Condensed Matter Physics, 2004, Vol. 7, No. 3(39), pp. 551–563

CONDENSED
MATTER
PHYSICS

# Pattern formation in neural dynamical systems governed by mutually Hamiltonian and gradient vector field structures

V.V.Gafiychuk [1], A.K.Prykarpatsky [2]

[1] Institute of computer modeling, Krakow University of Technology,
24 Warszawska Street, 31155, Krakow, Poland

[2] Department of Applied Mathematics
at the AGH University of Science and Technology,
30 Mickiewicz Al. Bl. A4, 30059 Krakow, Poland

Received April 29, 2004, in final form July 21, 2004

We analyze dynamical systems of general form possessing gradient (symmetric) and Hamiltonian (antisymmetric) flow parts. The relevance of such systems to self-organizing processes is discussed. Coherent structure formation and related gradient flows on matrix Grassmann type manifolds are considered. The corresponding graph model associated with the partition swap neighborhood problem is studied. The criterion for emerging gradient and Hamiltonian flows is established. As an example we consider nonlinear dynamics in a neuron network system described by a simulative vector field. A simple criterion was written in order to establish conditions for the formation of an oscillatory pattern in a model neural system under consideration.

**Key words:** *dynamical system, gradient flow, Hamiltonian flow, self-organization, neural network*

**PACS:** *05.45.-a, 07.05.Mh, 05.65.+b*

## 1. Introduction. Complex dynamics – immanent property of nonlinear systems

The emergence of spatio-temporal patterns in nonlinear dynamical systems is a very well established fact. Accounts of the state of art on the problem can be found in many books written recently on this field. In the last couple of decades especially popular became investigations of pattern formation and propagation phenomena in self-organizing systems (see for example [1–4]).

Self-organization phenomena are inherent to a wide class of nonlinear systems.

 

They are so popular that a new interdisciplinary science has been established [1,2]. The goal of investigations of these science centers is to describe the evolution of nonlinear systems and to explore general rules of the pattern formation and to reveale the structures arising in the process of this evolution. Diverse mathematical tools are used to connect general properties of these phenomena in all specific areas. Meanwhile there is no strict definition of the concepts of self-organization. In spite of many attempts to express exact meaning of this term different people imply by self-organization different meaning. By the term “self-organization” people consider patterns in dissipative and even Hamiltonian systems not caring that initially this term was introduced to specify certain phenomena in open dissipative systems. Such an approach frequently has got some sense because not all properties emerging in these systems are so easy to distinguish. It is surprising that the evolution of the system as a whole is difficult to predict by considering the local interaction of system components. At the same time, the local information available amongst components of the system is of major importance for self-organization and formation of the pattern.

In this case the problem of classification of possible evolutionary dynamics is still not resolved. The question whether the patterns arising in the physical system belong to self-organization phenomena or are just parts of nonlinear complex dynamics is far from being simple to answer. The matter is that there is no strict boundary between the phenomena emerging in the process of system evolution. The classification of self-organization phenomena is possible to perform by quantifying entropy as a criterion of the order of the system showing that self-organization is a result of entropy decreasing [1]. Such an approach was practically realized by Yu.L. Klimontovich in [5] where he showed that the turbulence is much more self-organized than the corresponding laminar flow. This result was obtained by calculating the entropy production in two regimes: laminar and turbulent. But, eventually, Klimontovich's result is still a unique one because it is very difficult to calculate the entropy production for real evolutionary systems due to their essential nonlinearity. In fact, in order to be synergetic these systems must be nonlinear and the behavior of the whole systems is not a simple sum of the behaviors of their parts. Moreover, the dynamics of the systems must be a consequence of the intrinsic properties governing the system. It should, naturally, possess feedbacks (feedback is an output sent back into a system as the input and can be positive leading to self-production and negative corresponding to the system stabilizing). As a result the dynamics leads to nonlinear emergent behavior. In this case finding exact special solutions of such a system is a challenge and usually the computer simulation methods are used to analyze the system dynamics. To get the solutions that have a physical meaning and have practical sense, the physical understanding of the phenomena and the mechanisms of formation of these solutions should be clearly realized. The situation becomes much easier if it is possible to develop some approximate methods of analyzing the possible solutions with special properties. Dealing with the evolution it is beneficial to make use of the general extremality principles which determine the direction of evolution. In finding the extremals with respect to certain variation principle such an approach often makes it possible to understand the system organization laws. In conservative systems invariants play a role of the main constraints involved on the system [6]. For dissipative systems some variation principles can be constructed as well. Some approaches in this direction were made in [7,8]. In this case using model functions for an approximate attractor present in the system appears to be very productive and makes it possible to find an approximate form of the governing solutions. In this case the evolution of the parameters characterizing the evolution of model functions is often of gradient type.

