THE USAGE OF MAXWELL FRACTIONAL EQUATIONS FOR THE INVESTIGATION OF THE WAVEGUIDE PROCESSES

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By means of nabla operator written down with using both of some differential operators with integer orders and fractional differential Caputo operators, gradient, divergence and rotor operators are determined, it is checked up the fulfillment of vector relations in fractional vector analysis, fractional Green's, Stocks' and Ostrogradsky-Gauss' formulas. For a specific expression of nabla operator (nabla components along x and y axes have a unit order and along z axis, correspondingly, a fractional value in the interval from zero till unit) Maxwell's fractional equations are written down. Based on the following from them some fractional wave equations, dissipative and polarization processes at electromagnetic waves distribution both in rectangular (planar) and in cylindrical waveguide structures are analyzed.

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INTRODUCTION

Some physical investigations last time often use the methods of fractional integro-differentiation (see, for example, [1]). Particularly, in periodic press there are various electrodynamics problems. For example, in [2] some polarization properties of fractional electric and magnetic fields are considered, in [3] microwaves distribution in rectangular waveguides is investigated, in [4] by means of a fractional wave equation it is made a theoretical and experimental prove of the fractality of the propagation of electromagnetic radiation in absorbing media.

Evidently, these problems (and many others) have to assume the existence of fractional Maxwell's equations which were introduced by Tarasov V.E. (see, for example, [5]). They may be written down in parallelepiped $W := \left(a \le x \le b, c \le y \le d, g \le z \le h\right)$ in the following way:

$$\mathrm{rot}_W^{\alpha_1}\, \vec{E}\left(\vec{r},t\right) = \frac{\mu}{c}\frac{\partial \vec{H}\left(\vec{r},t\right)}{\partial t}, \quad \mathrm{div}_W^{\alpha_2}\, \vec{E}\left(\vec{r},t\right) = 0, \qquad (1a)$$

$$\mathrm{rot}_W^{\alpha_3}\, \vec{H}\left(\vec{r},t\right) = -\frac{\varepsilon}{c}\frac{\partial \vec{E}\left(\vec{r},t\right)}{\partial t}, \quad \mathrm{div}_W^{\alpha_4}\, \vec{H}\left(\vec{r},t\right) = 0, \qquad (1b)$$

where the orders α_k $\left(k = 1,2,3,4\right)$ may be both integer and fractional, ε, μ – are correspondingly dielectric and magnetic permeability. Besides, div_W^{α} and rot_W^{α} operators are determined in a general way but with the help of the following nabla operator [5]:

$$\nabla_W^{\alpha} = \vec{e}_x\, {}_a^C D_x^{\alpha} + \vec{e}_y\, {}_c^C D_y^{\alpha} + \vec{e}_z\, {}_g^C D_z^{\alpha}, \ \left(n-1 < \alpha \le n\right), \qquad (2)$$

where ${}_{a,c,g}^{\ \ C}D_{x,y,z}^{\alpha}$ – is left-side operators of Caputo's fractional derivatives which action on real-value function of $f\left(x,y,z\right)$ is written down, for example, by the value *x* in the following way [6, 7]:

$${}_a^C D_x^{\alpha} f\left(x,y,z\right) = \frac{1}{\Gamma\left(n-\alpha\right)}\int_a^x \frac{f^{(n)}\left(t,y,z\right)dt}{\left(x-t\right)^{\alpha+1-n}}. \qquad (3)$$

Note that functions $f\left(x,y,z\right)$ should have continuous derivatives up to $\left(n-1\right)$ order and $\left(n-1\right)$ derivatives should be absolutely continuous, i.e. $f\left(x,y,z\right) \in AC^n\left[W\right]$ [1, 5].

The given paper shows that the further generalization of Maxwell equations on the basis of nabla operator proposed in [8] of so-called mixed orders can be carry out and the direct consideration of which follows.

1. NABLA OPERATOR OF MIXED ORDERS

As it was supposed in [8], we write down nabla operator of so-called mixed orders in the following way:

$$\nabla_W^{\alpha,\beta,\gamma} = \vec{e}_x\, {}_a^C D_x^{\alpha} + \vec{e}_y\, {}_c^C D_y^{\beta} + \vec{e}_z\, {}_g^C D_z^{\gamma}, \\ \left(m-1 < \alpha \le m, n-1 < \beta \le n, p-1 < \gamma \le p\right). \qquad (4)$$

From formula (4) it follows that unlike the expression (2) the orders of derivatives by the variables of x, y, z may have unequal arbitrary numerical values that significantly increase the possibilities of Maxwell's equations (1a, b) for receiving different fractional differential equations. It is also evidently that using Caputo's operators in formula (4) in the form of (3), the scalar functions $f\left(x,y,z\right)$ (and vector functions ~~of~~ $\vec{F}\left(x,y,z\right)$ as well) should belong to the class of the functions $AC^{\max\{m,n,p\}}\left[W\right]$.

