ON THE EMISSION SPECTRUM OF OSCILLATOR TRAPPED IN A POTENTIAL WELL

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We study the spectrum of electromagnetic waves emitted by oscillator, trapped in an external potential well. It is assumed that the natural frequency of the oscillator is much greater than the frequency of oscillations in the potential well. We consider the quantum model of emission with taking into account the recoil effect. The highest intensity of the absorption and emission lines is observed on the eigenfrequency of the oscillator when the recoil energy is equal to energy of the quantum of low-frequency oscillations in the potential well.

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INTRODUCTION

The scattering of high-energy photons by free electrons results in a decrease in energy of photons due to the recoil effect (the Compton effect). This fact together with the phenomenon of the photoelectric effect confirmed the basic principles of quantum theory of radiation [1 - 3]. The processes underlying the interaction of radiation and matter are characterized by an impressive variety and form the basis for many physical research directions [4 - 6]. One of the problems that appears when considering the processes of absorption and emission by a matter is the problem of interaction with the external radiation field of the oscillating particle trapped in the potential well formed by the spatial structure of the medium. This problem requires the use of methods of quantum electrodynamics for describing the behavior of an excited oscillator in a potential well.

The purpose of this work is to analyze the interaction between a charged particle-oscillator trapped in a potential well and external electromagnetic field. We found that the highest intensity of the absorption and emission lines is observed on the natural frequency of the oscillator ω_0 when the recoil energy E_r is equal to the energy of low-frequency quantum $\hbar\Omega$ (Ωis the frequency of oscillations in the potential well). We also discuss the role of relaxation of low-frequency motion in the potential well due to emission of sound waves and conditions under which the relaxation does not affect the considered emission and absorption of high-frequency quanta.

1. THE EMISSION OF VIBRATING OSCILLATOR

Following the method described in [1], we consider the quantum model of the oscillator with a charge e, mass m and natural frequency ω_0, which oscillates as a whole in the potential well aligned along axis OZ. Let the oscillator emits the electromagnetic wave in the same direction $\vec{k} = (0; 0; k_z)$. The components of the oscillator's velocity vector are:

$$v_x = v_{x0}\cos\omega_0 t = a\omega_0\cos\omega_0 t, \quad v_z = b\Omega\cos(\Omega t). \tag{1}$$

The electromagnetic field can be found from

$$\vec{E} = -\frac{1}{c}\dot{\vec{A}}, \ \vec{H} = \mathrm{rot}\,\vec{A}. \tag{2}$$

The vector potential has the components

$$A_x = q(t)\sqrt{2}\cos(kz+\delta), \ A_y = 0, \ A_z = 0. \tag{3}$$

Phase δ depends on the oscillator orientation. The form (3) choice is determined by normalizing condition, so that the integral of the vector potential squared in the unit volume equals to unity and $q(t)$ satisfies the equation

$$\ddot{q} + \omega^2 q = 0. \tag{4}$$

The total field energy inside V-volume is equal to

$$U = \frac{V}{4\pi c^2}\frac{1}{2}(\dot{q}^2 + \omega^2 q^2). \tag{5}$$

Let define the effective mass of the oscillator as

$$m_{eff} = V / 4\pi c^2. \tag{6}$$

As follows from Eqs. (1) and (3), the x-component of vector potential A_x in the location point of the oscillator is equal

$$A_x = \sqrt{2}q_0\cos\omega t\cdot\cos\left(kb\sin\Omega t + \delta\right). \tag{7}$$

Suppose the particle occupies the lowest energy level in an external potential well and consider the case, when it remains inside the well after the emission or absorption of a quantum. That means the recoil energy to be insufficient for the particle to leave the potential well it stays in. After absorbing a high-frequency quantum $E_v = \hbar(\omega_0 + \Omega)$, the particle gets the recoil momentum mV and begins to perform slow oscillations inside the potential well.