It should be noted that the development of variational principles is also beneficial in describing specific nonlinear properties of natural systems such as living systems, market structures etc. (see [9,10]). The matter is that natural systems evolve towards increasingly adapted states, building higher and higher degrees of order starting from almost nothing. Their behavior could be understood as adaptive self-organization and the dynamics could be frequently described as a gradient dynamics with respect to a certain potential. Herein below by considering a certain example of a nonlinear system we develop an approach which enables us to analyze the existence of stable patterns and periodic structures in the nonlinear system and extends computer assisted methods to its investigation. As an example of general system we consider a dynamical system of neurons which in reality has a very complex behavior. We only present our approach to the study of this neural system but do not claim a specific result which would explain the real neuron dynamics.

## 2. Physical models and related gradient and Hamiltonian flows

It is well known [11] that synaptic connections in biological neural networks are seldom symmetric since the signal sent by neurons along their axons are sharp spikes and the relevant information is not contained in the spikes themselves but in the so-called firing rates, which depends on the magnitude of the membrane potentials which governs all the process. On the other hand it should be pointed out that the recent neurophysiological observation of extremely low firing rates [12] without some doubt on the general usefulness of this notion is really the relevant neural variable. Thereby one can use some natural continuous variables to describe neural networks as dynamical systems of special structure similar to gradient (symmetric) and Hamiltonian skew-symmetrical flows. This gives rise to making use of a lot of methods and techniques of studying the structural stability of the networks and the existence of the so-called coherent temporal structures fitting for learning process.

Based on the considerations above one can introduce a class of nonlinear dynamical systems

$$\frac{\mathrm{d}u}{\mathrm{d}t} = K(u), \tag{1}$$

where $M \ni u$ is some smooth finite-dimensional metrizible manifold, $K : M \to T(M)$ is a smooth vector field on $M$ modeling the information transfer process in a biological neural network under regard. The question is what conditions should be involved on the flow (1) for it to be represented as follows:

$$K(u) = -\mathrm{grad}V(u) - \vartheta(u)\nabla H(u), \tag{2}$$

that is as a mixed sum of a gradient flow and a Hamiltonian flow on $M$. Here $V : M \to \mathbb{R}$ is the potential function and $H : M \to \mathbb{R}$ is the Hamiltonian function relevant to the flow (1), grad $:= g^{-1}(u)\nabla,\ \nabla := \partial/\partial u : u \in M\}$, $g : T(M)\times T(M) \to \mathbb{R}_+$ is a Riemannian metrics and $\vartheta : T^*(M) \to T(M)$ is a Poisson structure on $M$.

Thus we need to find the corresponding metrics and Poisson structure on $M$ subject to which the representation (2) holds on $M$. We shall dwell on this topic more in detail within the proceeding chapter.

## 3. Poissonian structure analysis

Assume first that the representation (2) holds, that is

$$-\vartheta(u)\nabla H(u) = K(u) + \mathrm{grad}V(u) := K_V(u) \tag{3}$$

for all $u \in M$ and some relevant $\vartheta$ and $g$ structures on $M$. This means therefore that the constructed vector field (3) is exactly Hamiltonian. Thereby one has ([13]) the expression

$$\vartheta^{-1}(u) = \varphi'(u) - \varphi'^{*}(u), \tag{4}$$

where $\varphi \in T^*(M)$ is some nonsymmetric solution to the linear determining equation

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} + K_V'^{*}\varphi = \nabla\mathcal{L}. \tag{5}$$

Here, by definition, the flow $K_V$ is defined as

$$\frac{\mathrm{d}u}{\mathrm{d}\tau} = K_V(u) \tag{6}$$

and $\mathcal{L} : M \to \mathbb{R}$ is a suitable smooth function chosen for convenience when solving (4). It is clear (see [13]) that the symplectic structure (4) does not depend on the choice of the function $\mathcal{L} : M \to \mathbb{R}$.

As the second step, assume that the metrics and Poisson structures on $M$ are given a priori. Then due to (5) the following equation on an element $\psi \in T(M)$ for determining the potential function $V : M \to \mathbb{R}$ holds:

$$\varphi' \cdot K + \varphi' \cdot \psi + K'^{*} \cdot \varphi + \psi'^{,*} \cdot \varphi = \nabla\mathcal{L}, \tag{7}$$

where the element $\varphi \in T^*(M)$ has been assumed also to be known a priori as a solution to the equation (4). The expression (7) is a linear second order equation in partial derivatives on the potential function $V : M \to \mathbb{R}$. If this equation is compatible, then its solution exists and the decomposition (2) holds.