It can be shown that there are following main vector relations:

$$\mathrm{Div}_W^{\alpha,\beta,\gamma}\, \mathrm{Grad}_W^{\alpha,\beta,\gamma}\, f\left(x,y,z\right) =$$

$$= \left[\left({}_{a}^{C}D_x^{\alpha} \right)^2 + \left({}_{c}^{C}D_y^{\beta} \right)^2 + \left({}_{g}^{C}D_z^{\gamma} \right)^2 \right] f(x,y,z), \quad (5a)$$

$$\mathrm{Rot}_W^{\alpha,\beta,\gamma}\, \mathrm{Grad}_W^{\alpha,\beta,\gamma}\, f(x,y,z) = 0, \quad (5b)$$

$$\mathrm{Div}_W^{\alpha,\beta,\gamma}\, \mathrm{Rot}_W^{\alpha,\beta,\gamma}\, \vec{F}(x,y,z) = 0, \quad (5c)$$

$$\mathrm{Rot}_W^{\alpha,\beta,\gamma}\, \mathrm{Rot}_W^{\alpha,\beta,\gamma}\, \vec{F}(x,y,z) = \mathrm{Grad}_W^{\alpha,\beta,\gamma}\, \mathrm{Div}_W^{\alpha,\beta,\gamma}\, \vec{F}(x,y,z) - \left[\left({}_{a}^{C}D_x^{\alpha} \right)^2 + \left({}_{c}^{C}D_y^{\beta} \right)^2 + \left({}_{g}^{C}D_z^{\gamma} \right)^2 \right] \vec{F}(x,y,z). \quad (5d).$$

Let's define the following integral vector operators of mixed orders:

$$I_L^{\alpha,\beta,\gamma} = \vec{e}_x I_L^{\alpha}[x] + \vec{e}_y I_L^{\alpha}[y] + \vec{e}_z I_L^{\gamma}[z], \quad (6a)$$

$$I_S^{\alpha,\beta,\gamma} = \vec{e}_x I_S^{\beta,\gamma}[y,z] + \vec{e}_y I_S^{\gamma,\alpha}[z,x] + \vec{e}_z I_S^{\alpha,\beta}[x,y]. \quad (6b)$$

Further, analogous to [5] we shall determine with (6a) a fractional circulation in the form of a fractional linear integral of the vector field ~~of~~ $\vec{F}$ along curve L

$$\varepsilon_L^{\alpha,\beta,\gamma}\left(\vec{F}\right) = I_L^{\alpha}[x] F_x + I_L^{\alpha}[y] F_y + I_L^{\gamma}[z] F_z, \quad (7a)$$

a fractional flux of a vector field ~~of~~ $\vec{F}$ through the surface S by the help of (6b)

$$\Phi_S^{\alpha,\beta,\gamma}\left(\vec{F}\right) = I_S^{\beta,\gamma}[y,z] F_x + I_S^{\gamma,\alpha}[z,x] F_y + I_S^{\alpha,\beta}[x,y] F_z, \quad (7b)$$

triple fractional integral on W region from the scalar function f:

$$V_W^{\alpha,\beta,\gamma}(f) = I_L^{\alpha}[x] I_L^{\alpha}[y] I_L^{\gamma}[z] f(x,y,z). \quad (7c)$$

Note, that in the formulas (7a, b, c) the integral operators act on the Lebesgue's measured functions, i.e. $f, \vec{F} \in L_1\left(R^3\right)$ [1].

At last, by means of the formulas of (7a, b, c) one may consequently formulate and prove (similarly to [5]) the fractional theorems of Green, Stocks and Ostrogradsky-Gauss.

Thus, in the frames of a fractional vector analysis it was shown that the generalization of nabla operator in the form (4) is mathematically correct.

