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BCS–BEC crossover and nodal points contribution in *p*-wave resonance superfluids

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We solve the Leggett equations for BCS–BEC crossover of the resonance *p*-wave superfluid. We calculate sound velocity, specific heat and the normal density for the BCS-domain ($\mu > 0$), BEC-domain ($\mu < 0$) as well as for the interesting interpolation point ($\mu = 0$) in the triplet A_1-phase in 3D. We are especially interested in the quasiparticle contribution coming from the zeroes of the superfluid gap in the A_1-phase. We discuss the spectrum of orbital waves and the superfluid hydrodynamics at temperature $T \to 0$. In this context we elucidate the difficult problem of chiral anomaly and mass-current nonconservation appearing in the BCS-domain. We present the different approaches to solve this problem. To clarify this problem experimentally we propose an experiment for the measurement of anomalous current in superfluid A_1-phase in the presence of aerogel for ^{3}He and in the presence of Josephson tunneling structures for the ultracold gases in magnetic traps.

PACS: 67.30.H– Superfluid phase of ^{3}He;
67.85.Lm Degenerate Fermi gases;
74.20.Rp Pairing symmetries (other than *s*-wave).

Keywords: BCS–BEC crossover, Feshbach resonance, superfluidity, ^{3}He-A, chiral anomaly.

1. Introduction

The first experimental results on *p*-wave Feshbach resonance [1–3] in ultracold fermionic gases ^{40}K and ^{6}Li make the field of quantum gases closer to the interesting physics of superfluid ^{3}He and the physics of unconventional superconductors such as Sr_2RuO_4. In this context it is important to try to build the bridge between the physics of ultracold gases and the physics of quantum liquids and to enrich both communities with the experience and knowledge accumulated in each of these fields. The purpose of the present paper is first of all to describe the transition from the weakly bound Cooper pairs with *p*-wave symmetry to strongly bound local *p*-wave pairs (molecules) and try to reveal the nontrivial topological effects connected with the presence of the nodes in the superfluid gap of the triplet *p*-wave A_1-phase. Note, that the A_1-phase symmetry is relevant both to ultracold Fermi-gases in the regime of *p*-wave Feshbach resonance and to superfluid ^{3}He-A in the presence of a magnetic field $B > H_c \sim T_c/\mu_B$, which is large enough to destroy isotropic B phase of ^{3}He already at $T = 0$. We pay the special attention to the spectrum of collective excitations and to the superfluid hydrodynamics of the A_1-phase, where the topological effects are very pronounced, especially in the BCS-domain. We propose the experimental verification of the different approaches connected with the problem of chiral anomaly and mass-current nonconservation in superfluid A_1-phase of ^{3}He in the presence of aerogel as well as for the A_1 *p*-wave condensates in magnetic traps in the presence of Josephson tunneling structures.

Our paper is organized as follows. Chapter 1 provides an Introduction. In Chapter 2 we briefly comment on the recent experiments on *p*-wave Feshbach resonance and describe the global phase-diagram for *p*-wave resonance superfluids. In Chapter 3 we describe the quasiparticle spectrum and nodal points in A_1-phase. In Chapter 4 we solve Leggett equations for triplet superfluids with the symmetry of A_1-phase and study the behavior of superfluid gap Δ, chemical potential μ, and sound velocity c_s in BCS- ($\mu > 0$) and BEC-domains ($\mu < 0$) as well as close to the interesting interpolation point $\mu = 0$. In Chapter 5 we study the temperature behavior of the normal density ρ_n and specific heat C_v in BCS-domain, in BEC-domain, and close to $\mu = 0$. In Chapter 6 we describe the orbital waves spectrum in BCS- and BEC-domains of the A_1-phase and describe the problem of chiral anomaly (mass-current nonconservation) which exists in the superfluid hydrodynamics of A_1-phase in BCS-domain at $T \to 0$. In Chapter

7 we present two different approaches to the calculation of anomalous current: first one based on supersymmetric hydrodynamics [4] and the second one on the analogy with Dirac equation in quantum electrodynamics (QED) [5,6]. Note that both approaches are very general. The first of them is based on the inclusion of fermionic goldstone mode in low-frequency hydrodynamic action [4]. It can be useful for all nodal superfluids and superconductors with zeroes of the superconductive gap such as ^{3}He-A, Sr_2RuO_4, UPt_3, UNi_2Al_3, $U_{1-x}Th_xBe_{13}$ and so on [7]. The second approach is connected with the appearance of the Dirac-like spectrum of fermions with zero mode [5,6] which also arises in many condensed-matter systems such as ^{3}He-A, chiral superconductor Sr_2RuO_4, organic conductor α-(BEDT-TTF)$_2I_3$, or recently discovered graphene [7–10]. In Chapter 8 we propose a scenario according to which the chiral anomaly exists only in high-frequency ballistic (or Knudsen) regime and is destroyed by damping in low-frequency hydrodynamic regime. In Chapter 9 we propose to use aerogel to increase the damping in superfluid ^{3}He-A_1-phase and thus to make the transition to hydrodynamic regime easier. We also provide a brief discussion on the role of Josephson tunneling structure in magnetic traps. Finally in Chapter 10 we present our conclusions and acknowledgements.

2. Feshbach resonance and phase-diagram for *p*-wave resonance superfluids

In the first experiments on p-wave Feshbach resonance the experimentalists measure the molecule formation in the ultracold fermionic gas of ^{6}Li-atoms close to resonance magnetic field B_0 [1,2].

In the last years the analogous experiments on p-wave molecules formation in spin-polarized fermionic gas of ^{40}K-atoms were started [3]. The lifetime of p-wave molecules is rather short yet [1–3]. However the physicists working in ultracold gases have started intensively to study the huge bulk of experimental and theoretical wisdom accumulated in the physics of superfluid ^{3}He in 1970-s–1980-s (see [11]).

