \left( \frac{1}{2^N} \right) + c^{p_0} \left( \frac{1}{2^{N+k}} \right). \]

In virtue of assumption b) we can choose \( \{h_n\}_{n=1} = \left\{ \frac{1}{2^m} \right\}_{m=1} \bigcap \Gamma \) such that \( c(h_n) \to 0 \) as \( n \to \infty \). So, the sequence \( \{u_{h_n}\}_{n=1} \) is fundamental in \( L_{p_0} (S; B_2) \).

Because of \( B_0 \subset B_1 \) with compact embedding, the sequence \( \{u_{h_n}\}_{n=1} \) is bounded in \( L_{min(p_0, p_1)} (S; B_0) \); due to corollary 2 it follows that \( \{u_{h_n}\}_{n=1} \) is fundamental in \( L_{min(p_0, p_1)} (S; B_1) \).
The proposition is proved.
Now we combine all results to obtain the necessary a priori estimate.

**Theorem 7.** Let all conditions of theorem 2 are satisfied and \( V \subset H \) with compact embedding. Then (4) be true and the set
\[
\bigcup_{n \geq 1} D_n
\]
is bounded in \( C(S;H) \) and precompact in \( L_p(S;H) \) for every \( p \geq 1 \).

**Proof.** Estimation (4) follows from theorem 2. Now we use compactness theorem 6 with \( B_0 = V \), \( B_1 = H \), \( B_2 = V^* \), \( p_0 = 1 \), \( p_1 = 1 \). Remark that \( X^* = L_1(S;V) \) and \( X = L_1(S;V^*) \) with continuous embedding. Hence, the set
\[
\bigcup_{n \geq 1} D_n
\]
is precompact in \( L_1(S;H) \).

In virtue of (4) and theorem 1 on continuous embedding of \( W^* \) in \( C(S;H) \), it follows that the set
\[
\bigcup_{n \geq 1} D_n
\]
is bounded in \( C(S;H) \).

Further, by using standard conclusions and the Lebesgue theorem we obtain the necessary statement.
The theorem is proved.

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