THE LARGE SCALE INSTABILITY AND NONLINEAR VORTEX STRUCTURES IN OBLIQUELY ROTATING FLUID WITH SMALL SCALE NON SPIRAL FORCE

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In this paper, we find a new large scale instability which appears in obliquely rotating flow with the small scale turbulence, generated by external force with small Reynolds number. The external force has no helicity. The theory is based on the rigorous method of multi scale asymptotic expansion. Nonlinear equations for instability are obtained in third order of the perturbation theory. In this article, we explain in details the nonlinear stage of the instability and we find the nonlinear periodic vortices and the vortex kinks of Beltrami type.

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INTRODUCTION

It is well known, that the rotating effects play an important role in many theoretical and practical applications for fluid mechanics [1] and are especially important for geophysics and astrophysics [2 - 4], when one have to deal with rotating objects such as the Earth, Jupiter, the Sun, etc. Rotating fluids could generate different wave and vortex motions, for example, gyroscopic waves, Rossby waves, internal waves, located vortices and coherent vortex structures [4 - 7]. Among the vortex structures, the most interesting are the large scale ones, since they carry out the efficient transport of energy and impulse. The structures which have characteristic scale much more than the scale of turbulence or the scale of external force which generates this turbulence are understood as large scale ones. In this paper we find a new large scale instability in obliquely rotating flow which is influenced by the small scale external force with zero helicity. Its axis of rotation does not coincide with the Z axis. This force supports small scale turbulent fluctuations in fluid. The nonlinear large scale helical vortex structures such as Beltrami vortices or localized kinks appear as a result of the development of this instability in rotating fluid. This supposes that the external small-scale force substitutes the action of small-scale turbulence. Further we consider that the external force acts in the plane (X, Y). Instability occurs only when the vector of angular velocity of rotation $\vec{\Omega}$ is inclined relatively to the plane (X, Y), as shown in Fig.1. If the fluid is rotating around the axis Z strictly, then instability does not occur. The helical 2D velocity field W_x, W_y turns around the axis Z, when Z changes in the periodic wave (Fig. 2) and makes one turn in the kink (Fig. 3). The found instability belongs to the class of instabilities called hydrodynamic α-effects. For these instabilities the positive feedback between velocity components is typical:

$$\partial_T W_x - \Delta W_x - \alpha_y \frac{\partial}{\partial Z} W_y = 0,$$

$$\partial_T W_y - \Delta W_y + \alpha_x \frac{\partial}{\partial Z} W_x = 0,$$

and leads to the instability. α-effect origins from magnetic hydrodynamics, where it engenders the increase of large scale magnetic fields (see for example [16]). Later it was extended to ordinary hydrodynamics. Several examples of hydrodynamics α-effect [8-15] are known for today. From this point of view, in this study we found a new example of the α-effect. The theory of this instability is based on a rigorous method of multi-scale development, which was proposed by Frisch, She and Sulem for the theory of the AKA effect [13]. This method allows to find the equations for large scale perturbations as the secular equations of the asymptotic theory, to calculate the Reynolds stress tensor and to find the instability. The small parameter of asymptotical development is the number of Reynolds $R, R \le 1.$ Our paper is organized as follows: in Section 2 we formulate the problem and the main equations in rotating system coordinates; in Section 3 we discuss the concept of multi-scale development and we give the secular equations. In Section 4 we calculate the velocity field of zero approximation. In Section 5 we describe the calculations of the Reynolds stress and find the large scale instability. In Section 6 we discuss the saturation of the instability and find the nonlinear stationary vortex structures. The results obtained are discussed in the conclusions given in Section 7.

1. THE MAIN EQUATIONS AND FORMULATION OF THE PROBLEM

Let us examine the equations of motion for non-compressible rotating fluid with the external force $\vec{F}_0$ in rotating coordinates system:

$$\frac{\partial \vec{V}}{\partial t} + \left(\vec{V}\nabla\right)\vec{V} + 2\vec{\Omega}\times\vec{V} = = -\frac{1}{\rho_0}\nabla P + \nu\Delta\vec{V} + \vec{F}_0, \quad (1)$$

$$div\vec{V} = 0. \quad (2)$$

The external force $\vec{F}_0$ is divergence-free. Here $\vec{\Omega}$ is angular velocity of fluid rotation, ν is viscosity, ρ_0 is constant fluid density. Let us design characteristic amplitude of force f_0, and its characteristic space and time scale λ_0 and t_0 respectively. Then $\vec{F}_0 = f_0 \vec{F}_0 \left(\frac{\vec{x}}{\lambda_0}, \frac{t}{t_0} \right)$.