As one can check, the equation (7) almost everywhere possesses a solution for the vector $\psi = \mathrm{grad} V$, that is the following expression

$$\mathrm{grad} V = \psi = g^{-1} \nabla V \tag{8}$$

holds on $M$ for some $\psi \in T(M)$. Thereby, one gets

$$\nabla V = g\psi. \tag{9}$$

Making use now of the well-known Volterra condition (see [13]), $(\nabla V)'^* \equiv (\nabla V)'$, we obtain the following criterion on the metrics $g : T(M) \times T(M) \to \mathbb{R}_+$ :

$$(g\psi)'^* = (g\psi)'. \tag{10}$$

Since from (9) one also follows that

$$\langle g\psi, u_x \rangle = \langle \nabla V, u_x \rangle = \frac{\mathrm{d}V}{\mathrm{d}x}, \tag{11}$$

the condition (11) is evidently equivalent to such one:

$$(g\psi)'^* u_x - \frac{\mathrm{d}}{\mathrm{d}x}(g\psi) = 0. \tag{12}$$

Calculating the left hand side expression of (12) one gets the following final result:

$$(g'^* u_x - g' u_x)\psi = g\psi' u_x - \psi'(g u_x), \tag{13}$$

which is feasible at check, if the metrics is given. Otherwise, if this is not the case, the linear expression (13) determines a suitable metrics as its solution with respect to the mapping $g : T(M) \times T(M) \to \mathbb{R}_+$. As soon as the equation (13) is compatible, its solution exists defining a suitable metrics on the manifold $M$.

## 4. Conditions of temporal pattern structure formation

Let us proceed to considering a network with two groups of neuron $\{x_i \in \mathbb{R} : i = \overline{1, n}\}$ and $\{y_j \in \mathbb{R} : j = \overline{1, m}\}$, connected in such a way that within both groups the synaptic strengths are symmetric, whereas between groups they are antisymmetric. That is, neurons $\{x\}$ are excitatory to $\{y\}$ and neurons $\{y\}$ are inhibitory to $\{x\}$ . This model is expressed in the form (2), where

$$\begin{aligned} V &= \sum_{i=1}^{n} \left( -\frac{1}{2}\beta_1 x_i^2 + \beta_2 \frac{x_i^4}{4} \right) + \frac{1}{2} \sum_{i,j=1}^{n} \beta_{i,j}^{(1)} x_i x_j \\ &\quad + \sum_{j=1}^{m} \left( -\frac{1}{2}\beta_4 y_j^2 + \beta_5 \frac{y_j^4}{4} \right) + \frac{1}{2} \sum_{i,j=1}^{m} \beta_{i,j}^{(2)} y_i y_j \, , \end{aligned} \tag{14}$$

$$H = \frac{1}{2} \left( \sum_{i=1}^{n} x_i^2 + \sum_{j=1}^{m} y_j^2 + \sum_{i=1}^{n} \sum_{j=1}^{m} w_{ij} x_i y_j \right) \tag{15}$$

with the standard metrics $g = \mathbf{1}$, a skew-symmetric Poisson structure $\vartheta = J \in Sp(\mathbb{R}^{(n+m)})$, $u = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m\}$,

$$g = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & I_{(n,m)} \\ -I_{(n,m)} & 0 \end{pmatrix} \tag{16}$$

or

$$J = \begin{pmatrix} J_{(n)} & 0 \\ 0 & J_{(m)} \end{pmatrix} \tag{17}$$

with constant $\beta$ and elements $w_{ij}$ being parameters, $I_{(n,m)} = \{\delta_{ij} : i = \overline{1,n},\ j = \overline{1,m}\}$, $J_{(n)}$ and $J_{(m)}$ being some skew-symmetric matrices.

It is worth mentioning here that the representation (2) with structures (16) is not unique and some other solutions to the equation (13) can be found.

This system (14) as we shall demonstrate below possesses a so-called coherent-temporal structure important in studying learning processes in biological neural networks.

Assume for simplicity that all $\beta$ - parameters are proportional to a small enough parameter $\varepsilon > 0$, that is $\{\beta\} \simeq \{\varepsilon\beta\}$ and consider first our flow (2) at $\varepsilon = 0$. It is easy to see that our model then possesses a closed orbit in the space of $\{x\}$ and $\{y\}$ - parameters, say $\sigma : \mathbb{S}^1 \to M = \mathbb{R}^n \times \mathbb{R}^m$, satisfying the equation

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\tau} = -J\nabla H(\sigma) \tag{18}$$

for all $\tau \in \mathbb{S}^1$. Moreover, the Hamiltonian function $H : M \to \mathbb{R}$ in (15) is a conservation law of (18). Take now $\varepsilon \neq 0$; then one can state ( [14]) that there exists a function $H_\varepsilon : M \to \mathbb{R}$, such, that for some closed orbit $\sigma_\varepsilon : \mathbb{S}^1 \to M$ this function $H_\varepsilon : M \to \mathbb{R}$ be a constant of motion (not a conservative quantity), that is for all small enough $\varepsilon > 0$

$$\frac{\mathrm{d}H_\varepsilon(\sigma_\varepsilon)}{\mathrm{d}t} = O(\varepsilon^2) \tag{19}$$

as $\varepsilon \to 0$. Then one can formulate the following proposition [15,16] about the existence of a limiting cycle in our model at $\varepsilon > 0$ small enough.