2. THE ANALYSIS OF THE PROCESSES IN RECTANGULAR WAVEGUIDE STRUCTURES BY USING OF MAXWELL's FRACTIONAL EQUATIONS

Let rewrite Maxwell**'s** equation of (1a, b) for the case when $\alpha_2 = \alpha_3 = 1$, and curl and divergence operators, correspondingly, in the first and the forth equations are determined with nabla operator in the form of $\nabla_W^{1,1,\alpha} \equiv \nabla_W^{\alpha} = \vec{e}_x \frac{\partial}{\partial x} + \vec{e}_y \frac{\partial}{\partial y} + \vec{e}_z \beta\, {}_{0}^{C}D_{\zeta}^{\alpha}$ (here $\zeta = \beta z$, where β – is a constant of the propagation, $W = (0 \le x \le a, 0 \le y \le b, 0 \le z \le c)$), i.e.

$$\mathrm{rot}_W^{\alpha}\, \vec{E}(\vec{r},t) = \frac{\mu}{c}\frac{\partial \vec{H}(\vec{r},t)}{\partial t}, \quad \mathrm{div}\, \vec{E}(\vec{r},t) = 0, \quad (8a)$$

$$\mathrm{rot}\, \vec{H}(\vec{r},t) = -\frac{\varepsilon}{c}\frac{\partial \vec{E}(\vec{r},t)}{\partial t}, \quad \mathrm{div}_W^{\alpha}\, \vec{H}(\vec{r},t) = 0. \quad (8b)$$

Acting the operator rot on the first equation of (8a) and taking into account the second equation of (8a), we get the following wave equations for Cartesian components of vector of the electric field intensity $\vec{E}(x,y,\zeta,\tau;\alpha)$:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \beta^2 \frac{\partial}{\partial \zeta}\, {}_{0}^{C}D_{\zeta}^{\alpha} \right) E_{x,y} - n^2 k^2 \frac{\partial^2}{\partial \tau^2} E_{x,y} = 0, \quad (9a)$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \beta^2\, {}_{0}^{C}D_{\zeta}^{\alpha} \frac{\partial}{\partial \zeta} \right) E_z - n^2 k^2 \frac{\partial^2}{\partial \tau^2} E_z = 0, \quad (9b)$$

where $\tau = \omega t$, $k = \omega / c$ – a wave number, $n = \sqrt{\varepsilon\mu}$ – an indicator of medium refraction.

Now proceed to the solution of the equations (9a,b). To separate the derivatives of x and y and eliminate a measureless time τ, we'll look for the solution of such equations in the form

$$E_{x,y,z}(x,y,\zeta,\tau;\alpha) = e(x,y)\, f_{x,y,z}^{(\alpha)}(\zeta) \exp(-i\tau). \quad (10)$$

After substituting of (10) in (9a, b), we get, first, Helmholtz equation for $e(x,y)$ functions $\Delta_{x,y} e(x,y) + \kappa^2 e(x,y) = 0$, methods of solution of which for the rectangular (in particular, planar) waveguides with account of various boundary conditions are set, for example, in [9]. Second, we get the following fractional wave equations for $f_{x,y,z}^{(\alpha)}(\zeta)$ functions:

$$\left(\frac{\partial}{\partial \zeta}\, {}_{0}^{C}D_{\zeta}^{\alpha} + 1 \right) f_{x,y}^{(\alpha)}(\zeta) = 0, \quad (11a)$$

$$\left(D_{\zeta}^{\alpha+1} + 1 \right) f_z^{(\alpha)}(\zeta) = 0, \quad (11b)$$

at solution of which a dispersion equality of $\kappa^2 = n^2 k^2 - \beta^2$ is used (see [9]), and also in (11b) it is taken into account that ${}_{0}^{C}D_{\zeta}^{\alpha} \frac{\partial}{\partial \zeta} = {}_{0}^{C}D_{\zeta}^{\alpha+1}$ [5].

By using of initial conditions $f_z^{(\alpha)}(0) = 1$ and $\partial f_z^{(\alpha)}(0) / \partial \zeta = i$, that follow from the solution $f_z(\zeta) = \exp(i\zeta)$ of the equation (11b) at $\alpha = 1$, then the solution (11b) may be presented in the form (see [10])

$$f_z^{(\alpha)}(\zeta) = E_{\alpha+1,1}\left(-\zeta^{\alpha+1}\right) + i\zeta E_{\alpha+1,2}\left(-\zeta^{\alpha+1}\right), \quad (12)$$

where $E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} x^k / \Gamma(\alpha k + \beta)$ – a Mittag-Leffler function (see [1]).

