Fizika Nizkikh Temperatur, 2009, v. 35, Nos. 8/9, p. 805–811

Superconducting properties of a boson-exchange model of doped graphene

V.M. Loktev

Bogolyubov Institute for Theoretical Physics, Metrologichna Str. 14-b, Kiev 03680, Ukraine
E-mail: vloktev@bitp.kiev.ua

V. Turkowski

Department of Physics and NanoScience Technology Center, University of Central Florida, Orlando, Florida 32816

Received February 16, 2009

We study the superconducting properties of a doped one-layer graphene by using a model in which the interparticle attraction is caused by a boson (phonon-like) excitations. We study the dependencies of the superconducting gap Δ and the mean-field critical temperature T_c^{MF} on the carrier density, attraction strength and the characteristic (Debye) bosonic frequency. In addition, we study the temperature-carrier density phase diagram of the model by taking into account the thermal fluctuations of the order parameter. We show that the fluctuations result in a significant suppression of T_c^{MF}, such that the real (Berezinskii–Kosterlitz–Thouless) critical temperature T_c is much lower than T_c^{MF}. The region $T_c < T < T_c^{MF}$ is characterized by a finite density of states at the Fermi level (the pseudogap phase). We show that the width of the temperature interval of the pseudogap phase strongly depends on the model parameters — carrier concentration, attraction amplitude, and boson frequency.

PACS: **74.20.–z** Theories and models of superconducting state;
74.78.–w Superconducting films and low-dimensional structures;
74.40.+k Fluctuations (noise, chaos, nonequilibrium superconductivity, localization, etc.);
73.63.–b Electronic transport in nanoscale materials and structures.

Keywords: boson excitations, graphene, superconducting properties of thin films, low-dimensional structures.

1. Introduction

Possibility of superconductivity in graphene is one of the most interesting and important problems of modern theory of superconductivity. This problem is a part of more general problem of understanding the physical, especially the electronic, properties of graphene, which is one of the most important basic units of the carbon-based structures. These structures have a great potential to be used in modern nanotechnologies, especially due to strong and flexible carbon bonds. The single layer of graphene can be considered as the basis for two-dimensional (one- to several layer-graphene), one-dimensional (carbon nanotubes and nanoribbons) and zero-dimensional (fullerene molecules, graphene flakes etc.) structures. Naturally, one needs to understand the low-temperature properties of such systems, when it is possible that the superconducting state is the ground state of the system. Moreover, the superconducting properties of graphene demostrate many interesting applications. During the last years, there has been made a great progress in understanding some of the electronic properties of graphene (for overview and references, see Ref. 1). Evidences of superconductivity in graphite and graphite-sulfur structures were found experimentally [2–5]. Theoretical studies of superconductivity in graphene were performed in Refs. 6–14. Since the Fermi surface of undoped graphene consists of Dirac points, the quantum critical point scenario of superconductivity in this case was analyzed in Refs. 6, 8. It was found that in the undoped case, superconductivity takes place when the coupling is bigger than a critical value. Superconductivity in graphene within different scenarios, like the phonon, plasmon [9], resonant-valence-bond and the density-wave scenario, was considered in Refs. 6,10,11. In the doped case, when the Fermi surface is finite, the superconductivity can take place at any value of coupling. The

doped case within the framework of different models was analyzed in Refs. 6, 7, 9, 10,12,13. In particular, a simple model with an electron-electron attraction which is defined by the amplitude and the range (or the Debye-like cut-off energy ω_0 in the boson-exchange model) was considered in Ref. 12. The authors solved the coupled system of the BCS gap equation and the particle number equation, in order to analize the coupling and doping dependencies of superconductivity. Possibility of strong enhancement of superconductivity with the critical temperature up to $T_c \sim 10$ K due to the van Hove singularities in the electron-density of states in graphene at energies of order 3 eV with respect to the band bottom was discussed in Ref. 13. One needs to realize in practice such highly doped samples of graphene in order to check the possibility of the strong enhancement of superconductivity. Possibility of superconductivity in the papers mentioned above was analized within the mean-field approximation, when the fluctuations were neglected. Such fluctuations cannot be neglected in 2D systems, since they can significantly modify the properties of the system. In particular, the thermal fluctuations in a two-dimensional system lead to a significant reduction of the mean-field superconducting critical temperature T_c^{MF} [15,17]. In this case, the real critical temperature is the Berezinskii–Kosterlitz–Thouless (BKT) temperature $T_c \equiv T_{BKT} < T_c^{MF}$, below which the order parameter is algebraically ordered. Recently, we have studied the superconducting properties of the model considered in Ref. 12 in the case of single- and double-layer graphene by taking into account the thermal fluctuations [14]. We have shown that the fluctuations lead to a drastic reduction of the superconducting critical temperature in both cases. We have also shown that at temperatures $T_c < T < T_c^{MF}$ both systems the fluctuations of the order parameter produce so-called pseudogap phase with a reduced DOS. In that paper, we have considered the limit $\omega_0 = \infty$, which corresponds to the case of local attraction. Here we consider the case of superconductivity for boson-exchange electron-electron attraction, and show that the behavior of the system strongly depends on the bosonic energy (Debye frequency in the phonon case). The results obtained in this paper can lead to a further clarification of the mechanism of possible superconductivity in graphene.