**Proposition.** Let our model possess at small enough $\varepsilon > 0$ a smooth constant of motion $H_\varepsilon : M \to \mathbb{R}$ and a closed $\varepsilon$-deformed orbit $\sigma_\varepsilon : \mathbb{S}^1 \to \mathbb{R}$. Moreover, at $\varepsilon = 0$ the constant of motion $H_0 : M \to \mathbb{R}$ is a first integral of the model in the neighborhood of the orbit $\sigma_0$. Then a necessary condition for the existence of a limiting cycle at $\varepsilon > 0$ small enough is vanishing of the following circular integral:

$$\oint_{\mathbb{S}^1} \langle \nabla H_0(\sigma_0), \mathrm{grad} V(\sigma_0)\rangle \,\mathrm{d}t = 0. \tag{20}$$

Having substituted expression (14) into (20), one finds numerical constraints on the parameters locating our closed orbit $\sigma_0 : \mathbb{S}^1 \to M$ in the phase space $M$. Thereby, we can localize possible coherent temporal patterns available in our neuron network under study.

Using this approach let us consider the equation of motion on the variables $(x, y) \in \mathbb{R}^{n+m}$. The Lagrangian equation corresponding to potential (14), Hamiltonian (15) and matrix (17) can be represented as

$$\begin{pmatrix} \ddot{\mathbf{x}} \\ \ddot{\mathbf{y}} \end{pmatrix} + W \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = 0, \tag{21}$$

where columns $\mathbf{x} = \{x_{1,}...x_n\}^T$ , $\mathbf{y} = \{y_{1,...,}y_m\}^T$, and matrix

$$\begin{aligned} W &= \begin{pmatrix} A_1 & B_1 \\ A_2 & B_2 \end{pmatrix}, \\ A_1 &= J_n^2 + J_n w J_m w^T, \qquad B_1 = J_n^2 w + J_n w J_m, \\ A_2 &= J_m^2 + J_m w^T J_n w, \qquad B_2 = J_m^2 w + J_m w^T J_n, \\ w &= \{w_{ik}\}. \end{aligned}$$

A solution to the matrix equation (21) can be represented as $(\mathbf{x}, \mathbf{y})^T = \mathbf{a} \exp(\mathrm{i}\lambda t)$, with $\lambda \in \mathbb{C}$ being nontrivial only if the following determinant

$$\left| -\lambda^2 g + W \right| = 0 \tag{22}$$

is equal to zero.

Equation (22) is one of the degree $n + m$ subject to $\lambda^2 \in \mathbb{C}$ and determines $2(n+m)$ eigenfrequencies $\omega_r = \{\pm\omega_1, ..., \pm\omega_{m+n}\}$ . In this case the solution gets the form

$$\begin{pmatrix} \mathbf{x}_r \\ \mathbf{y}_r \end{pmatrix} = \mathbf{a}_r \exp(\mathrm{i}\omega_r t) + \mathbf{a}_r^* \exp(-\mathrm{i}\omega_r t). \tag{23}$$

Amplitudes $\mathbf{a}_r = \{a_{r,1}, ..., a_{r,n+m}\}^T$ should satisfy the matrix equation

$$\left( -\omega_r^2 g + W \right) \mathbf{a}_r = 0. \tag{24}$$

If we take, for example, that amplitudes $a_{r1} = 1$ for any $r \in \overline{1, n+m}$ and solve (24), we can get coefficients $K_{ri}$ of the distribution of amplitudes relative to any frequency $\omega_r$. For this case the solution (23) can be represented as

$$x_{rj} = K_{r,j} \exp(\mathrm{i}\omega_r t) + K_{r,j} \exp(-\mathrm{i}\omega_r t) = 2K_{r,j} \cos(\omega_r t), \tag{25}$$
$$y_{rj} = K_{r,j+n} \exp(\mathrm{i}\omega_r t) + K_{r,j+n} \exp(-\mathrm{i}\omega_r t) = 2K_{r,j+n} \cos(\omega_r t). \tag{26}$$