We will design the characteristic amplitude of velocity, generated by external force as v_0. We choose the dimensionless variables $(t, \vec{x}, \vec{V})$

$$\vec{x} \to \frac{\vec{x}}{\lambda_0}, \quad t \to \frac{t}{t_0}, \quad \vec{V} \to \frac{\vec{V}}{v_0}, \quad \vec{F}_0 \to \frac{\vec{F}_0}{f_0}, \quad P \to \frac{P}{\rho_0 P_0},$$

$$t_0 = \frac{\lambda_0^2}{\nu}, \quad P_0 = \frac{v_0 \nu}{\lambda_0}, \quad f_0 = \frac{v_0 \nu}{\lambda_0^2}, \quad v_0 = \frac{f_0 \lambda_0^2}{\nu}.$$

Then, in dimensionless variables the equation (1) takes following forme:

$$\frac{\partial \vec{V}}{\partial t} + R(\vec{V} \cdot \nabla)\vec{V} + \vec{D} \times \vec{V} = -\nabla P + \Delta \vec{V} + \vec{F}_0, \quad (3)$$

$$R = \frac{\lambda_0 v_0}{\nu}, |D| = \sqrt{Ta}\,.$$

$R = \frac{\lambda_0 v_0}{\nu}, |D| = \sqrt{Ta}$. Where R and $Ta = \frac{4\Omega^2 \lambda_0^4}{\nu^2}$ are respectively the Reynolds number and the Taylor number on scale λ_0. Further we will consider the Reynolds number as small $R \le 1$ and will construct on this small parameter the asymptotical development. Concerning the parameter D, we do not choose any range of values for the moment. Let us examine the following formulation of the problem. We consider the external force as being of small scale and of high frequency. This force leads to small scale fluctuations in velocity. After averaging, these rapidly oscillating fluctuations vanish. Nevertheless, due to small nonlinear interactions in some orders of perturbation theory, nonzero terms can occur after averaging. This means that they are not oscillatory, that is to say, they are large scale. From a formal point of view, these terms are secular, i.e., they create the conditions for the solvability of large-scale asymptotic development. So, the purpose of this paper is to find and study the solvability equations, i.e., the equations for the large scale perturbations. Let us denote the small scale variables by $x_0 = (\vec{x}_0, t_0)$, and the large scale ones by $X = (\vec{X}, T)$. The small scale partial derivative operation $\frac{\partial}{\partial x_0^i}, \frac{\partial}{\partial t_0}$, and the large scale ones $\frac{\partial}{\partial \vec{X}}, \frac{\partial}{\partial T}$ are written, respectively, as $\partial_i, \partial_t, \nabla_i$ and ∂_T. To construct a multi-scale asymptotic development we follow the method which is proposed in [16].

2. THE MULTI-SCALE ASYMPTOTIC DEVELOPMENT

Let us search the solution to equations (2) and (3) in the following form:

$$\vec{V}(\vec{x},t) = \frac{1}{R}\vec{W}_{-1}(X) + \vec{v}_0(x_0) + R\vec{v}_1 + $$
$$+R^2\vec{v}_2 + R^3\vec{v}_3 + \cdots, \quad (4)$$

$$P(\vec{x},t) = \frac{1}{R^3}P_{-3}(X) + \frac{1}{R^2}P_{-2}(X) + \frac{1}{R}P_{-1}(X) + P_0(x_0) +$$
$$+R(P_1 + \overline{P}_1(X)) + R^2 P_2 + R^3 P_3 + \cdots. \quad (5)$$

We introduce the slow variables $\vec{X} = R^2 \vec{x}_0$ and $T = R^4 t_0$ which lead to the following expressions for the spatial and temporal derivatives:

$$\frac{\partial}{\partial x^i} = \partial_i + R^2 \nabla_i, \quad (6)$$

$$\frac{\partial}{\partial t} = \partial_t + R^4 \partial_T, \quad (7)$$

$$\frac{\partial^2}{\partial x^j \partial x^j} = \partial_{jj} + 2R^2 \partial_j \nabla_j + R^4 \partial_{jj}. \quad (8)$$

Using initial notation, the system of equations can be written as:

$$(\partial_t + R^4 \partial_T)V^i + R(\partial_j + R^2 \nabla_j)(V^i V^j) + D^j \varepsilon_{ijk} V^k =$$
$$= (\partial_j + R^2 \nabla_j)P + (\partial_{jj} + 2R^2 \partial_j \nabla_j + R^4 \nabla_{jj})V^i + F_0^i, \quad (9)$$

$$\left(\partial_i + R^2 \nabla_i\right) V^i = 0. \quad (10)$$

Substituting these expressions into the initial equations (2) and (3) and then gathering together the terms of the same order, we obtain the equations of the multiscale asymptotic development and write down the obtained equations up to order R^3 inclusive. In the order R^{-3} there is only one the equation:

$$\partial_i P_{-3} = 0, \Rightarrow P_{-3} = P_{-3}(X). \quad (11)$$

In order R^{-2} we have the equation:

