BEAM DYNAMICS OPTIMIZATION IN ELECTROSTATIC FIELD

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The problem of optimization of charged particle beam dynamics in an axially symmetric electric field is considered. The complex potential is represented as a Cauchy integral of a function defined on the boundary of the region and considered as the control function. Using a complex representation allows to get the explicit form of the field strength inside the area dependency on the control function and obtain the necessary optimality conditions for the entered functional.

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1. PROBLEM STATEMENT

Modern requirements to the accelerator technology and the parameters of the accelerated beam of charged particles require new approaches to the calculation of the accelerating and focusing structures at the design stage. Many works [1 - 8] are devoted to the problems of optimization of the dynamics of charged particles in electromagnetic fields. In particular, in works [9 - 10] electrostatic injectors for linear accelerator were investigated. Geometric parameters of systems and the potential values at the electrodes were considered as optimization parameters. However, these studies are not given analytical representations of variations in the optimized parameters. In this paper, the problem of optimization of beam dynamics of charged particles in the axial-symmetric electric field is considered. Analytical representation of variation is found and the optimality conditions are formulated.

In a simply connected bounded area G let us consider dynamics of charged particles described by a system of ordinary differential equations:

$$\ddot{r} = E_r(r, z, \varphi), \tag{1}$$

$$\ddot{z} = E_z(r, z, \varphi). \tag{2}$$

Note that the field intensity in the equations (1) and (2) is defined by specifying the function φ of the curve L. Here L – boundary of area G, assumed to be smooth closed curve, and the function φ is defined and continuous on the curve L, and satisfies the Hölder condition [11]:

$$|\varphi(\eta_1) - \varphi(\eta_2)| \le M|\eta_1 - \eta_2|^\nu,\ \nu > 0,\ M > 0. \tag{3}$$

In this case, the complex potential of the field is represented as a Cauchy integral [11]:

$$F(\xi, \varphi) = \frac{1}{2\pi i}\int_L \frac{\varphi(\eta)}{\eta - \xi} d\eta. \tag{4}$$

Here $\xi = r + i \cdot z \in G$, $\eta = x + i \cdot y \in L$. Here and further the real plane R^2 will be identified with the complex plane C.

The complex potential is an analytic function defined in a domain G, and its real and imaginary parts are harmonic functions of real variables r and z. Let us consider the function $U = \operatorname{Re} F$ as defining a potential electric field in the region G. Then the electric field intensity is given by

$$E_r - i \cdot E_z = -\frac{\partial F(\xi, \varphi)}{\partial \xi} = -\frac{1}{2\pi i}\int_L \frac{\varphi(\eta)}{(\eta - \xi)^2} d\eta. \tag{5}$$

Note that the dynamic equation (1), (2) may be converted to the form:

$$\ddot{\xi} = -\overline{\left(\frac{\partial F(\xi, \varphi)}{\partial \xi}\right)}, \tag{6}$$

where $\dot{\xi} = \dot{r} + i \cdot \dot{z}$ и $\ddot{\xi} = \ddot{r} + i \cdot \ddot{z}$, bar over the right-hand side denotes the complex conjugate.

For convenience we introduce the vector of phase variables $\mathbf{a} = (r, \dot{r}, z, \dot{z})^T$.

Equations (1) and (2) we will consider with the initial conditions

$$\mathbf{a}(0) = \mathbf{a}_0 = (r_0, v_{r0}, z_0, v_{z0})^T \in M_0 \subset R^4, \tag{7}$$

where M_0 a compact set such that for any point $\mathbf{a}(0) \in M_0$ satisfies $(r_0 + i \cdot z_0) \in G$.

Function φ will be referred to hereafter as boundary control or simply control. The class of admissible controls D is the set of continuous functions φ on a curve L satisfying the Hölder condition (3) and such that $\varphi(\eta) \in \Phi$ when $\eta \in L$, where Φ is a convex compact set in the complex plane.

We assume further that the solutions of system (1), (2) are defined and are unique to some fixed interval $[0, T]$, for all initial conditions (7) and for all admissible controls.

On the trajectories of the system (1), (2), we introduce the functional of quality of the form

$$I(\varphi) = \int_0^T \int_{M_{t,\varphi}} p(t, \mathbf{a}_t) d\mathbf{a}_t dt + \int_{M_{T,\varphi}} q(\mathbf{a}_T) d\mathbf{a}_T. \tag{8}$$

Here p and q are given non-negative, continuously differentiable functions, $\mathbf{a}_t = \mathbf{a}(t, \mathbf{a}_0, \varphi)$ is a vector of phase variables corresponding to the solution of system (1), (2) at the time t with the selected control function φ on a curve L and the initial condition (7). Set $M_{t,\varphi}$ is a section of the beam of trajectories of the system (1), (2) coming from the initial set M_0 at the time t with the given control function φ.