We dedicate this paper to an outstanding person, organizer and the first Director of the Institute for Low Temperature Physics and Engineering Prof. B.I. Verkin, for whom the physics of superconductivity was one of the main research interests.

2. The model

The effective superconducting Hamiltonian of doped single-layer graphene has the following form:

$$H(\tau) = \sum_{\sigma} \psi^{\dagger}_{\sigma}(\tau,\mathbf{r})[\varepsilon(-i\nabla) - \mu]\psi_{\sigma}(\tau,\mathbf{r}) -$$

$$-\frac{1}{2}\int d\mathbf{r}_1 \int d\mathbf{r}_2 \psi^{\dagger}_{\uparrow}(\tau,\mathbf{r}_2)\psi^{\dagger}_{\downarrow}(\tau,\mathbf{r}_1)V(\mathbf{r}_1-\mathbf{r}_2)\psi_{\downarrow}(\tau,\mathbf{r}_1)\psi_{\uparrow}(\tau,\mathbf{r}_2), \quad (1)$$

where $\psi^{\dagger}_{\sigma}(\tau,\mathbf{r})$ and $\psi_{\sigma}(\tau,\mathbf{r})$ are the fermionic operators of creation and annihilation of the quasiparticle on the site $\mathbf{r}$ at time τ with spin $\sigma = \uparrow,\downarrow$; $\varepsilon(-i\nabla)$ and μ are the free-fermion dispersion relation and the chemical potential, and $V(\mathbf{r}_1 - \mathbf{r}_2)$ is the interparticle attraction, respectively. The energy spectrum $\varepsilon(\mathbf{k})$ around the Dirac points can be approximated as $\varepsilon^{\pm}(\mathbf{k}) = \pm v_F|\mathbf{k}|$, where $v_F = 3ta/2 \sim 10^6$ m/s is the Fermi velocity, $t = 2.8$ eV is the nearest neighbor hopping and $a = 1.42$ Å is the distance between the nearest carbon atoms. In the undoped case, the band $\varepsilon^{-}(\mathbf{k})$ is filled. For definiteness, we consider the electron-doped case with the conduction band $\varepsilon^{+}(\mathbf{k}) \equiv \varepsilon(\mathbf{k})$.

We approximate the superconducting attraction by the following interparticle potential:

$$V_{\mathbf{p}} = V_0\theta(\omega_0 - |\varepsilon(\mathbf{p}) - \mu|), \quad (2)$$

where V_0 is the amplitude of the attraction and ω_0 is the bosonic (Debye-like) cut-off frequency, which corresponds to a half of the BCS attraction shell around the Fermi energy surface, which, in most cases, is proportional to μ (see a discussion below). We shall study the dependencies of the superconducting properties of the model on V_0 and ω_0 at different values of the doping, since the knowledge of such dependencies can help to establish the source of possible superconducting attraction in graphene (the coupling energy and boson frequency in the case of bosonic mechanism of superconductivity). We shall consider the isotropic *s*-wave pairing, when the gap $\Delta(\mathbf{k})$ is momentum-independent, however the results obtained below will remain qualitatively the same for more complicated interactions and different symmetries of the order parameter (see, e.g., Ref. 17).

3. The mean-field approximation

In order to study the superconducting properties of the system, we shall consider the partition function

$$Z = \int D\psi^{\dagger} D\psi \exp\left[-\int_0^{1/T} d\tau \left(\sum_{\sigma}\int d\mathbf{r}\psi^{\dagger}_{\sigma}(\tau,\mathbf{r})\partial_{\tau}\psi_{\sigma}(\tau,\mathbf{r}) + H(\tau)\right)\right], \quad (3)$$

which can be obtained by performing the path integration over the fermionic fields. To find Z, one can introduce the Nambu spinor operators:

$$\Psi(\tau,\mathbf{r})=\begin{pmatrix}\psi_{\uparrow}(\tau,\mathbf{r})\\ \psi^{\dagger}_{\downarrow}(\tau,\mathbf{r})\end{pmatrix}, \quad \Psi^{\dagger}(\tau,\mathbf{r})=(\psi^{\dagger}_{\uparrow}(\tau,\mathbf{r}),\psi_{\downarrow}(\tau,\mathbf{r})). \tag{4}$$