To understand the essence of p-wave Feshbach resonance we should recollect the basic formula on p-wave scattering amplitude from [12]:

$$f_{l=1} = \frac{p^2}{\frac{1}{V_p} + \frac{1}{2}\frac{p^2}{r_0} + ip^3}, \qquad (1)$$

where $l = 1$ is an orbital momentum in the p-wave channel, $V_p = r_0^2 a_p$ is scattering volume, a_p is p-wave scattering length, r_0 is the range of the interaction, p is the scattered momentum. For Feshbach resonance in fermionic systems $p \sim p_F$ and usually $p_F r_0 < 1$. The scattering

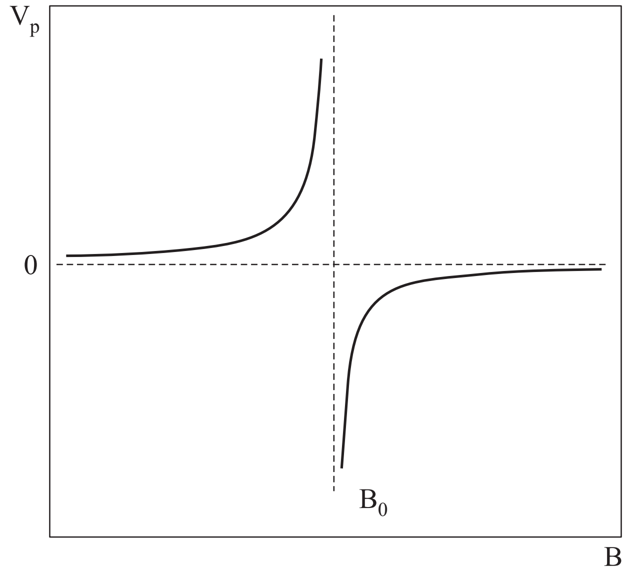

Fig. 1. The sketch of the p-wave Feshbach resonance.

length a_p and hence the scattering volume V_p are divergent in the resonance magnetic field B_0 (see Fig. 1): $1/V_p = 1/a_p = 0$. The imaginary part of the scattering amplitude f_p is small, so p-wave Feshbach resonance is intrinsically narrow.

The first theoretical review article on p-wave Feshbach resonance mostly deals with mean-field two-channel description of the resonance [13]. In our paper we will study p-wave Feshbach resonance in the framework of one-channel description, which is more close to the physics of superfluid ^{3}He and captures rather well the essential physics of BCS–BEC crossover in p-wave superfluids.

In magnetic traps (in the absence of the so-called dipolar splitting) people usually study fully (100%) polarized gas or more precisely — one hyperfine component of the gas. On the language of ^{3}He they study the pairs with $S_{\rm tot} = S_z^{\rm tot} = 1$, or $|\uparrow\uparrow$>-pairs. In our paper we consider p-wave triplet A_1-phase where just $S_{\rm tot} = S_z^{\rm tot} = 1$.

The qualitative picture of the global phase-diagram of the BCS–BEC crossover in A_1-phase is presented in Fig. 2.

BCS-domain where chemical potential $\mu > 0$, occupies on the global phase-diagram, the region of negative values of the gas parameter $\lambda_p = V_p p_F^3 < 0$ (or the negative values of the scattering length a_p). It stretches also to the small positive values of the inverse gas parameter $1/\lambda_p \le 1$ and is separated from the BEC-domain (where $\mu < 0$ and the inverse gas parameter is large and positive $1/\lambda_p \ge 1$) by the crossover line $\mu(T) = 0$. Usually in the regime of Feshbach resonance the density of «up» spins $n = p_F^3/6\pi^2$ is fixed. Deep inside BCS-domain (for small absolute values of the gas parameter $|\lambda_p| << 1$) we have

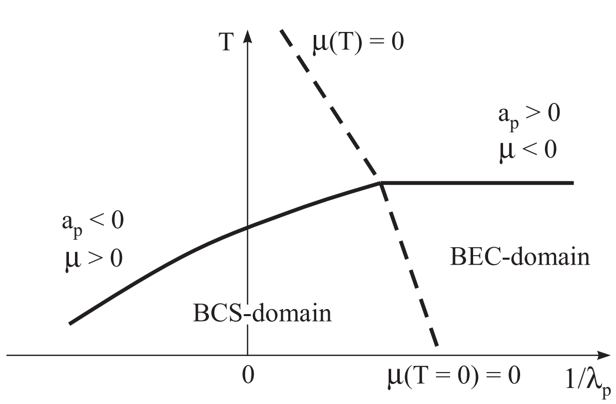

Fig. 2. Qualitative picture of the BCS–BEC crossover in A_1-phase in the axis temperature T and $1/\lambda_p$ — inverse gas parameter for p-wave superfluids ($\lambda_p = V_p\, p_F^3$, V_p is scattering volume).

the standard BCS-like formula for the critical temperature of the A_1-phase:

$$T_{Cp} = 0.1\varepsilon_F\, e^{-\pi/2|\lambda_p|}, \quad (2)$$

where the preexponential factor for the 100% polarized A_1-phase is defined by second order diagrams of Gor'kov and Melik-Barchudarov type [14] and approximately equals to $0.1\varepsilon_F$ [15].

Deep in BEC-domain ($\lambda_p << 1$) the well-known formula of Einstein is working in principal approximation for Bose-condensation of p-wave molecules with the density $n/2$ and the mass $2m$:

$$T_{Cp} = 3.31\frac{(n/2)^{2/3}}{2m}. \quad (3)$$

In the unitary limit $1/\lambda_p = 0$. Hence here $T_{Cp} \approx 0.1\varepsilon_F$ and we are still in BCS-regime. In the rest of the paper we will consider low temperatures $T << T_c$, so we will work deep in the superfluid parts of BCS- and BEC-domains of the A_1-phase.