Here $K_{r,j}$, $K_{r,j+n}$ are given constants depending on the frequencies $\omega_r$. Consider now the scalar product

$$\begin{aligned}\langle \nabla H_0(\sigma_0), \nabla V(\sigma_0)\rangle &= \left(x_1 + \sum_{j=1}^{m} w_{1j} y_j\right)\left(-\beta_1 x_1 + \beta_2 x_1^3 + \frac{1}{2}\sum_{j=1}^{n}\beta_{1,j}^{(1)} x_j\right) + \ldots \\ &+ \left(x_n + \sum_{j=1}^{m} w_{nj} y_j\right)\left(-\beta_1 x_n + \beta_2 x_n^3 + \frac{1}{2}\sum_{j=1}^{n}\beta_{n,j}^{(1)} x_j\right) \\ &+ \left(y_1 + \sum_{i=1}^{n} w_{i1} x_i\right)\left(-\beta_1 y_1 + \beta_2 y_1^3 + \frac{1}{2}\sum_{j=1}^{m}\beta_{1,j}^{(2)} y_j\right) + \ldots \\ &+ \left(y_n + \sum_{j=1}^{n} w_{in} x_i\right)\left(-\beta_1 y_m + \beta_2 y_m^3 + \frac{1}{2}\sum_{j=1}^{m}\beta_{n,j}^{(2)} y_j\right).\end{aligned}$$

Having substituted solution (25) into last expression and having integrated it along the period $T = 2\pi/\omega_r$ we get a hyperplane which determines the parameters of our neural network model:

$$\begin{aligned}&\left(1 + \sum_{j=1}^{m} w_{1j} K_{r,n+j}\right)\left(-\beta_1 + \frac{3}{4}\beta_2 + \frac{1}{2}\sum_{j=1}^{n}\beta_{1,j}^{(1)} K_{r,j}\right) + \ldots \\ &+ \left(K_{r,n} + \sum_{j=1}^{m} w_{nj} K_{r,n+j}\right)\left(-\beta_1 K_{r,n} + \frac{3}{4}\beta_2 K_{r,n}^3 + \frac{1}{2}\sum_{j=1}^{n}\beta_{n,j}^{(1)} K_{r,j}\right) + \ldots \\ &+ \left(K_{r,n+1} + \sum_{i=1}^{n} w_{i1} K_{r,i}\right)\left(-\beta_3 K_{r,1+n} + \frac{3}{4}\beta_4 K_{r,1+n}^3 + \frac{1}{2}\sum_{j=1}^{m}\beta_{1,j}^{(2)} K_{r,j+n}\right) + \ldots \\ &+ \left(K_{r,n+m} + \sum_{j=1}^{n} w_{in} K_{r,i}\right)\left(-\beta_3 K_{r,m+n} + \frac{3}{4}\beta_4 K_{r,m+n}^3 + \frac{1}{2}\sum_{j=1}^{m}\beta_{n,j}^{(2)} K_{r,j+n}\right) = 0.\end{aligned} \tag{27}$$

Thus, as the index $r$ changes from $r = 1$ to $n + m$, we can get in the general case $n + m$ constraints determining parameters $\{\beta_1, \beta_2, \beta_3, \beta_4, \beta_{n,j}^{(1)}, \beta_{n,j}^{(2)}\}$, at which the chosen oscillatory structure will persist for all $t \in \mathbb{R}_+$ thereby realizing a stable neural network and the temporal pattern under study related with it.

## 5. Coherent structure formation and related gradient flows on matrix Grassmann type manifolds

The results presented above can be successfully applied to many interesting dynamical system modeling information processes in neural networks, mentioned in the introduction. In section 3 above we have studied a many agent neural model allowing for the persistence of some coherent structures under applied small enough dissipative perturbations. In order to describe all of them one can make use of the analytical relationships (27) for some indices $r \in \{\overline{1, n+m}\}$ and find corresponding constraints on the set of external parameters$\{\beta_1, \beta_2,\ \beta_3,\ \beta_4,\ \beta_{n,j}^{(1)}\ \ \beta_{n,j}^{(2)},\ j=\overline{1,m}\}$ at which compact coherent structures persist. Unfortunatelly this problem is computationally too complicated and cumbersome. That is why we need to apply to its solving a more suitable approach based on a related statistical description of the available set of individual oscillatory orbits relating to two groups of $\{x\}$ and $\{y\}$ neurons.

Namely each possible oscillatory manymode orbit with phase variables $\{x_i, y_i\}$ $i=\overline{1,n},\ j=\overline{1,m}$ can be modeled by directed edge with vertices at $x_i$ and $y_i$ and directed, for instance, from $x_i$ to $y_i$. This edge is taken with a weight $C_{ij_{+n}}$, $i=\overline{1,n},\ j=\overline{1,m}$, depending both on the corresponding mode frequencies from the spectrum of the problem (24) and on the set of $\beta$-parameters mentioned above. Shortly speaking the weight matrix $C=\{C_{ij}: i,j=\overline{1,n+m}\}$ represents a vector space distribution of the persisting manymode oscillatory orbits taking part in the coherent structure formation permitted by the interacting neural network.