We use the Hubbard–Stratonovich transformation in order to decouple the four-fermion term in the Hamiltonian (see Eq. (1)). In this case, the partition function is equivalent to

$$Z=\int D\Psi^{\dagger}D\Psi D\Phi^{\dagger}D\Phi e^{-S(\Psi^{\dagger},\Psi,\Phi^{\dagger},\Phi)}, \tag{5}$$

where

$$S(\Psi^{\dagger},\Psi,\Phi^{\dagger},\Phi)=\int_0^{1/T}d\tau\int d\mathbf{r}_1\int d\mathbf{r}_2\Big\{\delta(\mathbf{r}_1-\mathbf{r}_2)\sum_{\sigma}\psi^{\dagger}_{\sigma}(\tau,\mathbf{r}_1)[\partial_{\tau}+\tau_z(\varepsilon(\nabla_{\mathbf{r}_2})-\mu)]\psi_{\sigma}(\tau,\mathbf{r}_2)+$$
$$+\frac{1}{2}\frac{|\Phi(\mathbf{r}_1,\mathbf{r}_2)|^2}{V(\mathbf{r}_1-\mathbf{r}_2)}-\frac{1}{2}\Psi^{\dagger}(\tau,\mathbf{r}_1)\tau_{+}\Psi(\tau,\mathbf{r}_2)\Phi(\tau,\mathbf{r}_1,\mathbf{r}_2)-\frac{1}{2}\Phi^{\dagger}(\tau,\mathbf{r}_1,\mathbf{r}_2)\Psi^{\dagger}(\tau,\mathbf{r}_1)\tau_{-}\Psi(\tau,\mathbf{r}_2)\Big\}, \tag{6}$$

is the effective action and $\tau_{\pm}=1/2(\tau_x\pm i\tau_y)$ and τ_z are the Pauli matrices. In Eq. (6), $\Phi(\tau,\mathbf{r}_1,\mathbf{r}_2)\sim V(\mathbf{r}_1-\mathbf{r}_2)\psi_{\downarrow}(\tau,\mathbf{r}_1)\psi_{\uparrow}(\tau,\mathbf{r}_2)$ is a complex function, which has the physical meaning of the superconducting order parameter. After integration over the Nambu spinors in Eq. (5) one can get the expression for the thermodynamical potential $\Omega(\Phi^{\dagger},\Phi)$:

$$e^{-\Omega(\Phi^{\dagger},\Phi)/T}=\int D\Psi^{\dagger}D\Psi e^{-S(\Psi^{\dagger},\Psi,\Phi^{\dagger},\Phi)}. \tag{7}$$

In the case of mean-field approximation, we assume that $\Phi(\tau,\mathbf{r}_1,\mathbf{r}_2)$ is momentum- and time-independent in the momentum representation, and is equal to $\Delta(\tau,\mathbf{k})\equiv\Delta=\text{const}$. In order to find the equations for the superconducting gap Δ and the chemical potential, one needs to minimize $\Omega(\Delta)$ with respect to Δ together with the particle-number equation $(1/V)\partial\Omega/\partial\mu=-n_f$ (V is the volume of the system). This gives:

$$1=\frac{1}{2}\sum_{\mathbf{k}}V(\mathbf{k})\tanh\left(\frac{E(\mathbf{k})}{2T}\right)\frac{1}{E(\mathbf{k})}, \tag{8}$$

$$n_f=2\sum_{\mathbf{k}}\left[1-\tanh\left(\frac{E(\mathbf{k})}{2T}\right)\frac{\varepsilon(\mathbf{k})-\mu}{E(\mathbf{k})}\right], \tag{9}$$

where $E(\mathbf{k})=\sqrt{(\varepsilon(\mathbf{k})-\mu)^2+\Delta^2}$ is the quasi-particle spectrum in the superconducting state. The solution of the system of equations (8) and (9) gives the doping- and the temperature-dependence of Δ, and also the doping-dependence of the mean-field critical temperature T_c^{MF} (at $\Delta=0$).

At $T=0$, the integration in Eqs. (8), (9) can be easily performed, and one gets the following system of equation for the superconducting gap and chemical potential as functions of the Fermi energy $\varepsilon_F=v_F\sqrt{8\pi n_f}$, coupling $\lambda=V_0/4\pi v_F^2$ and bosonic frequency ω_0:

$$\frac{1}{\lambda}=\theta(\omega_0-\mu)\left\{\sqrt{\omega_0^2+\Delta^2}-\sqrt{\mu^2+\Delta^2}+\mu\ln\left[(\omega_0+\sqrt{\omega_0^2+\Delta^2})(\mu+\sqrt{\mu^2+\Delta^2})/\Delta^2\right]\right\}+\theta(\mu-\omega_0)2\mu\ln\left((\omega_0+\sqrt{\omega_0^2+\Delta^2})/\Delta\right), \tag{10}$$

