

On the spectrum of facet crystallization waves at the smooth ^4He crystal surface

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The wavelike processes of crystallization and melting or crystallization waves are well known to exist at the ^4He crystal surface in the rough state. Much less is known about crystallization waves for the ^4He crystal surface in the smooth well-faceted state below the roughening transition temperature. To meet the lack, we analyze here the spectrum of facet crystallization waves and its dependence upon the wavelength, perturbation amplitude, and the number of possible facet steps distributed somehow over the wavelength. All the distinctive features of facet crystallization waves from conventional waves at the rough surface result from a nonanalytic cusplike behavior in the angle dependence for the surface tension of smooth crystal facets.

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1. Introduction

Since the prediction by Andreev and Parshin [1] in 1978, crystallization waves at the superfluid-solid ^4He interface have become a well-known phenomenon. At low temperature the ^4He crystal in contact with its liquid phase can support weakly damped oscillations of the interface due to processes of crystallization and melting. From the dynamical point of view such weakly damped crystallization waves at the rough crystal surface are an immediate counterpart of the familiar gravitational-capillary waves at the interface between two normal liquids (see review [2]).

To date, the crystallization wave phenomena in ^4He have extensively been studied for the rough state of crystal surfaces, but not much study has been made for the well-faceted and atomically smooth crystal surfaces which may have an infinitely large surface stiffness. The most distinctive feature of the smooth faceted surface from the rough one is the existence of nonanalytic cusplike behavior in the angle dependence for the surface tension, e.g., [2–4]. The presence of singularity leads to qualitative distinctions in a number of the phenomena at the smooth faceted crystal surface, e.g., amplitude dependence velocity of traveling waves [5], quantum fingering of the inverted liquid-crystal interface in the field of gravity [6], Rayleigh-Taylor instability with generation of crystallization waves [7].

In present work, we develop a theory on the spectrum of facet crystallization waves at the smooth faceted surface of a ^4He crystal in contact with its liquid phase. We con-

sider a few possible types of facet crystallization waves and determine the dispersion relation between the frequency and the wave vector, perturbation wave amplitude and the number of the crystal facet steps per wavelength. For simplicity, we discuss the basal plane of hexagonal ^4He crystal as an example of the crystal facet and disregard any anisotropy in the basal plane.

2. Lagrangian

Let us assume that the crystal surface is parallel to the xy plane, with vertical position at $z = 0$. In order to derive the oscillation spectrum of a facet surface, we proceed as follows. Let us call $\zeta = \zeta(t, \mathbf{r})$ the displacement of the surface from its horizontal position $z = 0$. In the lack of energy dissipation the surface oscillations can be described by the action

$$S = \int dt L[\zeta(t, \mathbf{r}), \dot{\zeta}(t, \mathbf{r})] \quad (1)$$

with the following Lagrangian

$$L[\zeta(t, \mathbf{r}), \dot{\zeta}(t, \mathbf{r})] = \frac{\rho_{\text{eff}}}{2} \int d^2r d^2r' \frac{\dot{\zeta}(t, \mathbf{r}) \dot{\zeta}(t, \mathbf{r}')}{2\pi |\mathbf{r} - \mathbf{r}'|} - \int d^2r \left(\alpha(\mathbf{n}) \sqrt{1 + (\nabla \zeta)^2} + \frac{1}{2} \Delta \rho g \zeta^2 \right).$$

Here $\mathbf{r} = (x, y)$ is a two-dimensional radius-vector. The first term in the Lagrangian represents the kinetic energy of

the interface having an effective density ρ_{eff} . We assume that both the liquid and the solid phases are incompressible. Because of low temperature consideration we will also neglect the normal component density in the superfluid phase or, equivalently, difference between the superfluid density ρ_s and the density of the liquid phase ρ . Then the effective interface density ρ_{eff} is given by

$$\rho_{\text{eff}} = (\rho' - \rho)^2 / \rho$$

and depends on the difference $\Delta\rho = \rho' - \rho$ between the solid density ρ' and the liquid density ρ . For our purposes, the exact magnitude of the effective density is inessential.