3. Quasiparticle energy and nodal points in A_1-phase

For standard s-wave pairing the quasiparticle spectrum reads:

$$E_p = \sqrt{\left(\frac{p^2}{2m} - \mu\right)^2 + \Delta_0^2}. \quad (4)$$

It has no zeroes (no nodes), so the topology of the s-wave pairing problem is trivial. For triplet A_1-phase however:

$$E_p = \sqrt{\left(\frac{p^2}{2m} - \mu\right)^2 + \frac{|\boldsymbol{\Delta}\cdot\mathbf{p}|^2}{p_F^2}}, \quad (5)$$

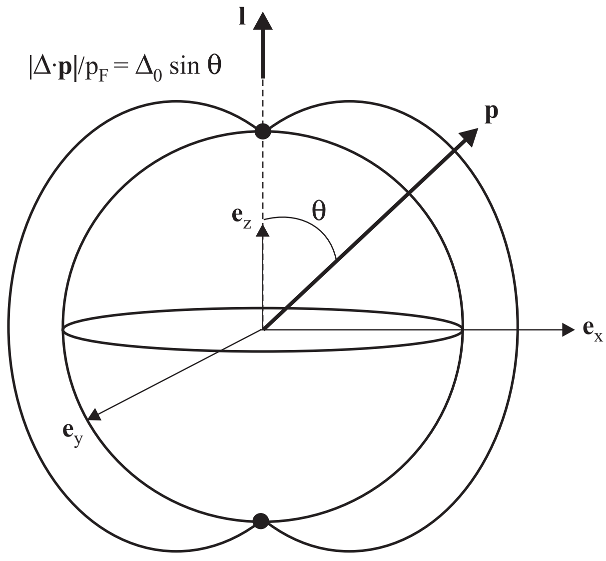

Fig. 3. The topology of the superfluid gap in A_1-phase. θ is the angle between momentum **p** and the axis of orbital momentum quantization $\mathbf{l} = \mathbf{e}_z$. There are two nodes in the quasiparticle spectrum corresponding to the south and north poles.

where $\boldsymbol{\Delta} = \Delta_0(\mathbf{e}_x + i\mathbf{e}_y)$ is the complex order parameter in A_1-phase, Δ_0 is the magnitude of the superfluid gap. In fact: $|\boldsymbol{\Delta}\cdot\mathbf{p}|^2 = \Delta_0^2 p^2 \sin^2\theta = \Delta_0^2(\mathbf{p}\times\mathbf{l})^2$, where $\mathbf{l} = \mathbf{e}_x \times \mathbf{e}_y$ is the unit vector of orbital momentum (see Fig. 3). Note that p_F is fixed by fixed density n. Angle θ is the angle between momentum **p** and the orbital momentum quantization axis $\mathbf{l} = \mathbf{e}_z$.

For $\mu > 0$ (BCS-domain) there are two nodes in the spectrum for $p^2/2m = \mu$ and for $\theta = 0$ or π. For $\mu < 0$ (BEC-domain) there are no nodes. The interesting point $\mu = 0$ is a boundary between the totally gapped BEC-domain and the BCS-domain with two nodes of the quasiparticle spectrum corresponding to the south and north poles of Fig. 3. Sometimes this point is called the point of *topological* phase transition. However, as we shall see below, this phase transition does not manifest itself in the Leggett equations for the superfluid gap Δ and the chemical potential μ, which are continuous functions for $\mu \to \pm 0$.

4. Leggett equations for A_1-phase

The Leggett equations for the triplet A_1-phase in 3D are the evident generalization of the standard Leggett equations for the s-wave BCS–BEC crossover. The first equation reads:

$$n = \frac{p_F^3}{6\pi^2} = \int_0^{1/r_0} \frac{p^2\,dp}{4\pi^2}\int_{-1}^{1} dx\, \frac{1}{2}\left(1 - \frac{\xi_p}{E_p}\right), \quad (6)$$

where

$$\xi_p = \left(\frac{p^2}{2m} - \mu\right), \quad E_p = \sqrt{\xi_p^2 + \frac{\Delta_0^2 p^2}{p_F^2} \sin^2\theta}$$

is a quasiparticle spectrum, $x = \cos\theta$. This equation defines the chemical potential μ for fixed density n.

The second self-consistency equation defines the magnitude of the superfluid gap Δ_0. It reads:

$$-\pi m \,\mathrm{Re}\, \frac{1}{f_{l=1}(2\mu)} = \int_{-1}^{1} \frac{dx}{2} \int_{0}^{1/r_0} p^4 dp \left\{\frac{1}{E_p} - \frac{1}{\xi_p}\right\}, \quad (7)$$

where

$$\mathrm{Re}\, \frac{1}{f_{l=1}(2\mu)} = \left(\frac{1}{V_p} + \frac{4m\mu}{\pi r_0}\right)$$

is a real part of an inverse scattering amplitude in p-wave channel for total energy $E = 2\mu$ of colliding particles. This energy is relevant for pairing problem.

The solution of Leggett equations yields for the BCS-domain:

$$\Delta_0 \sim \varepsilon_F \, \mathrm{e}^{-\pi/2|\lambda_p|} \sim T_{Cp} \; ; \; \mu \approx \varepsilon_F > 0 \,. \quad (8)$$

The sound velocity in 3D reads:

$$c_s = \left(\frac{n}{m}\frac{d\mu}{dn}\right)^{1/2} = \frac{v_F}{\sqrt{3}} \,. \quad (9)$$

For $1/|\lambda_p| = 0$: $\Delta_0 \sim \varepsilon_F$ and hence unitary limit is still inside BCS-domain.