$$\frac{\varepsilon_F^2}{4}=-\frac{\Delta^2}{2}+\mu^2+3\mu\sqrt{\mu^2+\Delta^2}+\Delta^2\ln\left((W-\mu+\sqrt{(W-\mu)^2+\Delta^2})(\mu+\sqrt{\mu^2+\Delta^2})/\Delta^2\right), \tag{11}$$

where $\theta(x)$ is step function and we assumed that the bandwidth W is much bigger than μ, ω_0, and Δ. One can find an approximate analytical solution of the equation for the zero-temperature superconducting gap:

$$\Delta\simeq\sqrt{2\omega_0\mu}e^{-(1/\lambda-\omega_0)/2\mu}\theta(\omega_0-\mu)+2\omega_0e^{-1/2\mu\lambda}\theta(\mu-\omega_0). \tag{12}$$

Deriving this result, we assumed $\omega_0>>\Delta$. One can show that at zero doping $\Delta=0$ when $\lambda<1/\omega_0$. In other words, in this case there is a critical value of the coupling below which there is no pairing. At finite doping, superconductivity takes place at any finite value of the coupling (see Ref. 12). One can compare the result of Eq. (12) with the corresponding expression for the two-dimensional system with the parabolic dispersion:

$$\Delta \simeq 2\sqrt{\omega_0}[\sqrt{\varepsilon_F}\,\theta(\omega_0-\mu)+\sqrt{\omega_0}\,\theta(\mu-\omega_0)]e^{-4\pi/mV_0}. \quad (13)$$

(see, e.g., Ref. 17). There are several important differences between the results (12) and (13). Namely, i) in the case of graphene, at low doping the gap is exponentially suppressed when μ is smaller then $1/2((1/\lambda)-\omega_0)$ (Figs. 1 and 2). This suppression is due to the μ factor in the exponent, which is proportional to the single-particle density of states (DOS) on the Fermi level. In the case with parabolic dispersion, the DOS on the Fermi level is a constant. ii) Another consequence of the difference of the density of states is increasing of the gap with the carrier density growth. In the case of Eq. (13), the gap is carrier density independent at large doping when approximately $\mu > \omega_0$. The doping dependence of the gap was analyzed by assuming that the doping is proportional to the chemical potential. Strictly speaking, in the two-dimensional system this is correct only for parabolic dispersion at large doping. As it follows from Eq. (11), in our case the particle density is proportional to μ^2 when $\mu >> \Delta, \Delta \ln W$, which is true when the doping is rather large and the coupling is weak. In other cases, the relation between n_f and μ is more complicated. Generally speaking, one can also obtain the crossover from superconductivity to superfluidity with doping decreasing or/and the coupling growth, when the chemical potential becomes negative (for over view, see Ref. 17). However, in actual paper we consider the superconductivity scenario only, by assuming that μ is positive (see Eq. (2), where the interaction is defined inside the belt $-\omega_0 < \varepsilon < \omega_0$ around the «Fermi energy» μ). The depencies of T_c^{MF} on n_f, V_0, and ω_0 are the same as those of $\Delta(T=0)$ (Figs. 3 and 4). Moreover, we have found that the ratio $2\Delta(T=0)/T_c^{MF}$ is close to the BCS value 3.52. In the next Section, we demonstrate that the real T_c is much lower than T_c^{MF} due to the thermal fluctuations in the two-dimensional system.

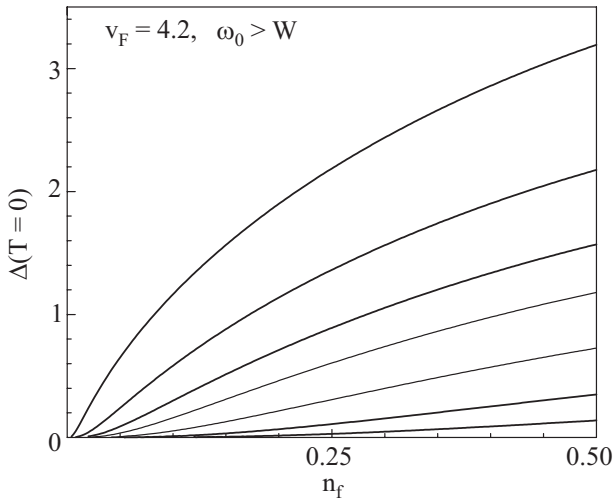

Fig. 2. The doping-dependence of the zero-temperature gap at different values of coupling V_0 = 3.2, 4, 5, 6, 6.8, 8 and 10.