This weight matrix $C$ can be effectively evaluated by means of an extremality graph problem related with the graph $G$ constructed above from the vertices and connecting their edges correspondingly to a chosen coherent structure with prescribed oscillatory modes. To proceed with, we describe our problem within the graph notions introduced above.

Let $S_l$ represent the symmetric matrix group on $l\in\mathbb{Z}_+$ symbols. Denote by $T_l$ the set of $(l\times l)$ matrices $l\in\mathbb{Z}_+$, realizing the correspondence between elements of $S_l$ and $T_l,\ l\in\mathbb{Z}_+$. By viewing $T_l,\ l\in\mathbb{Z}_+$, as the set of incident matrix for a related graph $G_l$, one also obtains a one-to-one correspondence between $T_l$ and $G_l,\ l\in\mathbb{Z}_+$. The set of directed graphs with the property that every vertex is both the source of directed edges and the sink of directed edges.

Proceed now to a definition of a neighborhood around an arbitrary element within an available matrix space modeling interaction of many agent system under regard above.

**Definition 1** *For any $l-$symbol extremality graph problem $G$ we define a $k-$ change neighborhood of $G$ as the set of elements from $G_l,\ l\in\mathbb{Z}_+$, that can be obtained by removing $k\in\mathbb{Z}_+$ directed edges from $G_l$ and then placing $k\in\mathbb{Z}_+$ alternative directed edges to the remaining graph.*

For example, if a matrix $\tau\in T_l,\ l\in\mathbb{Z}_+$, it is easy to see that the $2-$change neighborhood consists of the $l(l-1)/2$ permutation matrices of the form $\tau N_{ij}$, where $N_{ij},\ i,j=\overline{1,l}$, are permutation matrices that have only two nonzero entries at $(i,j)$ and $(j,i)$-places.

Let $G$ be a fully connected undirected graph with $l\in\mathbb{Z}_+$ vertices with some weights assigned to its undirected edges. The standard graph partition problem is to find a partition of a graph into subsets with $p$ and $q\in\mathbb{Z}_+$ elements such that the sum of the weights on the cut edges (that is, edges with their endpoints in different subsets of the partition) is minimized. To formulate this problem more analytically, let us denote by $C$ a cost matrix constructed in such a way that for any original weight assigned to an edge, we assign half of it to each of the two directed edges going to the same vertices in the undirected graph. Now define [17] the matrix

$$S_G(p,q) := \begin{bmatrix} O_{p,p} & I_{p,q} \\ I_{q,p} & O_{q,q} \end{bmatrix}, \tag{28}$$

where $q, p \in \mathbb{Z}_+$ are given, $O_{p,p}$ and $O_{q,q}$ are zero matrices and $I_{p,q}$ denotes the $(p \times q)-$matrix with all elements equal to 1. Then the graph partition problem with the swap neighborhood can be represented in the following way: find the infinum

$$\inf \operatorname{Sp}(C^T \tau^T S_G(p,q)\tau), \tag{29}$$

where $\tau \in T_l$, at which two different partitions appear to be neighbors if the one can be made identical to the other by swapping two vertices.

Now we proceed to embeding the set $T_l$ into $SO(l)$, $l \in \mathbb{Z}_+$, which makes it possible to construct the element corresponding to (29) in $T_l$, $l \in \mathbb{Z}_+$, related with the element in $SO(l)$, $l \in \mathbb{Z}_+$, by the following mapping:

$$i : \tau \to \begin{cases} \tau = A \in SO(l), & \det \tau = 1, \\ \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \tau := J\tau = A \in SO(l), & \det \tau = -1. \end{cases} \tag{30}$$

Then it is easy to see that (29) is equivalent to the problem on $SO(l)$, $l \in \mathbb{Z}_+$:

$$\inf \operatorname{Sp}(C^\intercal A^\intercal S_G(p,q)A), \tag{31}$$

where $A \in SO(l)$, $l \in \mathbb{Z}_+$, $A^\intercal A = \mathbf{1} = AA^\intercal$, $\det A = 1$.