In BEC-domain:

$$\Delta_0 \approx 2\varepsilon_F \sqrt{p_F r_0} \ll \varepsilon_F \quad \text{for} \quad p_F r_0 \ll 1 \,, \quad (10)$$

and chemical potential $\mu = -\frac{|E_b|}{2} + \frac{\mu_B}{2} < 0$, where

$$|E_b| = \frac{\pi}{2 m r_0 a_p} \quad (11)$$

is a binding energy of a triplet pair (molecule).

Accordingly:

$$\mu_B \approx \frac{4\varepsilon_F}{3} \sqrt{p_F r_0} \quad (12)$$

is a bosonic chemical potential which governs the repulsive interaction between two p-wave molecules.

The sound velocity in BEC-domain reads:

$$c_s = \left(\frac{n_B}{2m}\frac{d\mu_B}{dn_B}\right)^{1/2} \approx \frac{v_F}{\sqrt{3}} \sqrt{p_F r_0} \ll v_F \quad \text{for} \quad p_F r_0 \ll 1 \,, \quad (13)$$

where $n_B = n/2$ is bosonic density.

In the interesting interpolative region where $\mu \to 0$ (more rigorously $|\mu| < \Delta_0^2/\varepsilon_F$) we have:

$$\Delta_0(\mu = 0) = 2\varepsilon_F \sqrt{p_F r_0} \quad (14)$$

for the magnitude of the superfluid gap. It is possible to show that it behaves in a regular way (linearly in μ) for small μ.

For the gas parameter λ_p in the point $\mu = 0$ we have

$$\lambda_p(\mu = 0) = \frac{3}{4} > 0 \,. \quad (15)$$

Hence the interesting point $\mu = 0$ is effectively in BEC-domain (in the domain of positive p-wave scattering length $a_p > 0$).

The sound velocity (compressibility of the system) is also continuous close to $\mu = 0$:

$$c_s = \frac{v_F}{\sqrt{3}} \sqrt{p_F r_0} \quad (16)$$

as in BEC-domain (coincides with (16)). It means that the derivative

$$\left.\frac{\partial\mu}{\partial n}\right|_{\mu\to\pm 0} = \frac{2\pi^2 r_0}{m}$$

is continuous close to $\mu = 0$. The gap derivative $\partial\Delta_0/\partial\mu$ also behaves in a continuous manner close to $\mu = 0$:

$$\left.\frac{\partial\Delta_0}{\partial\mu}\right|_{\mu\to\pm 0} = \frac{3}{2\sqrt{p_F r_0}} \,.$$

We can conclude this chapter by the statement that inspite of nontrivial topology in A_1-phase, the BCS–BEC crossover in the framework of Leggett equations has a standard character and does not reveal any singularities.

5. Specific heat and normal density at temperatures $T \ll T_c$

In this chapter we study the thermodynamic functions namely, normal density ρ_n and specific heat C_v in resonance p-wave superfluids with A_1-symmetry at low temperatures $T \ll T_c$. Our goal is to try to find nontrivial contributions to ρ_n and C_v from the nodal points.

5.1. Specific heat in A_1-phase

The fermionic (quasiparticle) contribution to C_v yields (see [17]):

$$C_v = \int \frac{\partial n_0(E_p/T)}{\partial T} E_p \frac{d^3\mathbf{p}}{(2\pi)^3} \,, \quad (17)$$

where

$$n_0\left(\frac{E_p}{T}\right) = \frac{1}{(\mathrm{e}^{E_p/T} + 1)}$$

is quasiparticle distribution function, E_p is quasiparticle energy given by (5).

The results of the calculations yield:

$$C_v \sim N(0)\frac{T^3}{\Delta_0^2} \qquad (18)$$

for the BCS-domain, where

$$N(0) = \frac{mp_F}{2\pi^2}$$

is a density of states at Fermi-surface. In the BEC region we consider separately contribution of the fermionic and bosonic modes. The contribution of the fermionic excitations in the BEC-domain reads:

$$C_v \sim \frac{(2mT)^{3/2}}{2\pi}\frac{E_b^2}{4T^2}\,\mathrm{e}^{-|E_b|/2T}, \qquad (19)$$

where $|E_b|$ is given by (11).

Finally in the interesting region of small μ and low temperatures ($|\mu| < T < \Delta_0^2/\varepsilon_F$) we have

$$C_v \sim \frac{(2mT)^{3/2}}{2\pi^2}\frac{\varepsilon_F T}{\Delta_0^2}. \qquad (20)$$

For small $|\mu|$ but intermediate temperatures $|\mu| < \Delta_0^2/\varepsilon_F < T < \Delta_0$:

$$C_v \sim \frac{(2mT)^{3/2}}{2\pi}. \qquad (21)$$

Bosonic contribution (contribution from the sound waves) yields:

$$C_v^B \sim \frac{T^3}{c_s^3}\frac{1}{2\pi^2}, \qquad (22)$$

where the sound velocity c_s is given by (9) in BCS-domain and by (13), (16) in BEC-domain and close to $\mu = 0$.

We can see that it is possible to separate a power-law fermionic contribution $C_v \sim T^{5/2}$ at low temperatures and $C_v \sim T^{3/2}$ at intermediate temperatures from bosonic one $C_v^B \sim T^3$ close to the interesting point $\mu = 0$.

5.2. Normal density in A_1-phase

The quasiparticle contribution to normal density yields (see [14]):

$$\rho_n = -\frac{1}{3}\int p^2\,\frac{\partial n_0(E_p/T)}{\partial E_p}\frac{d^3\mathbf{p}}{(2\pi)^3}. \qquad (23)$$

In the BCS-domain the evaluation of ρ_n yields:

$$\rho_n \sim \rho\frac{T^2}{\Delta_0^2}, \qquad (24)$$

where $\rho = nm$ is a total mass-density.