4. Fluctuations

In order to study the fluctuation effects in the system, we represent the fermionic operators as the product of the neutral fermions and the phases:

$$\psi_\sigma(\tau,\mathbf{r}) = \chi_\sigma(\tau,\mathbf{r})\exp(i\theta(\tau,\mathbf{r})/2),$$
$$\psi^\dagger_\sigma(\tau,\mathbf{r}) = \exp(-i\theta(\tau,\mathbf{r})/2)\chi^\dagger_\sigma(\tau,\mathbf{r}), \quad (14)$$

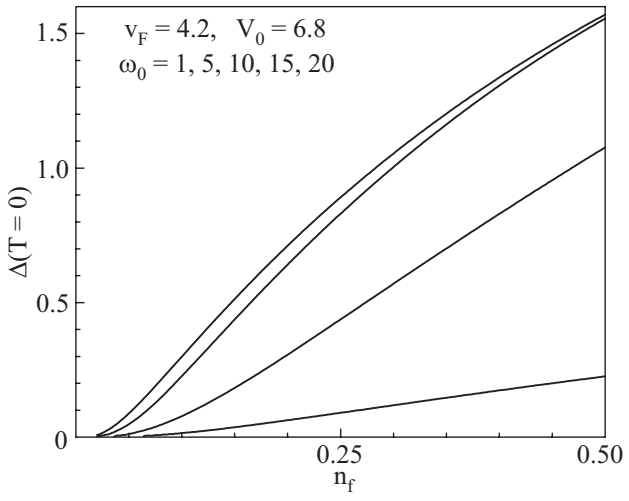

Fig. 1. The doping dependence of the zero-temperature gap at different values of the bosonic frequency ω_0. Here and in other Figures all parameters are given in eV, and the carrier density is divided by two.

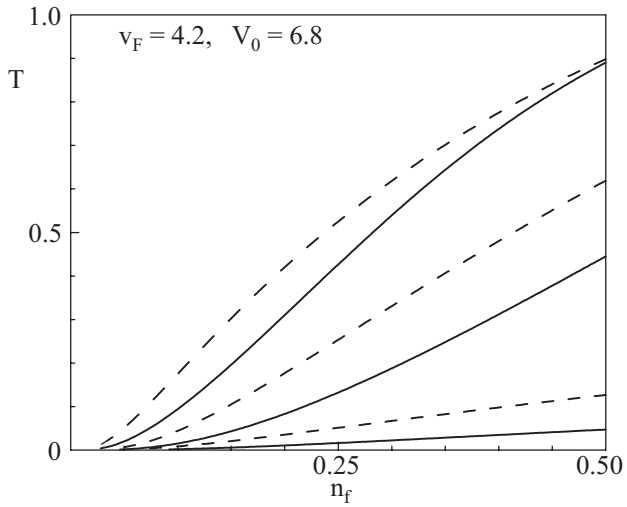

Fig. 3. The temperature-carrier density phase diagram at different values of frequency ω_0 = 1, 5, 20. The solid curves correspond to T_c and the dashed curves — to T_c^{MF}.

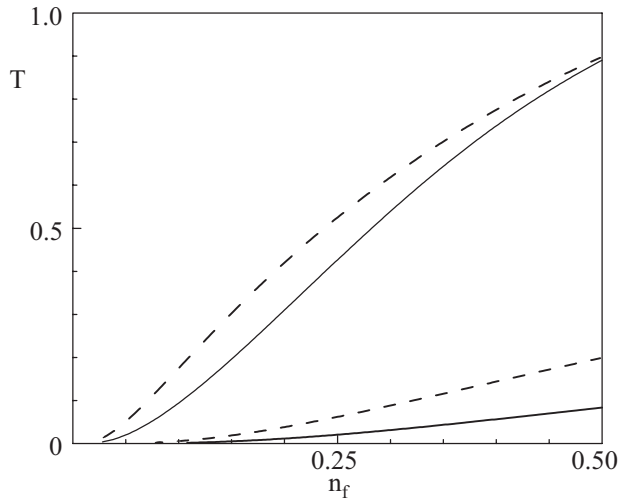

Fig. 4. The same as in the previous Figure, but for different values of the coupling $V_0 = 4$, 6.8, and $\nu_F = 4.2$, $\omega_0 = 20$.

such that

$$\Psi(\tau,\mathbf{r}) = e^{i\tau_z\theta(\tau,\mathbf{r})/2}\Upsilon(\tau,\mathbf{r}) = e^{i\tau_z\theta(\tau,\mathbf{r})/2}\begin{pmatrix}\chi_\uparrow(\tau,\mathbf{r})\\ \chi_\downarrow^\dagger(\tau,\mathbf{r})\end{pmatrix},$$

$$\Psi^\dagger(\tau,\mathbf{r}) = \Upsilon^\dagger(\tau,\mathbf{r})e^{-i\tau_z\theta(\tau,\mathbf{r})/2} = \begin{pmatrix}\chi_\uparrow^\dagger(\tau,\mathbf{r}),\\ \chi_\downarrow(\tau,\mathbf{r})\end{pmatrix}e^{-i\tau_z\theta(\tau,\mathbf{r})/2}. \tag{15}$$