$$\partial_i P_{-2} = 0, \Rightarrow P_{-2} = P_{-2}(X). \quad (12)$$

In order R^{-1} we get a system of equations:

$$\partial_t W_{-1}^i - \partial_{jj} W_{-1}^i + D^j \varepsilon_{ijk} W_{-1}^k = -(\partial_i P_{-1} + \nabla_i P_{-3}) - \partial_j W_{-1}^i W_{-1}^j, \quad (13)$$

$$\partial_i W_{-1}^i = 0. \quad (14)$$

The system of equations (13) and (14) gives the secular terms

$$-\nabla_i P_{-3} = D^j \varepsilon_{ijk} W_{-1}^k, \quad (15)$$

which corresponds to a geostrophic equilibrium equation. In zero order R^0, we have the following system of equations:

$$\partial_t v_0^i - \partial_{jj} v_0^i + \partial_j (W_{-1}^i v_0^j + v_0^i W_{-1}^j) + D^j \varepsilon_{ijk} v_0^k =$$
$$= -(\partial_i P_0 + \nabla_i P_{-2}) + F_0^i, \quad (16)$$

$$\partial_i v_0^i = 0. \quad (17)$$

These equations give the following secular equation:

$$\nabla P_{-2} = 0, \Rightarrow P_{-2} = Const. \quad (18)$$

Let us consider the equations of the first approximation R:

$$\partial_t v_1^i - \partial_{jj} v_1^i + D^j \varepsilon_{ijk} v_1^k + \partial_j (W_{-1}^i v_1^j + v_1^i W_{-1}^j + v_0^i v_0^j) =$$
$$= -\nabla_j (W_{-1}^i W_{-1}^j) - (\partial_i P_1 + \nabla_i P_{-1}), \quad (19)$$

$$\partial_i V_1^i + \nabla_i W_{-1}^i = 0. \quad (20)$$

Secular equations follow from this system of equations:

$$\nabla_i W_{-1}^i = 0, \quad (21)$$

$$\nabla_j (W_{-1}^i W_{-1}^j) = -\nabla_i P_{-1}, \quad (22)$$

The secular equation (21) and (22) are satisfied by choosing the following geometry for the velocity field (Beltrami field):

$$\vec{W} = (W^x_{-1}(Z), W^y_{-1}(Z), 0); T_{-1} = T_{-1}(Z); \quad (23)$$

$$\nabla P_{-1} = 0, \Rightarrow P_{-1} = Const.$$

In the second order R^2, we obtain the equations:

$$\partial_t v^i_2 - \partial_{jj} v^i_2 - 2\partial_j \nabla_j v^i_0 +$$
$$+\partial_j (W^i_{-1} v^j_2 + v^i_2 W^j_{-1} + v^i_0 v^j_1 + v^i_1 v^j_0) + D^j \varepsilon_{ijk} v^k_2 =$$
$$= -\nabla_j (W^i_{-1} v^j_0 + v^i_0 W^j_{-1}) - (\partial_i P_2 + \nabla_i P_0), \quad (24)$$

$$\partial_i v_2 + \nabla_i v_0 = 0. \quad (25)$$

It is easy to see that there are no secular terms in this order. Let us come now to the most important order R^3. In this order we obtain the equations:

$$\partial_t v^i_3 + \partial_T W^i_{-1} - (\partial_{jj} v^i_3 + 2\partial_j \nabla_j v^i_1 + \nabla_{jj} W^i_{-1}) +$$
$$+\nabla_j (W^i_{-1} v^j_1 + v^i_1 W^j_{-1} + v^i_0 v^j_0) +$$
$$+\partial_j (W^i_{-1} v^j_3 + v^i_3 W^j_{-1} + v^i_0 v^j_2 + v^i_2 v^j_0 + v^i_1 v^j_1) +$$
$$+D^j \varepsilon_{ijk} v^k_3 = -(\partial_i P_3 + \nabla_i \overline{P}_1), \quad (26)$$
$$\partial_i v_3 + \nabla_i v_1 = 0.$$

From this we get the main secular equation:

$$\partial_T W^i_{-1} - \Delta W^i_{-1} + \nabla_k (\overline{v^k_0 v^i_0}) = -\nabla_i \overline{P}_1, \quad (27)$$

There is also an equation to find the pressure P_{-3}:

$$-\nabla_i P_{-3} = D^j \varepsilon_{ijk} W^k_{-1}. \quad (28)$$

3. THE VELOCITY FIELD IN ZERO APPROXIMATION

It is clear that the most important is equation (27). In order to obtain these equations in closed form, we need to calculate the Reynolds stress $\nabla_k (\overline{v^k_0 v^i_0})$. First of all we have to calculate the fields of the zero approximation v^k_0. From the asymptotic development in zero order we have:

$$\partial_t v^i_0 - \partial_{jj} v^i_0 + W^k_{-1} \partial_k v^i_0 + D^j \varepsilon_{ijk} v^k_0 = -\partial_i P_0 + F^i_0, \quad (29)$$