One can show that the equation (11a) for functions $f_{x,y}^{(\alpha)}(\zeta)$ is equivalent the following nonhomogeneous Cauchy's problem for functions $g_{x,y}(\zeta)=\int_0^\zeta f_{x,y}(\zeta)d\zeta$:

$$\left({}_0^C D_\zeta^{\alpha+1}+1\right)g_{x,y}^{(\alpha)}(\zeta)=i,$$
$$g_{x,y}^{(\alpha)}(0)=0,\quad \partial g_{x,y}^{(\alpha)}(0)/\partial\zeta=1. \qquad (13)$$

The solution of (13) problem is given by the following formula [10]:

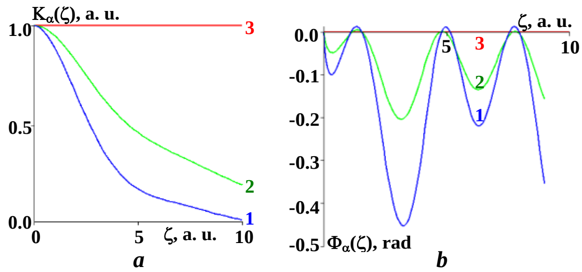
$$g_{x,y}^{(\alpha)}(\zeta)=i\int_0^\zeta(\zeta-t)^\alpha E_{\alpha+1,\alpha+1}\left[-(\zeta-t)^{\alpha+1}\right]dt+$$
$$+\zeta E_{\alpha+1,2}\left(-\zeta^{\alpha+1}\right). \qquad (14)$$

After integrating in (14) and further differentiation by ζ, we get:

$$f_{x,y}^{(\alpha)}(\zeta)=E_{\alpha+1,1}\left(-\zeta^{\alpha+1}\right)+i\zeta^\alpha E_{\alpha+1,\alpha+1}\left(-\zeta^{\alpha+1}\right). \qquad (15)$$

It is easy to prove that at $\alpha=1$ the solutions of (12) and (15) transit into the function of $\exp(i\zeta)$.

Note that for the magnetic field a *z*-component is proportional to the function of (15), and *x*- and *y*-components correspondingly to the function of (12).



The dependences of a relative coordinate of ζ *at* $\alpha=0{,}8;0{,}9;1$ *of the coefficient* $K_\alpha(\zeta)$ *of a power fading of – (a), phases difference of* $\Phi_\alpha(\zeta)$ *between a longitudinal and a transverse components of electric and magnetic fields – (b)*

Let's use now the obtained solutions for the investigation of dissipative and polarization processes. As it follows from Umov-Pointing theorem the expression for a mean flux of an active power going through cut S of a rectangular waveguide structure equals [9]

$$P_\alpha(\zeta)=\frac{c}{8\pi}\mathrm{Re}\int_S\left[\vec{E}(\zeta),\vec{H}^*(\zeta)\right]_z dS=$$
$$=P_\alpha(0)\mathrm{Re}\left[f_{x,y}^{(\alpha)}(\zeta)\,f_z^{(\alpha)*}(\zeta)\right], \qquad (16)$$

i.e. a power fading coefficient with a distance is determined by the following formula:

$$K_\alpha(\zeta)=\frac{P_\alpha(\zeta)}{P_\alpha(0)}=E_{\alpha+1,1}^2\left(-\zeta^{\alpha+1}\right)+$$
$$+\zeta^{\alpha+1}E_{\alpha+1,2}\left(-\zeta^{\alpha+1}\right)E_{\alpha+1,\alpha+1}\left(-\zeta^{\alpha+1}\right). \qquad (17)$$

In Figure there are graphs the function (17) at $\alpha=0{,}8;0{,}9;1$. Besides, in Fig. 1b there are graphs of phase shifts of $\Phi_\alpha(\zeta)=\arg f_z^{(\alpha)}(\zeta)-\arg f_{x,y}^{(\alpha)}(\zeta)$ between longitudinal and transverse components of electric and magnetic fields (it means that by means of fractional fields one may study some polarization phenomena).

3. THE INVESTIGATION OF ELECTROMAGNETIC RADIATION DISTRIBUTION IN CYLINDRICAL WAVEGUIDE STRUCTURES BY MEANS OF A FRACTIONAL WAVE EQUATION