The second and third terms in the Lagrangian are the surface energy and potential energy of the interface in the field of gravity with acceleration g .

Unlike the liquid-liquid interface, the surface tension $\alpha(\mathbf{n})$ for the crystal facet depends essentially on the direction of the normal \mathbf{n} to the interface. In our simple case this is a function of angle θ alone between the normal and

the crystallographic [0001] or c -axis of the crystal hcp structure with the geometric relation $|\tan \theta| = |\nabla\zeta|$.

For the crystal facet tilted by small angle θ from the basal plane, the expansion of surface tension $\alpha(\theta)$ in series in θ starts as, e.g., Refs. 2,4,

$$\alpha(\theta) = \alpha_0 + \alpha_1 \tan |\theta| + \dots; \quad |\tan \theta| = |\nabla\zeta|. \quad (2)$$

We do not write the next terms of expansion, e.g., cubic one due to step-step interaction, since we assume to study only small bending of the crystal surface. The angular behavior has a nonanalytic cusplike behavior at $\theta = 0$ due to $\alpha_1 = \alpha_1(T)$ representing a ratio of the linear facet step energy β to the crystallographic interplane spacing. Below the roughening transition temperature for the basal plane $T_R \sim 1.2$ K the facet step energy $\beta = \beta(T)$ is positive and vanishes for temperatures $T > T_R$.

To consider a travelling wave, e.g., propagating from the left to the right, we represent the interface perturbation as $\zeta(t, \mathbf{r}) = \zeta(\mathbf{r} - \mathbf{V}t)$. Here \mathbf{V} is the phase velocity of the wave. Then the action (1) can be rewritten as

$$S = \int dt \left\{ \frac{\rho_{\text{eff}}}{2} \int \frac{d^2r d^2r'}{2\pi |\mathbf{r} - \mathbf{r}'|} \left(\mathbf{V} \frac{\partial \zeta(\mathbf{r} - \mathbf{V}t)}{\partial \mathbf{r}} \right) \left(\mathbf{V} \frac{\partial \zeta(\mathbf{r}' - \mathbf{V}t)}{\partial \mathbf{r}'} \right) - \int d^2r (\alpha(\mathbf{n}) \sqrt{1 + (\nabla\zeta)^2} + \frac{1}{2} \Delta\rho g \zeta^2) \right\}.$$

Since the integration is performed over \mathbf{r} and \mathbf{r}' within the infinite limits and the kernel in the kinetic term depends on difference $|\mathbf{r} - \mathbf{r}'|$ alone, we can shift the argu-

ment in ζ by $\mathbf{V}t$. Next, by integrating twice by parts, we arrive at

$$S = \int dt \left\{ \int d^2r d^2r' \frac{\rho_{\text{eff}}}{2} \zeta(\mathbf{r}) \zeta(\mathbf{r}') \left(\mathbf{V} \cdot \frac{\partial}{\partial \mathbf{r}'} \right) \left(\mathbf{V} \cdot \frac{\partial}{\partial \mathbf{r}} \right) \frac{1}{2\pi |\mathbf{r} - \mathbf{r}'|} - \int d^2r (\alpha(\mathbf{n}) \sqrt{1 + (\nabla\zeta)^2} + \frac{1}{2} \Delta\rho g \zeta^2) \right\}.$$

In what follows, we will study sufficiently small bending of the crystal surface with the sufficiently small displacements ζ and small angles θ . Involving inequality $|\nabla\zeta| \ll 1$ and $|\tan \theta| = |\nabla\zeta|$, we take only first terms in the expansion of the surface energy

$$\alpha(\mathbf{n}) \sqrt{1 + (\nabla\zeta)^2} \approx \alpha_0 + \alpha_1 |\nabla\zeta| + \alpha_0 (\nabla\zeta)^2 / 2.$$