Let us consider further the minimization of the functional on the admissible class of controls. Let φ is an admissible control. Variation of the control $\Delta\varphi$ is admissible if the control $\tilde{\varphi} = \varphi + \Delta\varphi$ is also admissible control.

2. VARIATION OF FUNCTIONAL

Variation functional (8) with admissible variation of the control function φ such that $\|\Delta\varphi\| = \max_{\eta\in L}|\Delta\varphi(\eta)| \to 0$, can be represented as follows:

$$\delta I(\Delta\varphi)= = \mathrm{Re}\int_0^T \int_{M_{t,\varphi}} \lambda(t,\mathbf{a}_t)\cdot\left[\frac{1}{2\pi i}\int_L \frac{\Delta\varphi(\eta)}{(\eta-\xi(t,\mathbf{a}_t))^2}d\eta\right]d\mathbf{a}_t dt. \quad (9)$$

This complex function λ satisfies the following complex system defined on the trajectories of the system (1), (2)

$$\dot{\sigma} = \overline{\left(\frac{\partial^2 F(\xi,\varphi)}{\partial\xi^2}\lambda\right)} + \theta, \qquad \dot{\lambda} = -\sigma + \rho, \quad (10)$$

with terminal conditions

$$\sigma(T) = -\frac{\partial q(\mathbf{a}_T)}{\partial a_1} - i\frac{\partial q(\mathbf{a}_T)}{\partial a_3}, \qquad \lambda(T) = -\frac{\partial q(\mathbf{a}_T)}{\partial a_2} - i\frac{\partial q(\mathbf{a}_T)}{\partial a_4}, \quad (11)$$

where $\theta = \frac{\partial p}{\partial a_1} + i\cdot\frac{\partial p}{\partial a_3}$, $\rho = \frac{\partial p}{\partial a_2} + i\cdot\frac{\partial p}{\partial a_4}$.

3. NECESSARY EXTREMUM CONDITIONS

Let the boundary L has the following parameterization:

$$\eta = \chi(s) = x(s) + i\cdot y(s), \quad s\in[0,S]. \quad (12)$$

Then the integral over the complex circuit in the formula (9) may be replaced by definite integral. The result is

$$\delta I(\Delta\varphi)= \mathrm{Re}\int_0^T \int_{M_{t,\varphi}} \lambda(t,\mathbf{a}_t)\left[\frac{1}{2\pi i}\int_0^S \frac{\Delta\varphi(\chi(s))\dot{\chi}(s)ds}{(\chi(s)-\xi(t,\mathbf{a}_t))^2}\right]d\mathbf{a}_t dt. \quad (13)$$

By changing the order of integration in (13), we obtain:

$$\delta I(\Delta\varphi)= = \mathrm{Re}\int_0^S\left[\frac{1}{2\pi i}\int_0^T\int_{M_{t,\varphi}} \frac{\Delta\varphi(\chi(s))\dot{\chi}(s)\lambda(t,\mathbf{a}_t)}{(\chi(s)-\xi(t,\mathbf{a}_t))^2}d\mathbf{a}_t dt\right]ds \quad (14)$$
$$= \mathrm{Re}\int_0^S \Delta\varphi(\chi(s))\dot{\chi}(s)\omega(s)ds.$$

Here $\omega(s) = \frac{1}{2\pi i}\int_0^T\int_{M_{t,\varphi}} \frac{\lambda(t,\mathbf{a}_t)}{(\chi(s)-\xi(t,\mathbf{a}_t))^2}d\mathbf{a}_t dt$.

Theorem 1. Let φ_0 minimizes the functional (8). Then for any admissible variation of control function $\Delta\varphi$ the functional variation is non-negative

$$\delta I(\varphi_0,\Delta\varphi) = \mathrm{Re}\int_0^S \Delta\varphi(\chi(s))\dot{\chi}(s)\omega(s)ds \ge 0.$$

Proof. Assume that there is an admissible variation $\Delta\varphi$ such that $\delta I(\varphi_0,\Delta\varphi) < 0$. Variation of control is $\Delta\varphi = \tilde{\varphi} - \varphi_0$, where $\tilde{\varphi}$ is admissible control. Since Φ the convex set then the control $\varphi_\varepsilon = \varphi_0 + \varepsilon\Delta\varphi$ will also be permitted, where $\varepsilon\in[0,1]$. From the representation of variation (9) follows that

$$\delta I(\varphi_0,\varepsilon\Delta\varphi) = \varepsilon\delta I(\varphi_0,\Delta\varphi).$$

Thus

$$\Delta I(\varphi_0,\varepsilon\Delta\varphi) = \varepsilon\delta I(\varphi_0,\Delta\varphi) + o(\varepsilon).$$

It is clear that for sufficiently small ε, we obtain $\Delta I < 0$ that contrary to the assumption that the control φ_0 provides a minimum of the functional (8).