In the BEC-domain:

$$\rho_n \sim \frac{m}{\pi^2}(2mT)^{3/2}\,\mathrm{e}^{-|E_b|/2T}. \qquad (25)$$

Finally close to $\mu = 0$ at low temperatures ($|\mu| < T < \Delta_0^2/\varepsilon_F$):

$$\rho_n \sim \frac{m}{\pi^2}(2mT)^{3/2}\frac{\varepsilon_F T}{\Delta_0^2}. \qquad (26)$$

At intermediate temperatures $|\mu| < \Delta_0^2/\varepsilon_F < T < \Delta_0$:

$$\rho_n \sim \frac{m}{\pi^2}(2mT)^{3/2}. \qquad (27)$$

Bosonic (phonon) contribution from the sound waves yields (see Lifshitz, Pitaevskii [17]):

$$\rho_n^B \sim \frac{T^4}{c_s^5}, \qquad (28)$$

where c_s is again given by (5), (13) and (16) in BCS-, BEC-domain and close to $\mu = 0$, respectively. We can again separate a fermionic (quasiparticle) contribution to ρ_n ($\rho_n \sim T^{5/2}$ at low temperatures and $\rho_n \sim T^{3/2}$ at intermediate temperatures) from bosonic one ($\rho_n \sim T^4$) close to the point $\mu = 0$.

6. Orbital waves and the chiral anomaly in A_1-phase

The topological effects in A_1-phase are really pronounced in the spectrum of orbital waves and in the superfluid hydrodynamics at low temperatures $T \to 0$ especially in BCS-domain. Here by symmetry requirements we can write the following expression for the total mass-current:

$$\mathbf{j}_{\rm tot} = \mathbf{j}_B + \mathbf{j}_{\rm an}, \qquad (29)$$

where

$$\mathbf{j}_{\rm an} = -\frac{\hbar}{2m}C_0(\mathbf{l}\cdot\mathrm{rot}\,\mathbf{l})\mathbf{l} \qquad (30)$$

is an anomalous current. In BEC-domain $C_0 = 0$ and anomalous current is absent. However it is a difficult question whether $C_0 = 0$ or not in the BCS-domain. At the same time $\mathbf{j}_B$ in (29) is a total mass-current in BEC-domain for *p*-wave molecules. It reads:

$$\mathbf{j}_B = \rho\mathbf{v}_S + \frac{\hbar}{2m}\,\mathrm{rot}\,\frac{\rho\mathbf{l}}{2}, \qquad (31)$$

where $(\hbar\rho/2m)\mathbf{l}$ is the density of orbital momentum, $\mathbf{v}_S$ is a superfluid velocity.

Anomalous current $\mathbf{j}_{\rm an}$ violates conservation law for total mass-current (total linear momentum) $\mathbf{j}_{\rm tot}$ since it cannot be expressed as a divergence of some momentum tensor $\prod_{ik}$:

$$\frac{\partial j^i_{\text{an}}}{\partial t} \neq -\frac{\partial}{\partial x_k}(\Pi_{ik}) . \qquad (32)$$

Thus the presence of anomalous current destroys the superfluid hydrodynamics of the A_1-phase at $T \to 0$. Its contribution to the equation for total linear momentum (to $\partial j^i_{\text{tot}}/\partial t$) can be compensated only by adding the term with a normal velocity and normal density $\rho_n(T=0)\,\mathbf{v}_n$ to the total current $\mathbf{j}_{\text{tot}}$ already at $T = 0$ (see [5,6]). The anomalous current also changes significantly the spectrum of orbital waves. This additional Goldstone branch of collective excitations in the A_1-phase is connected with the rotation of **l**-vector around perpendicular axis. It is quadratic at low frequencies (A_1-phase is called an orbital ferromagnet). However, the coefficient in front of q^2 is drastically different in BCS- and BEC-domains.

In BEC-domain for small ω and **q**:

$$\omega \sim \frac{q^2}{m} . \qquad (33)$$

At the same time in BCS-domain:

$$\omega \sim \frac{q^2}{m}\frac{\rho}{(\rho - C_0)} . \qquad (34)$$

If according to (11), (12) the relative difference

$$\frac{(\rho - C_0)}{\rho} \sim \frac{\Delta_0^2}{\varepsilon_F^2} << 1$$

then the coefficient of q^2 in (34) is much larger in BCS-domain than in BEC-domain for instance in superfluid ^{3}He-A:

$$\frac{\Delta_0}{\varepsilon_F} \sim \frac{T_C}{\varepsilon_F} \sim 10^{-3}$$

and thus $(\rho - C_0)/\rho \sim 10^{-6}$.

There are two competing approaches how to deal with the complicated problem of anomalous current in BCS-domain at $T \to 0$.

7. Two different approaches to the problem of chiral anomaly in A_1-phase

The first approach [4] is based on supersymmetric hydrodynamics of the A_1-phase.

7.1. Supersymmetric hydrodynamics of the A_1-phase

The idea of [4] was to check whether the chiral anomaly is directly connected with the zeroes of the gap. The authors of [4] assumed that in condensed matter system at low frequencies the only physical reason for anomaly can be connected with the infrared singularity. Note that ultraviolet singularities are absent in condensed matter systems in contrast with quantum electrodynamics. Strong (critical) fluctuations are also suppressed in 3D system. Thus the main idea of [4] was to check the dangerous infrared regions where the gap is practically zero. To do that the authors of [4] consider the total hydrodynamic action S_{tot} of the A_1-phase for low frequencies and small q-vectors as a sum of bosonic and fermionic contributions:

$$S_{\text{tot}} = S_B + S_F , \qquad (35)$$

where $S_B(\rho, \mathbf{l}, \mathbf{v}_S)$ is a bosonic action and S_F is a fermionic action connected with the zeroes of a superfluid gap (see Fig. 4).

Generally speaking the idea of [4] was to describe by the supersymmetric hydrodynamics all the zero energy Goldstone modes including the fermionic Goldstone mode which comes from the zeroes of the gap.