Next, we will choose the x -axis as a direction of the wave propagation $\mathbf{V} = (V, 0, 0)$ and replace x' with x in the spatial derivatives. Finally, we arrive at

$$S[\zeta] - S[\zeta = 0] = - \int dt \left\{ \frac{\rho_{\text{eff}} V^2}{2} \int d^2r d^2r' \zeta(x) \zeta(x') \frac{\partial^2}{\partial x^2} \left(\frac{1}{2\pi |\mathbf{r} - \mathbf{r}'|} \right) + \int d^2r \left(\alpha_1 |\nabla\zeta| + \frac{\alpha_0}{2} (\nabla\zeta)^2 + \Delta\rho g \frac{\zeta^2}{2} \right) \right\}. \quad (3)$$

Variation $\delta S / \delta \zeta(x)$ yields the equation for interface oscillations

$$\int G(\mathbf{r} - \mathbf{r}') \zeta(x') d^2r' + \alpha_1 \frac{\partial}{\partial x} \left[\text{sgn} \left(\frac{\partial \zeta}{\partial x} \right) \right] = 0. \quad (4)$$

Here, for convenience, we have introduced the Green function according to

$$G(\mathbf{r} - \mathbf{r}') = -\rho_{\text{eff}} V^2 \frac{\partial^2}{\partial x^2} \left(\frac{1}{2\pi |\mathbf{r} - \mathbf{r}'|} \right) + \alpha_0 \frac{\partial^2}{\partial x^2} \delta(\mathbf{r} - \mathbf{r}') - \Delta\rho g. \quad (5)$$

The solution of Eq. (4) for $V = 0$ has been studied in Ref. 6.

Before solving Eq. (4), we make the following remarks. First, in the regions with $\zeta'(x) \neq 0$ the equation (4) reduces to a linear equation with the difference kernel

$$\int G(\mathbf{r} - \mathbf{r}') \zeta(x') d^2 r' = 0.$$

The solution can be found as a sum of independent Fourier harmonics $\zeta(x) = \sum_q \zeta_q \exp(iqx)$. For a single harmonic

$$\zeta(x) = \zeta_q \exp(\pm iqx) \text{ or } \zeta(x) = \zeta_q \cos q(x - x_0), \quad (6)$$

one should have $G(q)\zeta_q = 0$. For an existence of nontrivial solutions, it is necessary to put $G(q) = 0$. Thus, vector q must satisfy the condition

$$G(q) = \rho_{\text{eff}} V^2 \frac{q^2}{q} - \alpha_0 q^2 - \Delta \rho g = \rho_{\text{eff}} q [V^2 - V_0^2(q)] = 0. \quad (7)$$

Here we have introduced notation $V_0(q)$ for the phase velocity of crystallization waves at the rough crystal surface with the spectrum $\omega_0(q)$:

$$V_0(q) = \omega_0(q) / q, \quad \omega_0(q) = \sqrt{(\alpha_0 q^3 + \Delta \rho g q) / \rho_{\text{eff}}}.$$

Second, in the regions with $\zeta'(x) \equiv 0$ the solution is trivial, i.e.,

$$\zeta(x) = \text{const.}$$

And the last, since during the melting-crystallization process the total mass of the solid and liquid phases remains invariable, the solution of Eq. (4) must satisfy the following condition

$$\int \zeta(x) dx = 0.$$

As a result, the general solution for profile $\zeta(x)$ should represent a train of flat segments and half-sinusoids. The width of a half-sinusoid, which we denote $l/2$, is governed by the magnitude of vector q according to $G(q) = 0$ or $V_0(q) = V$ and is equal to $l/2 = \pi/q$. The half-sinusoid, which connects two neighboring flat segments, can be regarded as a macroscopic facet step in contrast to elementary steps of an atomic scale. To illustrate, we give two examples of such facet crystallization waves with alternation of one or two various flat segments in Figs. 1 and 2.