Theorem 2. Let the φ_0 minimizes the functional (8). Then for any admissible variation $\Delta\varphi$

$$\mathrm{Re}[\Delta\varphi(\chi(s))\cdot\dot{\chi}(s)\omega(s)] \ge 0$$

for all $s\in[0,S]$.

Proof. Suppose this is not the case. Then there is $s_0\in(0,S)$ that

$$\mathrm{Re}[\Delta\varphi(\chi(s_0))\cdot\dot{\chi}(s_0)\omega(s_0)] < 0$$

at some admissible variation $\Delta\varphi$. By continuity there will be an interval $[s_1,s_2]$ containing inside point s_0 and such that

$$\mathrm{Re}[\Delta\varphi(\chi(s))\cdot\dot{\chi}(s)\omega(s)] < 0, \quad (15)$$

For all $s\in[s_1,s_2]$.

We choose a neighborhood of point $\eta_0 = \chi(s_0)$ so small that to get into it could only points of the curve L corresponding to the parameter s values within the interval $[s_1,s_2]$, and the points of intersection of the circle $|\eta-\eta_0| = R$ with the curve there were only two. Denote the resulting neighborhood

$$S_R(\eta_0) = \{|\eta-\eta_0| \le R\}.$$

We construct the variation of the control function in the following way

$$\Delta\tilde{\varphi}(\eta(s)) = \Delta\varphi(s)K(\eta(s),\eta_0,R),$$

where

$$K(z,z_0,r_0) = \begin{cases} \frac{1}{r_0^4}\left(|z-z_0|^2 - r_0^2\right)^2, & |z-z_0| \le r_0; \\ 0, & |z-z_0| > r_0. \end{cases}$$

The function K takes real values of the range $[0,1]$. It is easy to see that the introduced variation will be valid.

With this choice of variation of control obviously be violated condition of Theorem 1:

$$\mathrm{Re}\int_0^S \Delta\tilde{\varphi}(\chi(s))\cdot\dot{\chi}(s)\omega(s)ds =$$
$$= \mathrm{Re}\int_0^S \Delta\varphi(\chi(s))\cdot\dot{\chi}(s)\omega(s)K(\chi(s),\chi(s_0),R)ds =$$
$$= \mathrm{Re}\int_{s_3}^{s_4} \Delta\varphi(\chi(s))\cdot\dot{\chi}(s)\omega(s)K(\chi(s),\chi(s_0),R)ds < 0,$$

where s_3 and s_4 correspond to the points of intersection of the curve L with the circle $|\eta-\eta_0| = R$. Hence φ_0 cannot deliver the minimum of the functional (8). This contradiction proves the theorem.

Remark. Obtained results obviously can be extended to the case of piecewise smooth boundary of area G.

CONCLUSIONS

In this paper a new approach to optimization problems in electrostatic axially symmetric field was proposed. Analytical representation for the variation of the optimized functional and the necessary optimality conditions are found. On the basis of the expression for the variation of the functional can be built directed methods of optimization. Various practical implementations of fields obtained in the optimization process are possible.

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ОПТИМИЗАЦИЯ ДИНАМИКИ ПУЧКА В ЭЛЕКТРОСТАТИЧЕСКОМ ПОЛЕ

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Современные требования к ускорительной технике, к параметрам ускоряемого пучка заряженных частиц требуют и новых подходов к расчету ускоряющих и фокусирующих структур на стадии проектирования. В данной работе рассматривается задача оптимизации динамики пучка заряженных частиц в аксиально-симметрическом электрическом поле. Комплексный потенциал представляется в виде интеграла типа Коши от функции, заданной на границе области и рассматриваемой в качестве управляющей функции. Использование комплексного представления позволяет получить явный вид зависимостей напряженности поля внутри области от управляющей функции и получить необходимые условия оптимальности для введенного функционала.

ОПТИМІЗАЦІЯ ДИНАМІКИ ПУЧКА В ЕЛЕКТРОСТАТИЧНОМУ ПОЛІ

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Сучасні вимоги до прискорювальної техніки, до параметрів прискорючого пучка заряджених частинок вимагають і нових підходів до розрахунку прискорючих і фокусуючих структур на стадії проектування. У даній роботі розглядається задача оптимізації динаміки пучка заряджених частинок в аксіально-симетричному електричному полі. Комплексний потенціал представляється у вигляді інтеграла типу Коші від функції, заданої на межі області і розглядається в якості керуючої функції. Використання комплексного уявлення дозволяє отримати явний вигляд залежностей напруженості поля в середині області від керуючої функції і отримати необхідні умови оптимальності для введеного функціоналу.