At that the conservation laws are kept:

$$\hbar(\omega_0 + \Omega)/v_s = mV, \tag{8}$$

$$\hbar\Omega = mV^2/2. \tag{9}$$

It is noticeable that the oscillation energy of the particle in the potential well equals to $\hbar\Omega$. That is why the exciting quantum energy has to exceed the oscillator`s energy $\hbar\omega_0$ by this value. Assuming the condition

$$\hbar\omega_0 << 2Mc^2, \tag{10}$$

is fulfilled and using Eqs. (8) - (9), we find the oscillation frequency inside the potential well

$$\Omega \approx \hbar\omega^2{}_0 / 2mc^2. \tag{11}$$

When the recoil effect is taken into account, the absorbed and emitted quantum frequencies are shifted on

$\pm\Omega$ and the expression (7) should be modified as follows:

$$A_x = \sqrt{2}q_0 \cos((\omega\pm\Omega)t)\cos(kb\sin\Omega t+\delta)\,. \qquad (12)$$

The Hamiltonian of the system including the oscillator and the field can be represented as [1]

$$H = \frac{1}{2m}\left(\vec{p}-\frac{e}{c}\vec{A}\right)\left(\vec{p}-\frac{e}{c}\vec{A}\right)+e\Phi\,. \qquad (13)$$

Thus, the interaction part can be determined as

$$H' = -ev_x\, A_x/c\,. \qquad (14)$$

Representing the system “oscillator in the potential well” as a set of oscillators with frequencies $\omega_m = \omega_0 + m\Omega$ and imposing the condition of temporal synchronism one can find that for the frequency of the external field

$$\omega = (m\mp 1)\Omega + \omega_0\,, \qquad (15)$$

the interaction Hamiltonian (14) takes the form

$$H' = -\frac{ev_{x0}}{c}q_0\sqrt{2}\sum_m J_m(kb)\cos\delta\,. \qquad (16)$$

The upper sign in Eq. (15) corresponds to the emission of the quantum with taking into account the recoil effect (the frequency of the external field at this is greater than the proper frequency of the oscillator on the value of $\Omega/2\pi$). The lower sign corresponds to the absorption (the frequency of the external field at this is less than the proper frequency of the oscillator on the value $\Omega/2\pi$). Note that the correction to the matrix element due to the recoil effect is proportional to the small parameter $\hbar\omega_0/mc^2$ [8] both for the individual processes of absorption and emission, as well as for the Compton scattering, which combines these two processes.

When the oscillator in rest, $b=0$, the only term in the sum (16) is different from zero ($m=0$):

$$H' = -\frac{ev_{x0}}{c}q_0\sqrt{2}\cos\delta. \qquad (17)$$

Thus, the frequency of the absorbed radiation $\omega=\omega_0+\Omega$ differs from the frequency of the emitted radiation $\omega=\omega_0-\Omega$ on the value of 2Ω that corresponds to the value of the double recoil energy.

For $b\neq 0$, the interaction Hamiltonian for the most interesting case of the absorption and emission on the proper frequency of the oscillator ω_0 becomes:

$$H' = -\frac{e\cdot v_x}{c}q\sqrt{2}J_{\pm 1}(kb)\cos\delta. \qquad (18)$$

The interaction Hamiltonian for emission on the frequency $\omega=\omega_0-\Omega$ and absorption on the frequency $\omega=\omega_0+\Omega$ is similar to Eq. (18), where $J_{\pm1}(kb)$ should be replaced by $J_0(kb)$.

Let estimate the value of the argument of Bessel’s functions. Since the frequency of low-frequency oscillations in the potential well is $\Omega = V_Q/b$ and the energy of quantum $\hbar\Omega$ is equal to the recoil energy then [7]

$$kb\approx\frac{\omega_0 b}{c}\approx 2. \qquad (19)$$

The matrix element corresponding to Eq. (18) can be written as

$$H_{if} = -\frac{e}{c}\sqrt{2}\omega_0 x_{ab}q_{nn'}J_{\pm1}(kb)\cos\delta. \qquad (20)$$

The subscripts a and b indicate two states of the emitted ($n, n+1$) or absorbed ($n, n-1$) field. For the absorption case $|q_{nn'}|^2 = |q_{n,n-1}|^2 = n|q_{01}|^2$ and for the emission case $|q_{nn'}|^2 = |q_{n+1,n}|^2 = (n+1)|q_{01}|^2$.