It is obvious that the wavelength λ cannot be less than a sum of two half-sinusoids $l = 2\pi/q$. In other words, the



Fig. 1. The wave is formed by alternating one flat segment of length $(\lambda - l)/2$ and kink of width $l/2$.



Fig. 2. The wave is composed of alternating two flat segments of different lengths and four kinks of width $l/2$ each.

wave number $k = 2\pi/\lambda$ does not exceed vector q , i.e., $k \leq q$. Later, it will be seen that always $\omega(k) \geq \omega_0(k)$.

Besides the wavelength, the wave spectrum at the crystal facet will also be characterized by the number of flat segments and their amplitudes. The width of macroscopic crystal steps $l/2 = \pi/q$ depends on a ratio of wave frequency ω to wave number k and can be determined using relation

$$\frac{\omega}{k} = \frac{\omega_0(q)}{q}. \quad (8)$$

Provided $l \ll \lambda$, the width of half-sinusoids can be neglected in first approximations and the macroscopic facet steps can be treated as a kink of zero width. Usually, this is a range of sufficiently high frequencies $\omega(k) \gg \omega_0(k)$ and small wave perturbation amplitudes $|\zeta| \lesssim l \ll \lambda$.

3. Simplest type of waves

Let us start from the simplest type of the crystallization wave which is formed by alternating a single flat segment of height ζ and a half-sinusoid of width $l/2$ so that $\int \zeta(x) dx = 0$ (Fig. 1). In this case the wave perturbation is specified by

$$\zeta(x) = \zeta \begin{cases} 1, & |x| < \frac{\lambda - l}{4} \\ \cos \left[\frac{2\pi}{l} \left(|x| - \frac{\lambda - l}{4} \right) \right], & \frac{\lambda - l}{4} < |x| < \frac{\lambda + l}{4} \\ -1, & \frac{\lambda + l}{4} < |x| < \frac{\lambda}{2} \end{cases} \quad (9)$$

The total length of the flat segments is equal to $\lambda - l$. For correctness of approximation used, we should assume that $|\zeta| \ll l/2\pi \leq \lambda/2\pi$.

To determine the spectrum for such type of the facet wave, we calculate the action as a function of the wave amplitude ζ and then minimize the action. The calculation is readily performed in the Fourier representation using

$$\zeta(x) = \sum_n \zeta_n \exp(ik_n x),$$

$$k_n = 2\pi n / \lambda = kn \quad (n = 0, \pm 1, \pm 2, \dots). \quad (10)$$

Hence we have for the variation of the action taken from the flat crystal surface and ascribed per wavelength

$$S[\zeta] - S[\zeta = 0] = \frac{1}{\lambda} \sum_{n=-\infty}^{\infty} G_{\omega}(k_n) \zeta_n \zeta_{-n} - 4\alpha_1 |\zeta|. \quad (11)$$

The second term is a contribution due to formation of the facet segment with the step amplitude 2ζ . The number of such steps over wavelength equals 2 in our specific case. The calculation of Fourier components is straightforward

$$\zeta_n = \int_{-\lambda/2}^{\lambda/2} \zeta(x) \exp(-ik_n x) dx = \zeta \lambda f_n, \quad (12)$$

$$f_n = \frac{\sin(\pi n / 2)}{(\pi n / 2)} \left(\frac{\cos(\pi n l / 2\lambda)}{1 - n^2 l^2 / \lambda^2} - \cos \frac{\pi n}{2} \right)$$

with the obvious properties $\zeta_{-n} = \zeta_n$ and $\zeta_{n=2m} = 0$. For $\lambda = l$, harmonics $f_1 = f_{-1} = 1/2$ alone remain nonzero. Inserting Eq. (12) into (11) and minimizing the action $\partial S / \partial \zeta = 0$, we arrive at the equation which determines the spectrum of facet crystallization wave