Based on the fact that nabla operator ∇_W^α in a cylindrical coordinate system is written down as

$$\nabla_W^{1,1,\alpha}\equiv\nabla_W^\alpha=\vec{e}_\rho\frac{\partial}{\partial\rho}+\vec{e}_\varphi\frac{\partial}{\rho\partial\varphi}+\vec{e}_z\beta\,{}_0^C D_\zeta^\alpha, \qquad (18)$$

one may get the following wave equations for the transverse $\left(E_{\rho,\varphi},H_{\rho,\varphi}\right)(\rho,\zeta,\tau;\alpha)$ and longitudinal $\left(E_z,H_z\right)(\rho,\zeta,\tau;\alpha)$ components of electric and magnetic fields:

$$\left(\Delta_\rho-\frac{1}{\rho^2}+\beta^2\frac{\partial}{\partial\zeta}\,{}_0^C D_\zeta^\alpha-n^2k^2\frac{\partial^2}{\partial\tau^2}\right)\left(E_{\rho,\varphi},H_z\right)=0, \qquad (19a)$$

$$\left(\Delta_\rho+\beta^2\,{}_0^C D_\zeta^{\alpha+1}-n^2k^2\frac{\partial^2}{\partial\tau^2}\right)\left(E_z,H_{\rho,\varphi}\right)=0, \qquad (19b)$$

where $\Delta_\rho=\partial(\rho\partial/\partial\rho)/\rho\partial\rho$ – a radial part of Laplace's operator in cylindrical coordinates (equations (19a, b) are written down for the case of axis symmetric modes distribution in a cylindrical region of $W:=(\rho\le a, 0\le z\le c)$).

Evidently that solution of (19a, b) equations are found in accordance with an algorithm used in previous paragraph. Writing down the solutions of these equations in the form

$$\left(E_{\rho,\varphi},H_z\right)(\rho,\zeta,\tau;\alpha)=\left[e_{\rho,\varphi}(\rho),h_z(\rho)\right]r_\alpha(\zeta)\exp(-i\tau),$$
$$\left(E_z,H_{\rho,\varphi}\right)(\rho,\zeta,\tau;\alpha)=\left[e_z(\rho),h_{\rho,\varphi}(\rho)\right]s_\alpha(\zeta)\exp(-i\tau),$$

we get that $r_\alpha(\zeta)$ function is given by the expression (15) and $s_\alpha(\zeta)$ function by the expression (12), accordingly. It means that the fading function in this case coincides with the expression (17) (polarization characteristics are analyzed analogically as well).

In the conclusion note that by means of this mathematical formalism in [11] it was studied the distribution of electromagnetic radiation in optical waveguides.

CONCLUSIONS

Thus, on the basis of introducing a new nabla operator of so-called mixed orders (as it was shown, not contradicting a fractional vector analysis) there were got mathematically strict fractional wave equations of fractional Maxwell equations. It was shown that using of these equations to rectangular and cylindrical waveguide structures allows to study their dissipative and polarization characteristics.

At last, one may do the assumption that there must be some geometric ways of a fractional derivative α indicator definition (then the problems of similar type will be mathematically and physically closed).

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ИСПОЛЬЗОВАНИЕ ДРОБНЫХ УРАВНЕНИЙ МАКСВЕЛЛА ДЛЯ ИССЛЕДОВАНИЯ ВОЛНОВОДНЫХ ПРОЦЕССОВ

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С помощью оператора набла, записанного с одновременным использованием как дифференциальных операторов с целочисленными порядками, так и дробных дифференциальных операторов Капуто, определяются операторы градиента, дивергенции и ротора, проверяется выполнимость векторных соотношений дробного векторного анализа, дробных формул Грина, Стокса и Остроградского-Гаусса. Для конкретного выражения оператора наблы (компоненты наблы вдоль осей x и y имеют единичный порядок, а вдоль оси z, соответственно, дробное значение в интервале от нуля до единицы) записываются дробные уравнения Максвелла. На основе следующих из них дробных волновых уравнений анализируются диссипативные и поляризационные процессы при распространении электромагнитных волн как в прямоугольных (планарных), так и в цилиндрических волноводных структурах.

ВИКОРИСТАННЯ ДРОБОВИХ РІВНЯНЬ МАКСВЕЛЛА ДЛЯ ДОСЛІДЖЕННЯ ХВИЛЕВОДНИХ ПРОЦЕСІВ

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За допомогою оператора набли, записаного за одночасного використання як диференціальних операторів з цілочисельними порядками, так і дробових диференціальних операторів Капуто, визначаються оператори градієнта, дивергенції та ротора, перевіряється виконуваність векторних співвідношень дробового векторного аналізу, дробових формул Гріна, Стокса та Остроградського-Гауса. Для конкретного виразу оператора набли (складові набли уздовж осей x та y мають одиничний порядок, а уздовж осі z, відповідно, дробове значення в інтервалі від нуля до одиниці) записуються дробові рівняння Максвелла. На основі випливаючих із них дробових хвильових рівнянь аналізуються дисипативні та поляризаційні процеси при розповсюдженні електромагнітних хвиль як у прямокутних (планарних), так і в циліндричних хвилеводних структурах.