Taking into account the expression for effective mass (6), we can write the matrix element for the spontaneous emission [1]

$$|q_{01}|^2 = \frac{hc^2}{V\omega_0}\,, \qquad (21)$$

and

$$|H_{if}|^2 = -\frac{2e^2}{c^2}\omega_0\left(x_{ab}^2+y_{ab}^2\right)\frac{hc^2}{V}\cdot J_1^2(kb)\cos^2\delta\begin{Bmatrix}n+1\\ n\end{Bmatrix}, \qquad (22)$$

where the upper value corresponds to emission and the lower value to absorption. The transition probability can be found by taking the product of Eq. (22) and $4\pi^2\rho(\nu_{ab})/h^2$, where $\rho(\nu_{ab})$ is the oscillation density, with taking into account the averaging over initial phases $\langle\cos^2\delta\rangle = 1/2$

$$P_{if} = \frac{4\pi^2}{h^2}|\mathrm{H}_{if}|^2\rho = \frac{8\pi e^2}{hc^3}\omega_0^2\left(|x_{ab}|^2+|y_{ab}|^2\right)\cdot J_1^2(kb)\cos^2\delta\begin{Bmatrix}n+1\\ n\end{Bmatrix}. \qquad (23)$$

Note that the probability of absorption on the frequency $\omega_0+\Omega$ and the probability of emission on the frequency $\omega_0-\Omega$ can be obtained by replacing $J_1^2(kb)$ by $J_0^2(kb)$ in Eq.(23). Multiplying Eq.(23) by $\hbar\omega_0$ one can obtain the intensity of radiation along axis OZ and integrating over the angle $\theta = \vec{k}\wedge\vec{OZ}$ – the total intensity over all directions.

It is easy to verify that since $J_1^2(kb) >> J_0^2(kb)$ the intensity of absorption and emission spectral lines on the proper frequency of the oscillator ω_0 exceeds the intensity on the frequency $\omega_0\pm\Omega$ by an order. In the case of captured oscillator emission, the possibility of such phenomena was discussed in [7].

Note that the width of the potential well should be greater than b and its depth should exceed the recoil energy. The proposed quantum-mechanical description may be more correctly clarify the mechanism of emission and absorption of γ -quanta without recoil [6].

2. ON RELACXATION OF LOW-FREQUENCY OSCILLATIONS

When the above-discussed system is placed in solid medium, it is necessary to consider a possibility of emission of a low-frequency phonon $\Omega = \omega_0(\hbar\omega_0/2M_Qc^2)$. The relatively low velocity acquired by an oscillator due to recoil is often significantly less than the phase velocity of phonons

$$v_s >> c(\hbar\omega_0 / 2M_Q c^2) . \qquad (24)$$

This makes impossible the direct transfer of kinetic energy to phonon. This is evidenced by the inability to fulfill the requirements of conservation of energy and momentum. For example, it can be shown for the momentum that

$$\hbar\omega_0 / c >> \hbar\Omega / v_s = = \hbar\omega_0 \left(\frac{\hbar\omega_0}{2mc^2} \right) \Big/ v_s = \frac{\hbar\omega_0}{c} \frac{\hbar\omega_0}{2mc^2} \frac{c}{v_s} . \qquad (25)$$

However, if the oscillator is trapped to the potential well, its movement becomes irregular and it becomes capable to emit phonons. At this, the spatial period of low-frequency oscillations in the potential well

$$b = \frac{V_Q}{\Omega} = \frac{\hbar\omega_0}{M_Q c} \frac{2M_Q c^2}{\hbar\omega_0{}^2} = \frac{c}{\omega_0} , \qquad (26)$$

is much less than the wavelength of phonon oscillations, which frequency is equal to the frequency of oscillations in the potential well

$$k_s b = \frac{\Omega}{v_s} \frac{c}{\omega_0} = \frac{\hbar\omega_0}{2mc^2} \frac{c}{v_s} << 1 . \qquad (27)$$

Let estimate the lifetime of the low-frequency oscillator. If the lifetime occurs much greater than the period of oscillations, the above consideration of the isolated system “oscillator in potential well” remain applicable for the case when this system is placed in medium. In other words, the relaxation processes caused by account of phonon spectrum can be neglected.

In one-dimensional case the perturbation of the medium density generated by low-frequency motion of the oscillator can be written as

$$\rho_{внешн}(t,x) = M_Q \cdot \delta[x - V_Q \cos(\Omega t)] . \qquad (28)$$

Its Fourier image is equal

$$\rho(\omega,k) = \frac{\omega^2}{v_s^2 k^2 - \omega^2} \frac{m}{4\pi} \frac{kb}{2} \cdot \qquad (29)$$
$$\cdot [\delta(\omega - \Omega) - \delta(\omega + \Omega)].$$

The inverse transform gives

$$\rho(x,t) = -\frac{mb\Omega^2}{8v_s^2} \{\sin(\Omega t + k_0 z) + \sin(\Omega t - k_0 z)\} . \qquad (30)$$