$$\sum_n G_{\omega}(k_n) f_n^2 = \frac{2\alpha_1}{\lambda |\zeta|}. \quad (13)$$

Then, using (12) and $l = 2\pi / q$, we have

$$\sum_{m=0}^{\infty} \left(\frac{\rho_{\text{eff}} \omega^2}{k} \frac{1}{2m+1} - \alpha_0 k^2 (2m+1)^2 - \Delta\rho g \right) f_{2m+1}^2 = \frac{\alpha_1 k}{2\pi |\zeta|},$$

$$f_{2m+1}^2 = \frac{4}{\pi^2} \frac{1}{(2m+1)^2} \frac{\cos^2 \left[\frac{\pi}{2} (2m+1) \frac{k}{q} \right]}{[1 - (2m+1)^2 k^2 / q^2]^2}, \quad m = 0, 1, 2, \dots \quad (14)$$

The magnitude of vector q is determined from the condition $\omega_0(q) / q = \omega / k$.

We first analyze the limiting case of infinitely narrow step $l \ll \lambda$ or $k \ll q$. In this approximation one can here neglect the contributions from the regular α_0 surface and gravitational $\Delta\rho g$ terms and estimate f_{2m+1} at point $k = 0$. Then, a sum in (14) reduces to

$$\rho_{\text{eff}} \frac{\omega^2}{k} \frac{4}{\pi^2} \sum_{m=0}^{\infty} \frac{1}{(2m+1)^3} = \frac{7\zeta(3) \omega^2}{2\pi^2 k},$$

where $\zeta(3) \approx 1.20$ is the Riemann zeta-function. Finally, we get the spectrum

$$\omega^2 = \frac{\pi}{7\zeta(3)} \frac{\alpha_1 k^2}{\rho_{\text{eff}} |\zeta|} = \frac{4\pi^3}{7\zeta(3)} \frac{\alpha_1}{\rho_{\text{eff}} \lambda^2 |\zeta|}. \quad (15)$$

The phase velocity V depends on the perturbation amplitude alone

$$V = \frac{\omega}{k} = \left(\frac{\pi}{7\zeta(3)} \frac{\alpha_1}{\rho_{\text{eff}} |\zeta|} \right)^{1/2}.$$

A special feature of the spectrum is its growing stiffness as $|\zeta| \rightarrow 0$ when the frequency of surface oscillations becomes infinitely large. Regardless of magnitudes α_0 and $\Delta\rho g$ the behavior $\omega \sim |\zeta|^{-1/2}$ is universal in the $|\zeta| \rightarrow 0$ limit.

To understand the typical magnitudes and applicability of our approximations, we start from the case when wavelength λ exceeds slightly the double width of crystal facet step, i.e., $(\lambda - l) / \lambda \ll 1$. Then, in Eq. (14) the main contribution results from the first term with $m = 0$. Approximately, one has

$$\omega(k) = \omega_0(k) + \frac{1}{\pi \rho_{\text{eff}} \omega_0(k)} \frac{\alpha_1 k^2}{|\zeta|}. \quad (16)$$

The effect of crystal facet smoothness on the wave spectrum due to nonzero α_1 is governed by a ratio

$$\frac{\alpha_1 k^2}{\alpha_0 k^2 + \Delta\rho g} \frac{1}{k |\zeta|} = \frac{2\pi\alpha_1 / \alpha_0}{1 + \lambda^2 / \lambda_0^2} \frac{\lambda}{|\zeta|},$$

where $\lambda_0 = 2\pi\sqrt{\alpha_0 / \Delta\rho g}$ is a usual capillary length. To have a strong effect on the spectrum in the sense $\omega(k) \gg \omega_0(k)$, we should satisfy the inequality

$$|\zeta| \ll \frac{2\pi\alpha_1}{\alpha_0} \frac{\lambda}{1 + \lambda^2 / \lambda_0^2}. \quad (17)$$