Before proceeding to solving the problem (31), observe that the cost matrix $C \in \operatorname{End} E^l$ satisfies the projection condition $C^2 = C$, entailing the following reformulation of the problem (31):

$$\inf \Psi(P),\ \Psi(P) = \operatorname{Sp}(PS_G(p,q)), \tag{32}$$

where, by definition $\Psi : \mathcal{P} \to \mathbb{R}$ is a Lyapunov function, $P := AC^\intercal A^\intercal \in \mathcal{P}$ and $\mathcal{P}$ is the standard compact Grassmann manifold [18] of projection matrices, satisfying the constraint

$$P^2 = P. \tag{33}$$

Owing to the fact that the matrix $S_G(p,q)$ is symmetric, the functional $\Psi : \mathcal{P} \to \mathbb{R}$ in (32) can be without changing its values replaced by a functional $\Psi^\intercal : \mathcal{P} \to \mathbb{R}$, where the Lyapunov function

$$\Psi^\intercal(P) = \operatorname{Sp}(S_G(p,q)P^\intercal) \tag{34}$$

satisfies the condition

$$\Psi^\intercal(P) = \Psi(P) \tag{35}$$

for any $P \in \mathcal{P}$. Thereby, the general true problem setting of (32) is as follows:

$$\frac{1}{2}\inf(\Psi(P) + \Psi^\intercal(P)) \tag{36}$$

for $P \in \mathcal{P}$. In a particular case when $C^{\intercal} = C \in \text{End}\ E^{l}$, the condition $P^{\intercal} = P$ follows for all $P \in \mathcal{P}$ and the problem (36) reduces to (32), which will be our main subject of the study below.

It is easy to show that solutions to the problem (32) are given by the critical points $\bar{P} \in \mathcal{P}$ of the following gradient vector field on $\mathcal{P}$ :

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \nabla_{\varphi}\Psi(P), \tag{37}$$

where the constraint functional $\varphi : \mathcal{P} \times \mathcal{P} \to \mathbb{R}$ is given as follows:

$$\varphi(X, P) := \mathrm{Sp}(X(P^2 - P)) = 0 \tag{38}$$

for all $X \in \text{End}\ E^{l}$. As a result one easily finds [19] that

$$\nabla_{\varphi}\Psi(P) = [[S_G, P], P] \tag{39}$$

for any $P \in \mathcal{P}$. Thus, one readily checks that the inequality

$$\frac{\mathrm{d}\Psi(P)}{\mathrm{d}t} = \mathrm{Sp}(\nabla_{\varphi}\Psi(P)S_G) = \mathrm{Sp}([[S_G, P], P]S_G) = \mathrm{Sp}([S_G, P]\ [P, S_G]) = \\ = -\mathrm{Sp}([S_G, P][S_G, P]^{\intercal}) \leqslant 0 \tag{40}$$

holds for all $P \in \mathcal{P}$. Thereby, the infinum of the problem (32) exists since the Grassmann manifold $\mathcal{P}$ is compact and the Lyapunov functional $\Psi : \mathcal{P} \to \mathbb{R}$ is decreasing along the orbits of the gradient vector field (37).

Now making use of the previous interpretation of the projector matrix $P \in \mathcal{P}$ related due to (30) with the corresponding transmutation matrix $\tau \in T_l,\ l \in \mathbb{Z}_+$, realizing the solutions to our partition problem with the swap neighborhood modeling a coherent structure formation within the multi-agent neural system described by the graph $G$. This coherent structure formation is now modeled by means of the gradient vector field (37) on the compact Grassmann manifold $\mathcal{P}$ describing the dynamics of a virtual "cost" matrix $P := AC^{\intercal}A^{\intercal} \in \mathcal{P}$ tending to the stable "cost" matrix $\bar{P} := \bar{A}\ C^{\intercal}\bar{A}^{\intercal} \in \mathcal{P}$ at some value $\bar{A} \in SO(l)$, or the same, at some transmutation $\bar{\tau} \in T_l$ due to (30). This interpretation also gives rise to some other interesting applications of this partition model, in particular, in many-agent market theory and others.

Another important aspect of our partition model is related with a possibility to describe our "cost" changing process (37) as a Hamiltonian flow on the Grassmann manifold $\mathcal{P}$. This aspect was just recently described in [19] and is based on the fact [20] that the Grassmann manifold $\mathcal{P}$ is also symplectic with the following non-degenerate symplectic structure $\omega^{(2)} \in \Lambda^2(\mathcal{P})$ on $\mathcal{P}$:

$$\omega^{(2)} = \mathrm{Sp}(P\mathrm{d}P \wedge \mathrm{d}P\ P) \tag{41}$$

for all points $P \in \mathcal{P}$, subject to which the gradient field (37) appears to be Hamiltonian on $\mathcal{P}$.

Let us also note that our gradient vector field (39) was derived above for the symmetric case when $P^{\intercal} = P \in \mathcal{P}$. If the condition $P = P^{\intercal}$ does not hold for $P \in \mathcal{P}$, then a new vector field expression must be derived for the resulting gradient flow (37). We plan to study these and related problems elsewhere.