Let us introduce the operator $\widehat{D}_0$:

$$\widehat{D}_0 \equiv \partial_t - \partial_{jj} + W^k \partial_k. \quad (30)$$

Using $\widehat{D}_0$, we rewrite equation (29) in the form:

$$\widehat{D}_0 v^i_0 + D^j \varepsilon_{ijk} v^k_0 = -\partial_i P_0 + F^i_0, \quad (31)$$

Pressure P_0 can be found from condition $div\vec{V} = 0.$

$$P_0 = \frac{\left[\vec{D} \times \vec{\partial}\right]_i v_0{}^i}{\partial^2} \quad (32)$$

Let us introduce designations for the operators:

$$\widehat{P}_{ij} = \partial_j \frac{\left[\vec{D} \times \vec{\partial}\right]_i}{\partial^2} \quad (33)$$

and for velocities: $v_0{}^x = u_0, v_0{}^y = v_0, v_0{}^z = w_0.$ Then excluding pressure from (31), we obtain the system of equations to find the velocity field of zero approximation:

$$\begin{aligned}
&\left(\widehat{D}_0 + \widehat{P}_{xx}\right) u_0 + \left(\widehat{P}_{yx} - D_z\right) v_0 + \left(\widehat{P}_{zx} + D_y\right) w_0 = F_0{}^x,\\
&\left(\widehat{P}_{xy} + D_z\right) u_0 + \left(\widehat{D}_0 + \widehat{P}_{yy}\right) v_0 + \left(\widehat{P}_{zy} - D_x\right) w_0 = F_0{}^y,\\
&\left(\widehat{P}_{xz} - D_y\right) u_0 + \left(\widehat{P}_{yz} + D_x\right) v_0 + \left(\widehat{D}_0 + \widehat{P}_{zz}\right) w_0 = F_0{}^z.
\end{aligned} \quad (34)$$

In order to solve this system of equations we have to set the force in the explicit form. Let us choose now the external force in the rotating system of coordinates in the following form:

$$F_0{}^z = 0, \vec{F}_{0\perp} = f_0\left(\vec{i}Cos\varphi_2 + \vec{j}Cos\varphi_1\right); \varphi_1 = \vec{k}_1\vec{x} - \omega_0 t, \varphi_2 = \vec{k}_2\vec{x} - \omega_0 t,$$
$$\vec{k}_1 = k_0(1,0,0), \vec{k}_2 = k_0(0,1,0).$$

It is obvious that divergence and helicity of this force us equal to zero: $\overrightarrow{F}_0 rot \overrightarrow{F} = 0.$ Thus, the external force is given in the plane (x, y), which is orthogonal to the projection of angular velocity $\vec{\Omega}$.

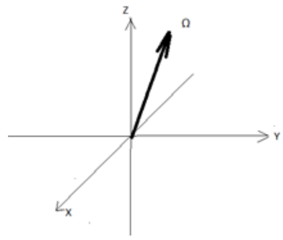

Fig. 1. In general, the angular velocity Ω is inclined relative to the plane (X, Y) in which there is an external force $\overrightarrow{F}_{0\perp}$

The solution for equations system (34) can be found easily in accordance with Cramer's Rule:

$$u_0 = \frac{\Delta_1}{\Delta}, v_0 = \frac{\Delta_2}{\Delta}, w_0 = \frac{\Delta_3}{\Delta}. \quad (35)$$

Here Δ – is the determinant of the system (34):

$$\Delta = \begin{vmatrix} \widehat{D}_0 + \widehat{P}_{xx} & \widehat{P}_{yx} - D_z & \widehat{P}_{zx} + D_y \\ \widehat{P}_{xy} + D_z & \widehat{D}_0 + \widehat{P}_{yy} & \widehat{P}_{zy} - D_x \\ \widehat{P}_{xz} - D_y & \widehat{P}_{yz} + D_x & \widehat{D}_0 + \widehat{P}_{zz} \end{vmatrix}, \quad (36)$$

$$\Delta_1 = \begin{vmatrix} F_0{}^x & \widehat{P}_{yx} - D_z & \widehat{P}_{zx} + D_y \\ F_0{}^y & \widehat{D}_0 + \widehat{P}_{yy} & \widehat{P}_{zy} - D_x \\ 0 & \widehat{P}_{yz} + D_x & \widehat{D}_0 + \widehat{P}_{zz} \end{vmatrix}, \quad (37)$$

$$\Delta_2 = \begin{vmatrix} \widehat{D}_0 + \widehat{P}_{xx} & F_0{}^x & \widehat{P}_{zx} + D_y \\ \widehat{P}_{xy} + D_z & F_0{}^y & \widehat{P}_{zy} - D_x \\ \widehat{P}_{xz} - D_y & 0 & \widehat{D}_0 + \widehat{P}_{zz} \end{vmatrix}, \quad (38)$$

$$\Delta_3 = \begin{vmatrix} \widehat{D}_0 + \widehat{P}_{xx} & \widehat{P}_{yx} - D_z & F_0{}^x \\ \widehat{P}_{xy} + D_z & \widehat{D}_0 + \widehat{P}_{yy} & F_0{}^y \\ \widehat{P}_{xz} - D_y & \widehat{P}_{yz} + D_x & 0 \end{vmatrix}. \quad (39)$$