Below, in detail, we will analyze most interesting region of sufficiently small perturbation amplitudes $\zeta \rightarrow 0$ and wavelengths smaller than capillary length $\lambda \ll \lambda_0$. Neglecting gravitational term in (14), we have

$$\sum_{m=0}^{\infty} \left(\frac{1}{(2m+1)^3} - \frac{k^3}{q^3} \right) \frac{\cos^2 \left[\frac{\pi}{2} (2m+1) \frac{k}{q} \right]}{[1 - (2m+1)^2 k^2 / q^2]^2} =$$

$$= \frac{\pi}{8} \frac{\alpha_1 k^2}{\rho_{\text{eff}} \omega^2 |\zeta|}.$$

Involving that $k / q = \alpha_0 k^3 / (\rho_{\text{eff}} \omega^2) \ll 1$ and estimating the above sum within logarithmic accuracy as

$$7\zeta(3) / 8 - (\pi^2 / 8 - 1)(k / q)^2 \ln(q / k),$$

we find the spectrum with the correction due to finiteness of α_0

$$\omega^2 \approx \frac{\pi}{7\zeta(3)} \frac{\alpha_1 k^2}{\rho_{\text{eff}} |\zeta|} \left[1 + \frac{\pi^2 - 8}{21\zeta(3)} \left(\frac{7\zeta(3) \alpha_1}{\pi \alpha_0} k |\zeta| \right)^{2/3} \times \right.$$

$$\left. \times \ln \left(\frac{\pi\alpha_1}{7\zeta(3)\alpha_0} \frac{1}{k |\zeta|} \right) \right].$$

The width of the kink between two flat segments can readily be estimated from $q = \rho_{\text{eff}} \omega^2(k) / (\alpha_0 k^2)$ as

$$\frac{l}{2} \approx 7\zeta(3) \frac{\alpha_0}{\alpha_1} |\zeta|. \quad (18)$$

Thus, the approximation of zero-width facet step $l \ll \lambda$ can be justified for the small amplitude perturbations if

$$|\zeta| \ll \frac{\lambda}{14\zeta(3)} \frac{\alpha_1}{\alpha_0}.$$

On the whole, the spectrum of facet crystallization waves can qualitatively be described by introducing effective surface tension or stiffness dependent on both wave vector and perturbation amplitude [6]

$$\alpha_{\text{eff}} \rightarrow \alpha_0 + \frac{\pi}{7} \zeta(3) \frac{\alpha_1}{k|\zeta|}. \quad (19)$$

4. Waves with a few crystal facet steps

Here we consider another type of facet crystallization waves with an arbitrary number of crystal facet steps per wavelength λ . The wave amplitude ζ is assumed to be sufficiently small in order to neglect the regular surface α_0 and gravitational $\Delta\rho g$ terms. This limit corresponds to high frequencies $\omega(k) \gg \omega_0(k)$ and zero width of the kink between two flat segments. Thus, the profile of the perturbed crystal surface represents a broken line consisting of vertical steps and horizontal segments. For the definiteness, we consider the surface profile from $4N$ vertical steps linking the same $M = 4N$ flat facet segments with length $\Delta x = \lambda / M$ (Fig. 3).

The vertical steps, having the same height of ζ / N , are located at the points

$$x_k = \pm \frac{2k-1}{2} \Delta x, \quad k = 1, 2, \dots, N.$$

The points x_{k+1} and x_k are connected with a horizontal segment of length Δx with the vertical amplitude

$$\zeta_k = \pm \zeta \frac{N - |k - N|}{N}, \quad k = 0, 1, 2, \dots, 2N.$$

In accordance with (10) we find the Fourier components for $\zeta(x)$

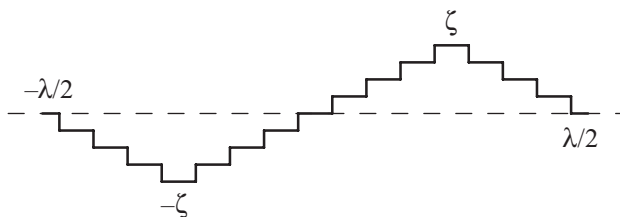


Fig. 3. The sketch of crystallization wave of the total perturbation amplitude ζ and wavelength λ .