## 6. Conclusion

Conditions for emerging of different type solutions in a dynamical system is an issue of practical importance. From the standpoint of such a science like the theory of self-organization one needs a reliable mathematical theory which can classify the possible solutions depending on a general form of available nonlinearities. The coherent structure formation and related gradient flows on matrix Grassmann type manifolds were considered as well as the corresponding graph model associated with the partition swap neighborhood problem was studied. In the paper we have also derived the conditions for a system vector field to be separated into two different flows: gradient and Hamiltonian. In the general case this problem is still far from being ultimately resolved and is under study.

## References


1. Nicolis G., Prigogine I. Self-organization in Non-equilibrium Systems. Wiley, New York, 1977.
2. Haken H. Synergetics. Heidelberg, Springer, 1978.
3. Ebeling W., Klimontovich Yu.L. Self-organization and Turbulence in Liquids. Leipzig, Teubner-Verlag, 1984.
4. Kerner B.S., Osipov V.V. Autosolitons. Dordrecht, Kluwer, 1994.
5. Klimontovich Y.L. Turbulent Motion and the Structure of Chaos: A New Approach to the Statistical Theory of Open Systems. Dordrecht and Boston, Kluwer Academic Publishers, 1991.
6. Landau L.D., Lifshitz E.M. Mechanics Course of Theoretical Physics, vol. 1. Pergamon Press, 1960.
7. Lubashevskii I.A., Gafiychuk V.V. The projection dynamics of highly dissipative system // Phys. Rev. E, 1994, vol. 50, No. 1, p. 171–181.
8. Gafiychuk V.V., Lubashevskii I.A. Variational representation of the projection dynamics and random motion of highly dissipative systems // J. Math. Phys., 1995, vol. 36, No. 10, p. 5735–5752.
9. Gafiychuk V.V., Lubashevsky I.A., Klimontovich Yu.L. Self-regulation in a simple model of hierarchically organized market // Complex Systems, 2000, vol. 12, No. 1, p. 103–126.
10. Lubashevsky I.A., Gafiychuk V.V. Cooperative mechanism of self-regulation in hierarchical living systems // SIAM J. Appl. Math., 2000, vol. 60, No. 2, p. 633–663.
11. Amit D.J. Modeling Brain Functions. Cambridge, University Press, 1989.
12. Abeles M., Vaadia E., Bergman H. // Networks, 1990, vol. 1, p. 13–15.
13. Prykarpatsky A.K., Mykytyuk I.V. Algebraic Integrability of Nonlinear Dynamical Systems on Manifolds. The Netherlands, Kluwer Acad. Publ., 1998.
14. Verhulst R. Ordinary Differential Equations. NY, Springer, 2001.
15. Guckenheim J., Holmes P. Nonlinear oscillations, dynamical systems and bifurcation of vector fields // Appl. Math. Sci., 1983, vol. 42, No. 1, p. 28–63.
16. Duarte T.R., Mendes V.R. Deformation of Hamiltonian dynamics and constant of motion in dissipative systems // J. Math. Phys., 1983, vol. 24, No. 7, p. 1772–1778.

17. Wang W.S. Gradient flows for local minima and combinatorial optimization problems // Field structures communications, 1994, vol. 3, p. 145–155.
18. Wels R.O. Differential Analysis on Complex Manifolds. N.Y., Prentice-Hall Inc., 1973.
19. Gafiychuk V.V., Prykarpatsky A.K. Replicator dynamics and mathematical description of multi-agent interaction in complex systems // Journal of Nonlinear Mathematical Physics, 2004, vol. 11, No. 1, p. 113–122.
20. Blackmore D.L., Prykarpatsky A.K., Zagrodzinski J.A. Lax-type flows on Grassmann manifolds and dual momentum mappings // Report on Math.Phys., 1997, vol. 40, p.539–549.

## Формування структур в нейронних динамічних системах обумовлене взаємодією гамільтонових і градієнтних векторних полів

В.Гафійчук [1], А.Прикарпатський [2]

[1] Краківський технологічний університет, Краків, Польща
[2] Академія гірництва і металургії, Краків, Польща

Отримано 29 квітня 2004 р., в остаточному вигляді – 21 червня 2004 р.

Аналізуються динамічні системи загального виду, векторні поля яких складаються з градієнтної (симетричної) та Гамільтонової (антисиметричної) складових. Дискутується відповідність таких систем процесам самоорганізації. Розглядається виникнення когерентних структур і відповідних градієнтних потоків на грасманових многовидах, а також моделювання таких структур відповідною моделлю графа, який виникає в результаті такого формування. Встановлено критерій виникнення гамільтонових і градієнтних векторних полів. Розглядається модельний приклад нейронної динамічної системи, для якої встановлені умови виникнення осциляційних структур.

**Ключові слова:** *Гамільтонове векторне поле градієнтне векторне поле динамічна система самоорганізація*

**PACS:** *05.45.-a, 07.05.Mh, 05.65.+b*