Expanding the determinant, we obtain:

$$u_0 = \frac{1}{\Delta}\left[\left(\widehat{D}_0 + \widehat{P}_{yy}\right)\left(\widehat{D}_0 + \widehat{P}_{zz}\right) - \left(\widehat{P}_{yz} + D_x\right)\left(\widehat{P}_{zy} - D_x\right)\right] F_0{}^x + \quad (40)$$
$$+\frac{1}{\Delta}\left[\left(\widehat{P}_{zx} + D_y\right)\left(\widehat{P}_{yz} + D_x\right) - \left(\widehat{P}_{yx} - D_z\right)\left(\widehat{D}_0 + \widehat{P}_{zz}\right)\right] F_0{}^y,$$

$$v_0 = \frac{1}{\Delta}\left[\left(\widehat{P}_{xz} - D_y\right)\left(\widehat{P}_{zy} - D_x\right) - \left(\widehat{P}_{xy} + D_z\right)\left(\widehat{D}_0 + \widehat{P}_{zz}\right)\right] F_0{}^x + \quad (41)$$
$$+\frac{1}{\Delta}\left[\left(\widehat{D}_0 + \widehat{P}_{xx}\right)\left(\widehat{D}_0 + \widehat{P}_{zz}\right) - \left(\widehat{P}_{xz} - D_y\right)\left(\widehat{P}_{zx} + D_y\right)\right] F_0{}^y,$$

$$w_0 = \frac{1}{\Delta}\left[\left(\widehat{P}_{xy} + D_z\right)\left(\widehat{P}_{yz} + D_x\right) - \left(\widehat{P}_{xz} - D_y\right)\left(\widehat{D}_0 + \widehat{P}_{yy}\right)\right] F_0{}^x + \quad (42)$$
$$+\frac{1}{\Delta}\left[\left(\widehat{P}_{xz} - D_y\right)\left(\widehat{P}_{yx} - D_z\right) - \left(\widehat{D}_0 + \widehat{P}_{xx}\right)\left(\widehat{P}_{yz} + D_x\right)\right] F_0{}^y.$$

$$\Delta=\left(\widehat{D_0}+\widehat{P_{xx}}\right)\left[\left(\widehat{D_0}+\widehat{P_{yy}}\right)\left(\widehat{D_0}+\widehat{P_{zz}}\right)-\left(\widehat{P_{yz}}+D_x\right)\left(\widehat{P_{zy}}-D_x\right)\right]-$$
$$-\left(\widehat{P_{yx}}-D_z\right)\left[\left(\widehat{P_{xy}}+D_z\right)\left(\widehat{D_0}+\widehat{P_{zz}}\right)-\left(\widehat{P_{xz}}-D_y\right)\left(\widehat{P_{zy}}-D_x\right)\right]+$$
$$+\left(\widehat{P_{zx}}+D_y\right)\left[\left(\widehat{P_{xy}}+D_z\right)\left(\widehat{P_{yz}}+D_x\right)-\left(\widehat{D_0}+\widehat{P_{yy}}\right)\left(\widehat{P_{xz}}-D_y\right)\right]. \quad (43)$$

In order to calculate the expressions (40) - (43) we present the external force in complex form:

$$F_0{}^x=\frac{f_0}{2}\left(e^{i\varphi_2}+e^{-i\varphi_2}\right),F_0{}^y=\frac{f_0}{2}\left(e^{i\varphi_1}+e^{-i\varphi_1}\right). \quad (44)$$

Then all operators in formulae (40) - (42) act from the left on their eigen function. In particular:

$$\widehat{D_0}e^{i\varphi_2}=e^{i\varphi_2}\widehat{D_0}\left(k_2,-\omega_0\right),\widehat{D_0}e^{i\varphi_1}=e^{i\varphi_1}\widehat{D_0}\left(k_1,-\omega_0\right),$$
$$\Delta e^{i\varphi_2}=e^{i\varphi_2}\Delta\left(k_2,-\omega_0\right),\Delta e^{i\varphi_1}=e^{i\varphi_1}\Delta\left(k_1,-\omega_0\right) \quad (45)$$