$$\begin{aligned} \zeta_n &= \int_{-\lambda/2}^{\lambda/2} dx \zeta(x) \exp(-2\pi i n x / \lambda) = \\ &= -2i \sum_{k=1}^{2N-1} \zeta \frac{N - |k - N|}{N} \int_{x_k}^{x_{k+1}} \sin(2\pi n x / \lambda) dx = \\ &= -i \frac{i\lambda}{\pi n N} \zeta \sum_{k=1}^N (\cos y_k - \cos y_{k+N}), \quad y_k = \frac{\pi n}{4N} (2k-1). \end{aligned}$$

Summation, in essence, is reduced to summing geometric series. As a result, we obtain

$$\zeta_n = \zeta \lambda f_n, \quad f_n = \frac{i}{2\pi n} \frac{1 - (-1)^n}{N} \frac{\sin(\pi n / 2)}{\sin(\pi n / 4N)} \quad (20)$$

with the obvious properties $\zeta_{-n} = \zeta_n$ and $\zeta_{n=2m} = 0$.

Unlike derivation of Eq. (11), we must take here into account that each crystal step contributes $\alpha_1 |\zeta| / N$ into the action and the number of steps equals $4N$ at the wavelength. Since $4N\alpha_1 |\zeta| / N = 4\alpha_1 |\zeta|$, we have the same contribution to the action from the steps and thus the same form (13) of the equation to determine the wave spectrum

$$\sum_{m=0}^{\infty} \frac{\rho_{\text{eff}} \omega^2}{k} \frac{|f_{2m+1}|^2}{(2m+1)} = \frac{\alpha_1 k}{2\pi |\zeta|} \quad (21)$$

but with another Fourier component f_{2m+1} . Applying f_{2m+1} from (20), we have

$$\omega^2 \sum_{m=0}^{\infty} \frac{1}{(2m+1)^3} \frac{1}{N^2 \sin^2[\pi(2m+1)/4N]} = \frac{\pi}{2} \frac{\alpha_1 k^2}{\rho_{\text{eff}} |\zeta|}.$$

Finally, the spectrum of facet crystallization waves is determined by

$$\omega_N = s_N \omega_1, \quad \omega_1^2 = \frac{2\pi}{7\zeta(3)} \frac{\alpha_1 k^2}{\rho_{\text{eff}} |\zeta|}, \quad (22)$$

where s_N is given by

$$s_N^2 = \frac{4 \sum_{m=0}^{\infty} (2m+1)^{-3}}{N^2 \sum_{m=0}^{\infty} (2m+1)^{-3} \sin^{-2}[\pi(2m+1)/4N]}$$

and

$$s_{\infty}^2 = \frac{7\pi^2 \zeta(3)}{62\zeta(5)}.$$

The factor s_N varies insignificantly within the range from 1 for $N = 1$ to 1.136 for $N = \infty$ (Table 1).

We have analyzed above the wave spectrum for the regular arrangement and identical height of crystal steps over wavelength. This is not, of course, solely possible structure with $4N$ crystal steps. The location of crystal

Table 1. The ratio of frequencies $s_N = \omega_N / \omega_1$ for facet crystallization waves with the different number N of crystal facet steps per wavelength

N	1	2	3	4	5	6	7	∞
s_N	1.000	1.103	1.122	1.128	1.131	1.133	1.134	1.136

steps and their heights can have an arbitrary and disordered structure. However, the dimensional estimate (22) for the spectrum holds for. As concerns the factor s_N , it varies slightly as a function of the perturbation profile.