Since the perturbation of the medium speed $u = v_s \rho / \rho_0$, the ratio of emission intensity to the energy of the quantum is equal

$$I / \hbar\Omega = \frac{1}{16} \frac{m}{\rho_0 \lambda_s} \Omega . \qquad (31)$$

The small parameter $m/\rho_0\lambda_s$ here is equal to the ratio of oscillator’s mass to the total mass of similar particles located on the wavelength of sound wave. The relaxation time of low-frequency oscillations

$$\tau_{LF} \approx 8\rho_0 \lambda_s / \pi m \Omega , \qquad (32)$$

occurs, as was supposed earlier, much greater than the period of low-frequency oscillations in the potential well. Moreover, if τ_{LF} far exceeds the lifetime of high-frequency quantum, the relaxation process does not affect the character of emission and absorption of photons considered above. Note, that in the three-dimensional case the characteristic relaxation time of low-frequency motion

$$\tau_{LF} \approx 3(\rho_0 \lambda_s^3 / m)(\omega_0 / \pi^2 \Omega_0^2) , \qquad (33)$$

is proportional to a large parameter $\rho_0 \lambda_s^3 / m$. This parameter is equal to the ratio of total mass of atoms within a cube with edge λ to the mass of single atom. The factor (ω_0 / Ω) is large too. In this case the lifetime of high-frequency oscillator may occur less than the relaxation time of low-frequency motion caused by emission of sound.

CONCLUSIONS

We considered the quantum-mechanical model of electromagnetic emission by one-dimensional oscillator, trapped in the external potential well. On the assumption of the energy of the emitted quantum is much less that the rest energy of the oscillator ($\hbar\omega_0 / mc^2 << 1$), the perturbation of the Hamiltonian matrix elements caused by the recoil effect can be neglected, that simplifies their calculation.

It is shown that the intensity of the absorption and emission lines at the natural frequency ω_0 of the oscillator significantly exceeds the intensity of other spectral lines, particularly at frequencies $\omega_0 \pm \Omega$. This is caused by the equality of the oscillation energy in the potential well to the recoil energy, and the fact that the amplitude of the oscillation in the potential well in this case is comparable to the wavelength of the radiation.

Such a quantum-mechanical description may be more correctly explains the phenomenon of emission and absorption of electromagnetic field quanta without recoil by the matter [6]. Really, the emission of the oscillator with Eigen frequency (natural frequency) ω_0 produces the equidistant spectrum $\omega_0 + n\Omega$ that can be considered as result of emission of a set of oscillators with oscillation amplitude proportional to $J_n^2(kb)$. Moreover, since the amplitude of low-frequency oscillations in the potential well b is comparable with the emission wavelength $\lambda = 2\pi/k$, the emission and absorption on the proper frequency ω_0 dominates in this case. The nature of the high-frequency oscillator is not critical for the manifestation of the above-described properties of the system.

Note, that the application of this model to description of emission and absorption of γ-quanta in crystal structures is possible when the relaxation time of low-frequency perturbations exceeds the period of oscillations in the potential well and the lifetime of high-frequency oscillator.

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ИЗЛУЧЕНИЕ ЗАХВАЧЕННОГО В ПОТЕНЦИАЛЬНУЮ ЯМУ ОСЦИЛЛЯТОРА

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Обсуждается характер излучения осциллятора, захваченного во внешнюю потенциальную яму осциллятора. Собственная частота осциллятора значительно превосходит частоту его колебаний в потенциальной яме. Рассмотрена квантово-механическая модель излучения такого осциллятора в случае отличной от нуля энергии отдачи. В том случае, если энергия отдачи равна энергии кванта низкочастотных колебаний в потенциальной яме, наблюдается наибольшая интенсивность линий поглощения и излучения на собственной частоте покоящегося осциллятора.

ВИПРОМІНЮВАННЯ ОСЦИЛЯТОРА, ЗАХОПЛЕНОГО В ПОТЕНЦІЙНУ ЯМУ

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Обговорюється характер випромінювання осцилятора, який захоплений в зовнішню потенційну яму. Власна частота осцилятора значно перевершує частоту його коливань у потенційній ямі. Розглянуто квантово-механічну модель випромінювання такого осцилятора у випадку відмінної від нуля енергії віддачі. У тому випадку, якщо енергія віддачі дорівнює енергії кванта низькочастотних коливань у потенційній ямі, спостерігається найбільша інтенсивність ліній поглинання і випромінювання на власній частоті осцилятора, що знаходиться у стані спокою.