The authors of [4] were motivated by the nice paper of Volkov and Akulov [18] who for the first time included massless fermionic neutrino in the effective infrared Lagrangian for electro-weak interactions.

After the integration over the fermionic variables the authors of [4] got the effective bosonic action and checked what infrared anomalies were present in it.

As a result they obtained:

$$S_B^{\text{eff}} = S_B + \Delta S_B , \qquad (36)$$

where a nodal contribution to the liquid–crystal like part of the effective action, which is connected with the gradient orbital energy, reads:

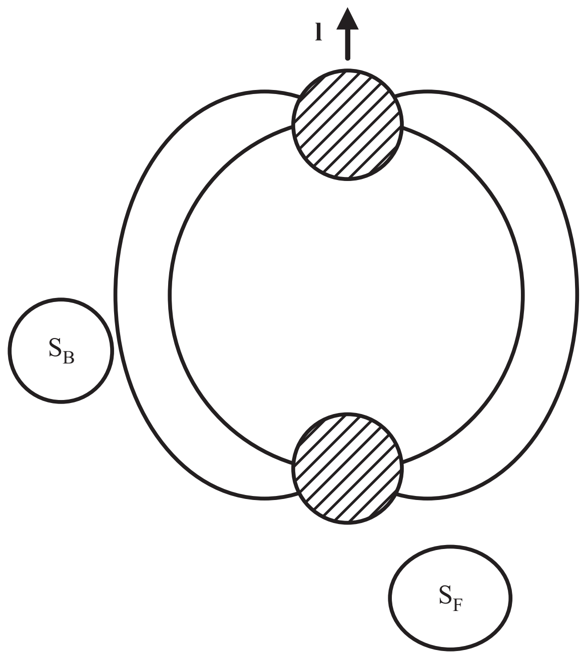

Fig. 4. The qualitative illustration of fermionic (S_F) and bosonic (S_B) contributions to the total hydrodynamic action S_{tot} of the A_1-phase at $T \to 0$.

$$\Delta S_B = -\frac{p_F^2 v_l}{32\pi^2}\int d^4x\left[(\mathbf{l}\times\mathrm{rot}\,\mathbf{l})^2 + \frac{v_t^2}{v_l^2}(\mathbf{l}\cdot\mathrm{rot}\,\mathbf{l})^2\right]\left(\ln\frac{l_{MF}^2}{r^2}\right). \tag{37}$$

Note, that in weak-coupling case

$$v_t \sim v_F\frac{\Delta_0}{\varepsilon_F} << v_F,\; v_l \sim v_F .$$

In Eq. (37), $x = (\mathbf{r},t)$, l_{MF} is a mean-free path, $\xi_0 < r < l_{MF}$, $\xi_0 \sim v_F/\Delta_0$ is a coherence length.

We can see that only weak logarithmic singularities are present in ΔS_B. They yield logarithmic renormalization of the orbital wave spectrum:

$$\omega \sim \frac{q^2}{m}\ln\frac{\Delta_0}{v_F\, q} \tag{38}$$

for small ω and $\mathbf{q}$.

However we do not observe any sign of strong singularity (actually it should be δ-functional since the fermionic density ρ_F coming from the nodal regions in S_F is small in comparison with the total density ρ). In other words, we do not see any trace of the anomalous contribution:

$$\mathbf{j}_{\rm an}\cdot\mathbf{v}_S = -\frac{\hbar}{2m}C_0(\mathbf{l}\cdot\mathrm{rot}\,\mathbf{l})(\mathbf{l}\cdot\mathbf{v}_S) \tag{39}$$

in ΔS_B.

Hence even if chiral anomaly exists in the BCS-domain of the A_1-phase, it is not directly connected with the dangerous regions of momentum space near zeroes of the gap (it does not have an infra-red character).

7.2. The approach based on the formal analogy with quantum electrodynamics

The authors of [5,6] have the different also rather nice approach based on the formal analogy between the anomalous current in ^{3}He-A and chiral anomaly in quantum electrodynamics (QED-theory). They assume that anomalous current with the coefficient $C_0 \sim \rho$ in BCS-domain of the A_1-phase is not directly connected with the zeroes of the gap (thus it is not contained even in the supersymmetric hydrodynamics). They assume that it is connected with the global *topological* considerations. To illustrate this point they solve microscopic Bogolubov–de-Gennes (BdG) equations for fermionic quasiparticles in a given twisted texture ($\mathbf{l}\,||\,\mathrm{rot}\,\mathbf{l}$) of the $\mathbf{l}$-vector. To be more specific they consider the case:

$$\mathbf{l} = \mathbf{l}_0 + \delta\mathbf{l}, \tag{40}$$

where

$$l_z = l_{0z} = e_z\;;\; l_y = \delta l_y = Bx\;;\; l_x = 0\,. \tag{41}$$

In this case:

$$\mathbf{l}\cdot\mathrm{rot}\,\mathbf{l} = l_z\frac{\partial l_y}{\partial x} = B = \mathrm{const} \tag{42}$$

and accordingly

$$\mathbf{j}_{\rm an} = -\frac{\hbar}{2m}C_0 B\,\mathbf{e}_z\,. \tag{43}$$

After linearization BdG equations become equivalent to Dirac equation in magnetic field $B = \mathbf{l}\cdot\mathrm{rot}\,\mathbf{l}$. Its solution yields the following level structure for fermionic quasiparticles:

$$E_n(p_z) = \pm\sqrt{\xi^2(p_z) + \tilde{\Delta}_n^2}\,, \tag{44}$$

where $\xi(p_z) = \frac{p_z^2}{2m} - \mu$; $e = \frac{p_z}{p_F} = \pm 1$ is an electric charge and

$$\tilde{\Delta}_n^2 = 2n v_t^2 p_F|eB| \tag{45}$$

is a gap squared $v_t \sim v_F\,\Delta_0/\varepsilon_F$.