Provided the regular part of surface tension α_0 differs from zero, the maximum number of crystal steps $4N_m$ is limited. Using magnitude l for the width of the kink between two flat segments

$$l \sim \frac{\alpha_0}{\alpha_1} |\zeta|,$$

we estimate the number $4N_m$ of possible crystal steps for the given frequency ω according to

$$4N_m \sim \frac{\lambda}{l} \sim \frac{\alpha_1}{\alpha_0} \frac{\lambda}{|\zeta|}. \quad (23)$$

Obviously, the smaller the perturbation amplitude $|\zeta|$, the larger the number of possible crystal steps.

On the whole, the frequency of crystallization waves at a smooth crystal facet proves to be dependent not only on the wavelength, but also on the wave amplitude and the number of crystal facet steps which can be placed within the wavelength. From the experimental point of view this means that the excitation of crystallization waves with a fixed frequency will result in exciting some train of waves with different wavelengths, amplitude, and the number of crystal facet steps. In this connection the shape of the perturbed crystal facet will resemble rather an irregular and ill-defined profile with some elements of irregular-like character. A weak dependence of the wave frequency on the number of steps facilitates such phenomenon. In some sense one might say about transition to a rough state of the surface and destruction of the crystal faceting [7].

5. Conclusion

The crystallization waves at the smooth crystal facets are expected to demonstrate a more varied and complicated picture than those at the rough crystal surfaces. The plane crystallization wave represents an alternation of flat crystal facets linked via macroscopic crystal steps with the width dependent on the wave velocity. Most striking phenomena should appear in the limit of sufficiently small perturbation amplitudes. The frequency spectrum of facet waves depends significantly on the perturbation amplitude. The dependence on the structure of wave perturbation and the number of crystal facet steps is not so drastic. Excitation of waves at a given frequency should produce a train consist-

ing of waves differing in wavelength, structure and the number of facet steps and interacting nonlinearly with each other. This all is in contrast to harmonic waves which exist at the rough crystal surfaces and have the lower frequencies at the same wavelengths. Evidently, the distinction results from a singularity in the angular behavior of the surface tension for the smooth crystal facets.

Let us estimate typical frequencies for the short wavelength range $\lambda \ll \lambda_0 \sim 6$ mm. Taking $\rho_{\text{eff}} \sim 2$ mg/cm³, $\alpha_1 \sim 0.014$ erg/cm² and $\alpha_0 \sim 0.16$ – 0.18 erg/cm², e.g., [2] for the (0001) ⁴He facet, we have the frequency $\omega \sim 10$ kHz and velocity $V \sim 2$ m/s for the wavelength $\lambda \sim 1$ mm and perturbation amplitude $|\zeta| \sim 1$ μ m. In this case one may expect the maximum number of possible steps over wavelength to ~ 100 . If the perturbation amplitude for the same λ approaches $|\zeta| \sim 0.1$ mm, only one or two steps become possible. The wave frequency reduces to about $\omega \sim 2$ kHz which insignificantly exceeds the magnitude at the rough surface. Note that, for the perturbation amplitude of a crystal lattice spacing in height, the propagation velocity V reaches the magnitudes of ~ 100 m/s comparable with the sound velocity.

In principle, one can find a few casual mentions about phenomena similar to the formation of crystallization waves at the crystal ⁴He facets under heavy shake of an experimental cell [8] or in the process of anomalously fast growth of a ⁴He crystal under high overpressures [9, 10]. More convincing observation in favor of an existence of progressive facet waves has recently been found [11] at the crystal (001) facet in ³He. Apparently, one of complicating factors in exciting and studying facet crystallization waves may be associated with the threshold character for most of phenomena occurring at the smooth crystal facets. In particular, it may require a sufficiently large size of the facet and sufficiently high amplitudes of driving perturbation. In this connection it may be helpful to employ the conditions close to an onset of some instability, e.g., electrocapillary one in an electric field across the interface[6], Rayleigh-Taylor [7] or Faraday instabilities [12].

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