In this case, the order parameters become

$$\Phi(\tau,\mathbf{r}_1,\mathbf{r}_2) = \Delta(\tau,\mathbf{r}_1,\mathbf{r}_2)\exp(i\theta(\mathbf{R})),$$
$$\Phi^\dagger(\tau,\mathbf{r}_1,\mathbf{r}_2) = \Phi^*(\tau,\mathbf{r}_1,\mathbf{r}_2) = \Delta(\tau,\mathbf{r}_1,\mathbf{r}_2)\exp(-i\theta(\mathbf{R})). \tag{16}$$

In the last equations, $\Delta(\tau,\mathbf{r}_1,\mathbf{r}_2) = |\Phi(\tau,\mathbf{r}_1,\mathbf{r}_2)|$ is the modulus of the order parameter, and $\theta(\mathbf{R}) \simeq [\theta(\mathbf{r}_1) + \theta(\mathbf{r}_2)]/2$, where $\mathbf{R} = (\mathbf{r}_1 + \mathbf{r}_2)/2$ is the center-of-mass coordinate of two electrons, is its phase. This approximation corresponds to the limit of weak thermal fluctuations, when the space dependence of the phase is not strong. Below, we neglect the time-(quantum-) fluctuations, which become important only at very low temperatures, and also assume that the gap depends only on the relative electron coordinate $\mathbf{r}_1 - \mathbf{r}_2$, i.e.

$$\Phi(\tau,\mathbf{r}_1,\mathbf{r}_2) \simeq \Delta(\mathbf{r}_1 - \mathbf{r}_2)e^{i\theta(\mathbf{R})}. \tag{17}$$

The approximation given by Eq. (17) means that the dynamics of Cooper pairs is described by the order-parameter modulus, and its symmetry depends on the relative pair coordinate, whereas the motion of the superconducting condensate is defined by the slowly varying function $\theta(\mathbf{R})$, which depends on the center-of-mass coordinate of electrons.

Substitution of Eqs. (15) and (17) into Eq. (7) and integration over the fermionic fields $\Upsilon^\dagger$ and Υ yield the following equation for the thermodynamic potential as a function of θ:

$$\Omega(\Delta,\partial_\tau\theta,\nabla\theta) = T\int_0^{1/T} d\,\tau\int d\mathbf{r}_1\int d\mathbf{r}_2\,\frac{1}{2}\frac{|\Delta(\mathbf{r}_1-\mathbf{r}_2)|^2}{V((\mathbf{r}_1-\mathbf{r}_2)} - \mathrm{Tr}\ln[G^{-1}-\Sigma] + \mathrm{Tr}\ln G^{-1}, \tag{18}$$

where

$$G_{\mathbf{k}}(t,t') = -i\langle T(\Upsilon_{\mathbf{k}}(t)\Upsilon_{\mathbf{k}}^\dagger(t')\rangle \tag{19}$$

is the time-ordered mean-field matrix Nambu Green's function. It has the following form in the Matsubara frequency-momentum representation:

$$G_{\mathbf{k}}(i\omega_n) = \frac{1}{i\omega_n - \tau_z(\varepsilon(\mathbf{k})-\mu) - \tau_x\Delta(\mathbf{k})}. \tag{20}$$

In Eq. (18), Σ is the self-energy of the nonhomogeneous Green's function, which depends on the gradients of the phase of the order parameter:

$$\Sigma(\mathbf{r}_1,\mathbf{r}_2) = \delta(\mathbf{r}_1-\mathbf{r}_2)\times$$
$$\times[e^{-i\tau_z\theta(\mathbf{r}_1)/2}\tau_z\varepsilon(-i\nabla_{\mathbf{r}_2})e^{i\tau_z\theta(\mathbf{r}_2)/2} - \tau_z\varepsilon(-i\nabla_{\mathbf{r}_2})]. \tag{21}$$

In Eq. (18), Tr means the space-time integration and the matrix trace (for details, see, e.g., Ref. 18). In order to find the thermodynamical potential, it is convenient to expand the loragithm in terms of the powers of Σ:

$$\mathrm{Tr}\ln[G^{-1}-\Sigma] = \mathrm{Tr}\ln G^{-1} + \mathrm{Tr}\sum_n\frac{1}{n}(G\Sigma)^n. \tag{22}$$

This allows to get the following expression for the second order expansion of the thermodynamical potential in the limit of small fluctuations of the phase of the order parameter:

$$\Omega(\Delta,\theta) = \frac{\mathcal{J}}{2}\int d^2r(\nabla\theta)^2, \tag{23}$$

where

$$\mathcal{J} = \frac{v_F^2}{8T} \int \frac{d^2k}{(2\pi)^2} \frac{1}{\cosh^2[E(\mathbf{k})/2T]} \tag{24}$$

is the superconducting stiffness (see Appendix).