For $n \neq 0$ (see Fig. 5) all the levels are gapped $\tilde{\Delta}_n \neq 0$ and doubly degenerate with respect to $p_z \to -p_z$. Their contribution to total mass-current is zero for $T \to 0$.

For $n = 0$ there is no gap $\tilde{\Delta}_0 = 0$ and we have an asymmetric chiral branch which exists only for $p_z < 0$. The energy spectrum for $n = 0$ yields:

$$E_0 = \xi(p_z)\,. \tag{46}$$

We can say that there is no gap for zeroth Landau level. Moreover in BCS-domain $E_0 = 0$ for $|p_z| = p_F$ — the chiral level crosses the origin in Fig. 5.

Note that in BEC-domain $E_0 \geq |\mu|$ and the zeroth Landau level does not cross the origin.

The zeroth Landau level gives an anomalous contribution to the total current in BCS-domain:

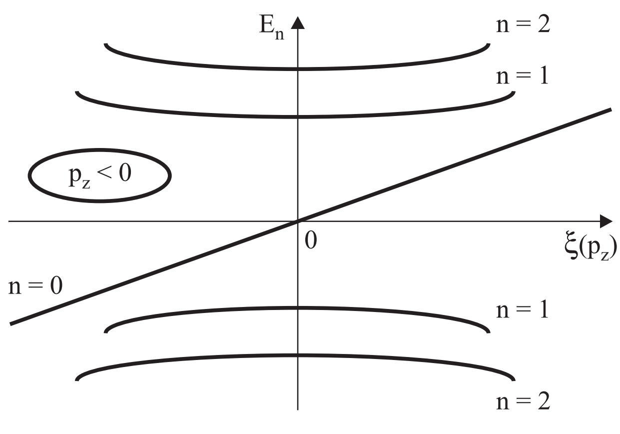

Fig. 5. The level structure of the Dirac equation in magnetic field $B = \mathbf{l}\cdot\mathrm{rot}\,\mathbf{l}$. All the levels with $n \neq 0$ are doubly degenerate. The zeroth level is chiral. It crosses the origin for $|p_z| = p_F$ in BCS-domain ($\mu > 0$).

$$\mathbf{j}_{\rm an}(\mathbf{r}=0) = -\mathbf{e}_z(\mathbf{l}\cdot\mathrm{rot}\,\mathbf{l}) \int\limits_{p_z<0} \frac{p_z}{2\pi^2} d\xi(p_z) = -\frac{\hbar C_0}{2m}(\mathbf{l}\cdot\mathrm{rot}\,\mathbf{l})\mathbf{l}, \quad (47)$$

where

$$\frac{(\mathbf{l}\cdot\mathrm{rot}\,\mathbf{l})p_z}{2\pi^2 p_F} = \frac{eB}{2\pi^2} = \int |f_0|^2 \frac{dp_y}{2\pi}, \quad (48)$$

and as a result:

$$C_0 \approx \frac{mp_F^3}{6\pi^2} \approx \rho \quad (49)$$

in BCS-domain.

Note that $f_0(x - p_y/eB)$ in (48) is an eigenfunction for zeroth Landau level.

It is easy to observe that the integral for C_0 in (47), (48) is governed by the narrow cylindrical tube inside the Fermi-sphere (see Fig. 6) with the length p_F parallel to the **l**-vector and radius of the cylinder squared given by:

$$\langle p_y^2 \rangle \sim p_F|eB|. \quad (50)$$

8. Whether the chiral anomaly can be destroyed by damping

The authors of [4] expressed their doubts and worries with respect to the calculation of C_0 based on Dirac equation in magnetic field $B = \mathbf{l}\cdot\mathrm{rot}\,\mathbf{l}$. From their point of view the calculation of C_0 from (48), (49) is an oversimlification of a difficult many-particle problem. In particular they emphasized the role of the finite damping $\gamma = 1/\tau$ to destroy the chiral anomaly at low frequencies $\omega < \gamma$, thus restoring the superfluid hydrodynamics (without normal velocity $\mathbf{v}_n$ and normal density ρ_n). Indeed, if the damping γ is larger than the level spacing of the Dirac equation:

$$\omega_0 = v_t\, p_F \sqrt{\frac{|\mathbf{l}\cdot\mathrm{rot}\,\mathbf{l}|}{p_F}} \quad (51)$$

in case when $\xi(p_z) = 0$, then the contribution from the zeroth Landau level should be washed out by damping (see Fig. 7) and the chiral anomaly should be destroyed. The damping γ for chiral fermions (for fermions living close to the nodes) in a very clean A_1-phase without impurities is defined at $T = 0$ by the different decay processes (see [17]).

It is natural to assume that the only parameter which defines γ at $T = 0$ for chiral fermions is

$$\Delta_0\langle\theta\rangle = \Delta_0 \frac{\langle p_\perp \rangle}{p_F}.$$

The leading term in decay processes is given by emission of an orbital wave (see Fig. 8). It reads

$$\gamma \propto \left[\frac{\Delta_0^2 p_\perp^2 / p_F^2 + v_F^2(p_z - p_F)^2}{\varepsilon_F}\right]. \quad (52)$$

For $p_z = p_F$ ($\xi(p_z) = 0$):

$$\gamma \sim \frac{\Delta_0^2}{\varepsilon_F}\frac{p_\perp^2}{p_F^2}. \quad (53)$$

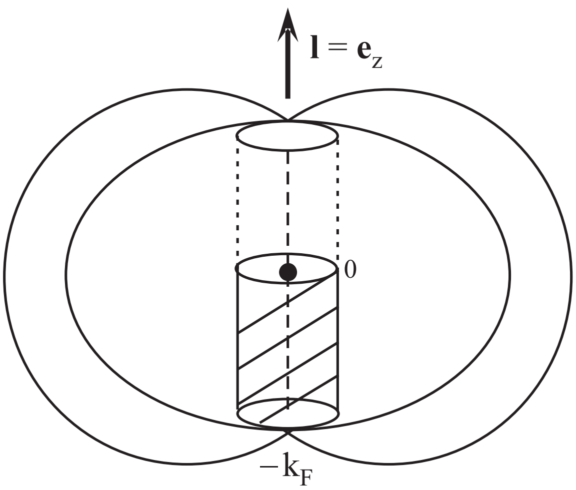

Fig. 6. The contribution to the coefficient C_0 is governed by the narrow cylindrical tube of the length p_F and the width $\langle p_y^2 \rangle \sim p_F|eB|$ inside the Fermi-sphere.