To simplify the formulae, let us choose $k_0=1,\omega_0=1, f_0=1.$

We will designate:

$$\widehat{D_0}\left(k_2,-\omega_0\right)=1-i\left(1-w_y\right)=A_y,\widehat{D_0}\left(k_1,-\omega_0\right)=1-i\left(1-w_x\right)=A_x. \quad (46)$$

Before doing further calculations, we have to note that some components of tensors $\widehat{P}_{ij}\left(k_1\right)$ and $\widehat{P}_{ij}\left(k_2\right)$ vanish. Let us write the non-zero components only:

$$\widehat{P}_{yx}\left(k_1\right)=D_z,\widehat{P}_{zx}\left(k_1\right)=-D_y,\widehat{P}_{xy}\left(k_2\right)=-D_z,\widehat{P}_{zy}\left(k_2\right)=D_x. \quad (47)$$

Taking into account the formulae (45) - (47), we can find the determinant:

$$\Delta\left(k_1\right)=A^3{}_x+D^2{}_xA_x,\Delta\left(k_2\right)=A^3{}_y+D^2{}_yA_y. \quad (48)$$

In a similar way we find velocity field of zero approximation:

$$u_0=\frac{1}{2}\frac{e^{i\varphi_2}A_y}{A^2{}_y+D^2{}_y}+C.C., \quad (49)$$

$$v_0=\frac{1}{2}\frac{e^{i\varphi_1}A_x}{A^2{}_x+D^2{}_x}+C.C., \quad (50)$$

$$w_0=\frac{1}{2}\frac{e^{i\varphi_2}D_y}{A^2{}_y+D^2{}_y}-\frac{1}{2}\frac{e^{i\varphi_1}D_x}{A^2{}_x+D^2{}_x}+C.C.. \quad (51)$$

We note that the angular velocity D_z component disappears from the expression for the velocity field of zero approximation, which is a consequence of the properties of an external force.

4. REYNOLDS STRESS AND LARGE SCALE INSTABILITY

To close the equations (27) we have to calculate the Reynolds stresses $\overline{w_0u_0}$ and $\overline{w_0v_0}$. These terms are easily calculated with help of formulae (49) - (51). As a result we obtain:

$$\overline{w_0u_0}=\frac{1}{2}\frac{D_y}{\left|A^2{}_y+D^2{}_y\right|^2},\overline{w_0v_0}=-\frac{1}{2}\frac{D_x}{\left|A^2{}_x+D^2{}_x\right|^2}. \quad (52)$$

Now equations (27) are closed and take form:

$$\partial_TW_x-\Delta W_x+\frac{\partial}{\partial z}\frac{1}{2}\frac{D_y}{\left|A^2{}_y+D^2{}_y\right|^2}=0,$$
$$\partial_TW_y-\Delta W_y-\frac{\partial}{\partial z}\frac{1}{2}\frac{D_x}{\left|A^2{}_x+D^2{}_x\right|^2}=0. \quad (53)$$

We calculate the modules and write the equations (53) in the explicit form:

$$\partial_TW_x-\Delta W_x+\frac{1}{2}\frac{\partial}{\partial z}\frac{D_y}{4\left(1-w_y\right)^2+\left[D^2{}_y+w_y\left(2-w_y\right)\right]^2}=0,$$
$$\partial_TW_y-\Delta W_y-\frac{1}{2}\frac{\partial}{\partial z}\frac{D_x}{4\left(1-w_x\right)^2+\left[D^2{}_x+w_x\left(2-w_x\right)\right]^2}=0. \quad (54)$$

With small W_x,W_y we obtain the linear zed equations (54)

$$\partial_TW_x-\Delta W_x-\alpha_y\frac{\partial}{\partial z}W_y=0,$$
$$\partial_TW_y-\Delta W_y+\alpha_x\frac{\partial}{\partial z}W_x=0. \quad (55)$$

$$\alpha_y=2\frac{D_y\left(D^2{}_y-2\right)}{\left(4+D^2{}_y\right)^2},\alpha_x=2\frac{D_x\left(D^2{}_x-2\right)}{\left(4+D^2{}_x\right)^2}.$$

The system (55) describes the positive feedback between the components of velocity. We will look for the solution of linear system (55) in the following form:

$$W_x,W_y\sim\exp\left(\gamma T+ikZ\right). \quad (56)$$

Substituting (56) in equation (55), we obtain the dispersion equation:

$$\gamma=\pm\sqrt{\alpha_x\alpha_y}k-k^2. \quad (57)$$

The dispersion equation (57) shows the existence at $\alpha_x\alpha_y\succ 0$ the large scale instability with maximum growth rate $\gamma_{\max}=\frac{\alpha_x\alpha_y}{4},$ at the wave vector $k_{\max}=\frac{1}{2}\sqrt{\alpha_x\alpha_y}.$ As a result of the development of instability the large scale helical Beltrami vortices are generated in the system. When $\alpha_x\alpha_y\prec 0$, damped oscillations with a frequency $\omega_0=\sqrt{\alpha_x\alpha_y}k$ arise instead of instability. In fact the behavior of γ depends on how is located the external force $F_0{}^x,F_0{}^y$ with respect to the perpendicular projections of the angular velocity of rotation and the values of components D_x,D_y. If one of the component D_x,D_y is zero or equal to $\sqrt{2}$, then the instability is absent. Instability exists in the following cases:

1. $D_x\succ\sqrt{2},D_y\succ\sqrt{2};$
2. $D_x,D_y\succ 0,D_x\prec\sqrt{2},D_y\prec\sqrt{2};$
3. $D_x\prec 0,D_y\prec 0,D^2{}_x\succ 2,D^2{}_y\succ 2;$
4. $D_x\prec 0,D_y\prec 0,D^2{}_x\prec 2,D^2{}_y\prec 2;$
5. $D_x\prec 0,D_y\succ 0,D^2{}_y\succ 2,D^2{}_x\prec 2; or D^2{}_y\prec 2,D^2{}_x\succ 2;$
6. $D_x\succ 0,D_y\prec 0,D^2{}_y\succ 2,D^2{}_x\prec 2; or D^2{}_y\prec 2,D^2{}_x\succ 2;$

In all other cases damped oscillations occur.

5. SATURATION OF INSTABILITY AND NONLINEAR VORTEX STRUCTURES

It is clear that with increasing of amplitude nonlinear terms decrease and instability becomes saturated. Consequently stationary nonlinear vortex structures are formed. To find these structures let us choose for equations (54) $\frac{\partial}{\partial T}=0$ and integrate equations one time over Z. We obtain the system of equations:

$$\frac{d}{dZ}W_x = \frac{1}{2}\frac{D_y}{4\left(1-w_y\right)^2+\left[D^2{}_y+w_y\left(2-w_y\right)\right]^2}+C_1, \qquad (58)$$

$$\frac{d}{dZ}W_y = -\frac{1}{2}\frac{D_x}{4\left(1-w_x\right)^2+\left[D^2{}_x+w_x\left(2-w_x\right)\right]^2}+C_2.$$

Let's take for this system new variables: $1-w_x = u_x, 1-w_y = u_y$. Then we obtain:

$$\frac{du_x}{dZ} = -\frac{1}{2}\frac{D_y}{\left(D^2{}_y+1\right)^2+2\left(1-D^2{}_y\right)u^2{}_y+u^4{}_y}+C_1, \qquad (59)$$

$$\frac{du_y}{dZ} = \frac{1}{2}\frac{D_x}{\left(D^2{}_x+1\right)^2+2\left(1-D^2{}_x\right)u^2{}_x+u^4{}_x}+C_2.$$

The system of equations (59) can be written in Hamiltonian form:

$$\frac{du_x}{dZ} = -\frac{\partial H}{\partial u_y},$$

$$\frac{du_y}{dZ} = \frac{\partial H}{\partial u_x}.$$

Where Hamiltonian H has the form:

$$H = h\left(D_x, u_x\right)+h\left(D_y, u_y\right), \qquad (60)$$

with function $h(D,u)$:

$$h(D,u) = \frac{D}{2}\int\frac{du}{\left(D^2+1\right)^2+2\left(1-D^2\right)u^2+u^4}+Cu. \qquad (61)$$

Integral in expression (61) is calculated in elementary functions [17]. Let us choose for simplicity $D_x = D_y = D = 1$. In this case, the function (61) is equal [17]:

$$h(u) = \frac{1}{16}\left\{\ln\frac{u^2+2u+2}{u^2-2u+2}+arctg\frac{2u}{2-u^2}\right\}+Cu. \qquad (62)$$

The sum $h(u_x)+h(u_y)$ can be write down as one formula. Then Hamiltonian is equal:

$$H = \frac{1}{16}\ln\frac{\left(u_x{}^2+2u_x+2\right)\left(u_y{}^2+2u_y+2\right)}{\left(u_x{}^2-2u_x+2\right)\left(u_y{}^2-2u_y+2\right)}+ \qquad (63)$$

$$+\frac{1}{16}arctg\frac{2u_y\left(u^2{}_x-2\right)+2u_x\left(u^2{}_y-2\right)}{2\left(u_x+u_y\right)^2-u^2{}_x u^2{}_y-4}+C_1u_x+C_2u_y$$

It is easy to construct the phase portrait of Fig. 4. for Hamiltonian (63) and specific values C_1 = 0.1, C_2 = 0.1.