One can find the equation for the critical temperature of the BKT transition T_c, below which the phases of order parameter become algebraically ordered, by using the analogy with the 2D spin *XY*-model, where the spin orientation correspond to the vector $\nabla\theta$ in our consideration. This equation has the following form:

$$T_c = \frac{\pi}{2} \mathcal{J}(\Delta(\mu, T_c), \mu, T_c), \tag{25}$$

where the function $\mathcal{J}$ is defined by Eq. (24). The doping-, interaction- and bosonic frequency-dependencies of T_c can be found by solving the system of Eqs. (8), (9), and (25).

The numerical analysis shows that T_c is significantly lower than T_c^{MF} (Figs. 3, 4). The critical temperature decreases with the bosonic frequency decreasing (Fig. 3). When the coupling is not too strong, T_c is exponentially suppressed (Fig. 4), similar to T_c^{MF} and $\Delta(T=0)$. Our analysis shows that T_c starts to grow rapidly at densities n_f higher than the critical value $n_f^{\rm cr} \sim 1/V_0^2$ at large ω_0. In particular, is this case in order to get the critical temperature bigger than 1 K, one needs to have the effective coupling $\sqrt{n_f} V_0 \sim 1$ eV (see Ref. 14). It is possible that T_c can be higher than the values obtained in this paper due to a van Hove singularity, if rather high values of doping ($\varepsilon_F \sim 3$ eV) can be obtained. We have found also that the ratio $2\Delta(T=0)/T_c$ is bigger than the BCS value 3.52, and it increases with doping decreasing and acquires values approximately two times bigger those of BCS.

Besides the reduction of the value of the critical temperature, another important consequence of the thermodynamic fluctuations is a finite DOS in the gap region at temperatures $T_c < T < T_c^{MF}$, or the pseudogap phase [14], which can be observed experimentally in graphene systems. Similar phase was observed in high-temperature superconductors, where the thermal fluctuations can be also responsible for its formation (see, e.g., Ref. 19 and references therein).

5. Conclusions

We have studied the doping-, coupling- and boson frequency-dependencies of the superconducting properties of a model of the doped single-layer graphene by taking into account the thermodynamical fluctuations of the superconducting order parameter. It is shown that due to the form of the doping-dependence of the free-electron density of states the superconductivity is suppressed at low doping, and at high values of doping the superconducting gap and critical temperature increase with doping growth, which is different from the two-dimensional system with the parabolic dispersion. We have found that the fluctuations lead to a significant reduction of the critical temperature. Namely, for the realistic values of the model parameters, the critical temperature T_c is exponentially suppressed at doping $n_f < n_f^{\rm cr} \sim 1{\rm eV}/V_0$ at large values of the bosonic frequency. The critical value $n_f^{\rm cr}$ decreases with ω_0 decreasing. The results obtained in the present paper can help to establish the mechanism of possible superconductivity in graphene, since the effective electron-electron coupling and the bosonic frequency can be estimated by comparing the theoretical results with the experimentally measured temperature-carrier density phase diagram.

Acknowledgements

V.M.L. acknowledges a support through the Special program of the NAS of Ukraine. V.T. thanks the Department of Energy for a partial support under grant number DOE-DE-FG02-07ER15842.

Appendix A: The thermodynamical potential for the phases of the order parameter

In this Appendix, we derive the expression for the low-energy effective action for the phases of the superconducting order parameter, Eqs. (23) and (24).

In order to find the thermodynamical potential within the $(\nabla\theta)^2$-approximation, one needs to consider the first two terms in the expansion of Eq. (22):

$$\mathrm{Tr}\ln[G^{-1} - \Sigma] \simeq \mathrm{Tr}\ln G^{-1} + \mathrm{Tr}[G\Sigma] + \frac{1}{2}\mathrm{Tr}[G\Sigma G\Sigma], \tag{A.1}$$

where the θ-dependent part of the self-energy given by Eq. (21) is

$$\Sigma(\mathbf{r}_1, \mathbf{r}_2) = \delta(\mathbf{r}_1 - \mathbf{r}_2) \times \left[e^{-i\tau_z\theta(\mathbf{r}_1)/2} \tau_z v_F \sqrt{-\nabla^2_{\mathbf{r}_2}}\, e^{i\tau_z\theta(\mathbf{r}_2)/2} - \tau_z v_F \sqrt{-\nabla^2_{\mathbf{r}_2}} \right]. \tag{A.2}$$