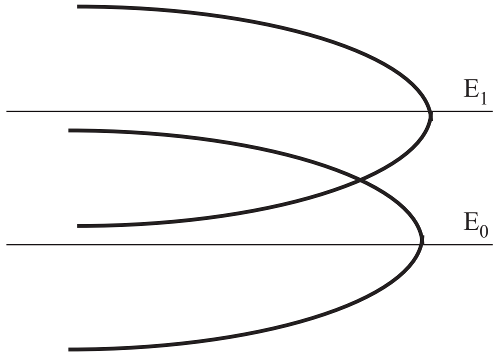

Fig. 7. The possible role of the damping in destruction of the chiral anomaly at low frequencies and small **k**-vectors when $\gamma > \omega_0$ ($\omega_0 = E_1 - E_0$ is level spacing).

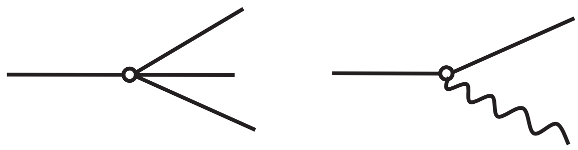

Fig. 8. Different decay processes for damping of chiral fermions at $T = 0$: the standard three-fermion decay process and a decay process with the emission of orbital wave.

Note that for chiral fermions on zeroth Landau level

$$\frac{\langle p_\perp \rangle}{p_F} = \left(\frac{|\mathbf{l} \cdot \operatorname{rot} \mathbf{l}|}{p_F} \right)^{1/2} \qquad (54)$$

and the level spacing for $\xi(p_z) = 0$ reads:

$$\omega_0 \sim \Delta_0 \frac{\langle p_\perp \rangle}{p_F}. \qquad (55)$$

Hence $\gamma/\omega_0 << 1$ for these two decay processes. Thus it is not obvious could we wash out the contribution form the zeroth Landau level by the different decay processes in superclean ^{3}He-A$_1$-phase at $T = 0$.

9. Discussion

The inequality $\omega_0 << \gamma$, which is sufficient to destroy the chiral anomaly can be definitely fulfilled in a moderately clean ^{3}He-A$_1$-phase, that is in the presence of the sufficient amount of aerogel.

The possible role of aerogel. Nowadays a lot of physicists working in the field of superfluid ^{3}He concentrate their efforts on studying the phase-diagram of superfluid ^{3}He in the presence of impurities which partially suppress the *p*-wave pairing. Namely they study the superfluid ^{3}He in the presence of aerogel, which constitutes the system of SiO_2-cylindrical filaments (strands) with the diameter 30 Å forming the network in ^{3}He [19,20].

It is possible nowadays to create experimentally rather substantial amount of aerogel in ^{3}He and have the damping γ as much as $0.1T_C$. Note, that effectively in the presence of aerogel γ is an *external* parameter which depends only on the concentration of aerogel x.

Moderately clean case means:

$$\omega_0 < \gamma < \Delta_0, \qquad (56)$$

and can be achieved experimentally. Thus a very interesting experimental proposal is to check our conjecture that an anomalous coefficient $C_0(\omega_0 << \gamma)$ is small by creating a twisted texture $\mathbf{l} \parallel \operatorname{rot} \mathbf{l}$ and then varying the aerogel concentration x. After that it is interesting to decrease the aerogel concentration drastically and to answer experimentally the question whether $\gamma(x \to 0)$ is larger or smaller then ω_0 in A$_1$-phase.

The similar project with the impurities can be also proposed for magnetic traps if it will be possible experimentally to get a sufficient lifetime for *p*-wave resonance condensate in BCS-domain. Note, that experimentally we can measure either the anomalous current (30) directly or the spectrum of orbital waves (33), (34), which is usually easier to do.

Note also, that in magnetic traps the textures of **l**-vector can be prepared with the help of «cutting» the condensate by the laser beam on two parts and then creating the Josephson tunneling structures for studying the circulation of anomalous current along the contour which contains one or two Josephson junctions.

10. Conclusion

In conclusion we would like to point out that in this paper we solve the Leggett equations for BCS–BEC crossover in A$_1$-phase of the resonance *p*-wave superfluid. As a result we found the behavior of the superfluid gap Δ, the chemical potential μ, and the sound velocity c_S in BCS-domain ($\mu > 0$), in BEC-domain ($\mu < 0$) and close to the interesting interpolation point $\mu = 0$.

We observed that fermionic (quasiparticle) contribution to the normal density ρ_n and the specific heat C_v have power-law temperature dependences in BCS-domain, while they are exponential in BEC-domain. Close to the interesting point $\mu = 0$ the power-law fermionic contributions to ρ_n and C_v can be separated rather reliably from the bosonic (phonon) contributions.

We review the two different approaches to the interesting problem of the chiral anomaly in BCS A$_1$-phase and propose a scenario for the crossover from the high-frequency ballistic regime with the presence of chiral anomaly to a low-frequency hydrodynamic regime with its absence.

Finally we stress the role of aerogel to create the different textures of the **l**-vector and increase the damping in superfluid A$_1$-phase of ^{3}He.

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