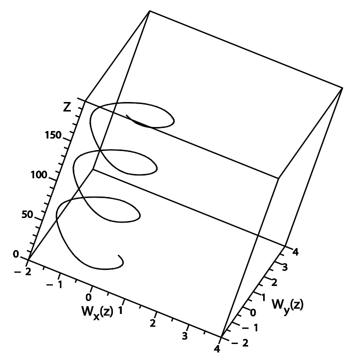

Fig. 2. Nonlinear spiral wave Beltrami, which corresponds to a closed trajectory in the phase plane (C_1 =0.1, C_2 = 0.1). The spiral is oriented along Z-axis and inclined relative to the axis of rotation

The phase portrait shows the presence of closed trajectories in the phase plane around elliptic points and separatrix that connect hyperbolic points. It is obvious that the closed trajectories correspond to nonlinear periodic solutions. The separatrix correspond to localized solutions of kink type.

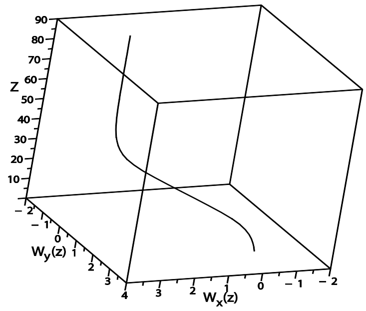

Fig. 3. Localized solution (kink), which corresponds to the separatrix in the phase plane (C_1=0.1, C_2= 0.1)

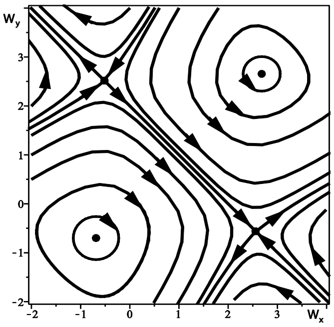

Fig. 4. Phase plane for Hamiltonian (63) (C_1=0.1, C_2= 0.1). We see the presence of closed trajectories around the elliptic points and separatrix that connect hyperbolic points. Phase portrait is typical for Hamiltonian systems

CONCLUSIONS

In this work we found new large scale instability in rotating fluid. It is supposed that the small scale vortex external force in rotating coordinates system acts on fluid which maintains the small velocity field fluctuations (small-scale turbulence with low Reynolds number $R, R \ll 1$). For the real applications this Reynolds number should be calculated with help of the turbulent viscosity. The asymptotic development of motion equations by small Reynolds number allows obtaining motion equations for the large scale. These equations are of the hydrodynamic α- effect type, in which velocity components W_x, W_y are connected by the positive feedback. This may result in the appearance of the large scale vortex instability. This instability is responsible for the formation in rotating fluid with small scale external force of large scale Beltrami vortices. With further increase of amplitude the instability stabilizes and passes to a stationary mode. In this mode the nonlinear stationary vortex structures are formed. The most interesting structures belong to a variety of vortex kinks. These kinks connect stationary hyperbolic points of the dynamical system (58).

Note that in contrast to previous work on the hydrodynamic α- effect in rotating fluid, the method enables us to construct an asymptotic development in a natural way and to explore non-linear theory of nonlinear stationary vortex kinks.

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КРУПНОМАСШТАБНАЯ НЕУСТОЙЧИВОСТЬ И НЕЛИНЕЙНЫЕ ВИХРЕВЫЕ СТРУКТУРЫ В НАКЛОННО ВРАЩАЮЩЕЙСЯ ЖИДКОСТИ С МЕЛКОМАСШТАБНОЙ ВНЕШНЕЙ НЕСПИРАЛЬНОЙ СИЛОЙ

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Найдена новая крупномасштабная неустойчивость, которая возникает в наклонно вращающейся жидкости с мелкомасштабной турбулентностью. Турбулентность генерируется мелкомасштабной внешней силой с малым числом Рейнольдса. Внешняя сила не имеет спиральности. Теория построена строгим методом многомасштабного асимптотического разложения. Нелинейные уравнения для неустойчивости получены в третьем порядке теории возмущений. Проведено детальное исследование нелинейной стадии неустойчивости и найдены нелинейные периодические вихри Бельтрамиевского типа и вихревые кинки.

ВЕЛИКОМАСШТАБНА НЕСТІЙКІСТЬ ТА НЕЛІНІЙНІ ВИХОРОВІ СТРУКТУРИ В РІДИНІ, ЩО ОБЕРТАЄТЬСЯ ПІД НАХИЛОМ, З МАЛОМАСШТАБНОЮ ЗОВНІШНЬОЮ НЕСПІРАЛЬНОЮ СИЛОЮ

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Знайдено великомасштабну нестійкість, яка виникає в рідині, що обертається під нахилом, у маломасштабній турбулентності. Турбулентність генерується маломасштабною зовнішньою силою з малим числом Рейнольдса. Зовнішня сила не має спіральності. Теорія побудована з використанням послідовного багатомасштабного асимптотичного методу. Нелінійні рівняння для нестійкості отримано в третьому порядку теорії збурень. Проведено детальне дослідження нелінійної стадії нестійкості та знайдено нелінійні періодичні вихори Бельтрамієвського типу та вихорові кінки.