Since the last equation contains the square roots of the Laplace operators, it is more convenient to calculate the trace of the operators in Eq. (A1) in the momentum space. In this case, the action of the operator $\sqrt{-\nabla^2_{\mathbf{r}}}$ has a simple form:

$$\sqrt{-\nabla^2_{\mathbf{r}}} f(\mathbf{r}) = \sum_{\mathbf{k}} |\mathbf{k}| e^{i\mathbf{kr}} f_{\mathbf{k}}, \tag{A.3}$$

where $f_{\mathbf{k}}$ is the Fourier transform of $f(\mathbf{r})$. In the momentum representation, the effective action Eq. (23) has the following form:

$$\Omega = \frac{J}{2} \sum_{\mathbf{k}} \mathbf{k}^2 \theta_{\mathbf{k}} \theta_{-\mathbf{k}} . \qquad \text{(A.4)}$$

Therefore, one needs to find the coefficient in front of $\theta_{\mathbf{k}} \theta_{-\mathbf{k}}$, which is the half of the the superconducting stiffness J in Eq. (23). Since the linear in Σ term leads to the term proportional to $|\mathbf{k}| \theta_{\mathbf{k}}$ in Eq. (A1), we neglect it. In fact, the linear in θ term can be absorbed into the bilinear term by defining a new phase θ shifted by the corresponding factor. Therefore, we consider only the second-order term:

$$\frac{1}{2} Tr[G\Sigma G\Sigma] = -\frac{v_F^2}{8} \mathrm{Tr} \left[\sum_{\mathbf{p}} G_{\mathbf{p}}^2 \right] \sum_{\mathbf{k}} \mathbf{k}^2 \theta_{\mathbf{k}} \theta_{-\mathbf{k}} , \qquad \text{(A.5)}$$

which gives

$$J = -\frac{v_F^2 T}{4} \mathrm{Tr} \sum_{\mathbf{p}} [G_{\mathbf{p}}^2]. \qquad \text{(A.6)}$$

In the derivation of Eq. (A5), we only kept the term proportional to $\mathbf{k}^2 \theta_{\mathbf{k}} \theta_{-\mathbf{k}}$. Equation (A5) gives the expression for the superconducting stiffness (24).

1. A.H. Castro Neto, F. Guinea, N.M.R. Peres, K.S. Novoselov, and A.K. Geim, *Rev. Mod. Phys.* (in press); *preprint arXiv:0709.1163v2*.
2. Y. Kopelevich, P. Esquinazi, J.H.S. Torres, and S. Moehlecke, *J. Low Temp. Phys.* **119**, 691 (2000).
3. R.R. da Silva, J.H.S. Torres, and Y. Kopelevich, *Phys. Rev. Lett*. **87**, 147001 (2001).
4. S. Moehlecke, Y. Kopelevich, and M.B. Maple, *Phys. Rev.* **B69** , 134519 (2004).
5. Y. Kopelevich, S. Moehlecke, and R.R. da Silva, in: *Carbon Based Magnetism*, T. Makarova and F. Palacio (eds.), Elsevier Science (2006), Chap. 18.
6. B. Uchoa, G.G. Cabrera, and A.H. Castro Neto, *Phys. Rev.* **B71**, 184509 (2005).
7. E. Zhao and A. Paramekanti, *Phys. Rev. Lett*. **97**, 230404 (2006).
8. E.C. Marino and Lizardo H.C.M. Nunes, *Nucl. Phys.* **B741**, 404 (2006); *Physica*, Amsterdam **460C–462C**, 1101 (2007); *Nucl. Phys*. **B769**, 275 (2007).
9. B. Uchoa and A.H. Castro Neto, *Phys. Rev. Lett.* **98**, 146801 (2007).
10. M. Black-Schaffer and S. Doniach, *Phys. Rev*. **B75**, 134512 (2007).
11. C. Honerkamp, *Phys. Rev. Lett.* **100**, 146404 (2008).
12. N.B. Kopnin and E.B. Sonin, *Phys. Rev. Lett*. **100**, 246808 (2008).
13. J. Gonzales, *Phys. Rev.* **B78**, 205431 (2008).
14. V.M. Loktev and V.M. Turkowski, *Phys. Rev. Lett.* **B79**, 233402 (2009).
15. V.M. Loktev and V.M. Turkowski, *JETP* **87**, 329 (1998).
16. V.M. Loktev, S.G. Sharapov, and V.M. Turkowski, *Physica* **C296**, 84 (1998).
17. V.M. Loktev and V.M. Turkowski, *Fiz. Nizk. Temp.* **30**, 247 (2004) [*Low Temp. Phys*. **30**, 179 (2004)].
18. S.G. Sharapov, H. Beck, and V.M. Loktev, *Phys. Rev.* **B64**, 134519 (2001).
19. P.A. Lee, N. Nagaosa, and X.G. Wen, *Rev. Mod. Phys*. **78**, 17 (2006).
20. V.M. Loktev, R.M. Quick, and S.G. Sharapov, *Phys. Rep.* **349**, 1 (2001).