ON THE KRAMERS-KRONIG RELATION WITH ACCOUNT OF SPATIAL DISPERSION

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Following the ideas of D.A. Kirzhnitz [1], the $\vec{E},\vec{B},\vec{D}-$ representation of the electrodynamic Kramers-Kronig relation is used to study the longitudinal and transverse permittivity properties of response functions of the isotropic medium. Also, the $\vec{E},\vec{B},\vec{D},\vec{H}-$representation permits studying the properties of the magnetic permeability.

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INTRODUCTION

The electrodynamics without account of spatial dispersion makes use of the Kramers-Kronig relation formulated for the permittivity $\varepsilon(\omega)$. It is assumed that the permittivity is exactly the function representing the electrodynamic response to external disturbances, and hence should be analytical in the upper half-plane of the complex frequency plane. For the case of non-conducting media the relation was analyzed in detail by Landau and Lifshitz [2, para 82]. The real part of the Kramers-Kronig relation for non-conducting media yields

$$\mathrm{Re}\,\varepsilon(0)-1=\frac{2}{\pi}\int_0^\infty\frac{d\omega'}{\omega'}\mathrm{Im}\,\varepsilon(\omega')>0,\quad \mathrm{Re}\,\varepsilon(0)>1. \qquad (1)$$

In the case of conducting media the imaginary part of the $\varepsilon(\omega)$ for possesses a pole in the low frequency range, $\mathrm{Im}\,\varepsilon(\omega)=4\pi\sigma/\omega$, where $\sigma=const$ is the d.c. conductivity of the medium. According to [2], this pole can be taken into account through a simple substitution in Eq.(1), namely

$$\mathrm{Re}\,\varepsilon(0)-1=\frac{2}{\pi}\int_0^\infty\frac{d\omega'}{\omega'}(\mathrm{Im}\,\varepsilon(\omega')-\frac{4\pi\sigma}{\omega'})=-\beta_1^2<0,$$
$$\mathrm{Re}\,\varepsilon(0)=1-\beta_1^2 \qquad (2)$$

The value β_1^2 can be either greater or smaller than one, and therefore $\varepsilon(0)$ can assume both positive and negative values. It is this property that was stressed in paper [3] where a Lorentz approximation for the $\mathrm{Re}\,\varepsilon(0)$ of plasma-like media was first derived,

$$\mathrm{Re}\,\varepsilon(0)=1-\frac{\omega_p^2}{v^2}. \qquad (3)$$

Here $\omega_p=\sqrt{4\pi e^2n/m}$ is the plasma frequency for charge carriers (*n* is their number density; *e* is charge, and *m* mass), and *v* the inverse time of momentum relaxation. The electron gas of metals is always characterized by $\mathrm{Re}\,\varepsilon(0)<0$, while electrolites with heavy weight of carriers probably $\mathrm{Re}\,\varepsilon(0)>0$, i.e. the size $\mathrm{Re}\,\varepsilon(0)$ can be both positive, and negative.

A Kramers-Kronig relation for the magnetic permeability $\mu(\omega)$ is written in [2] in a quite similar way, with the assumption that

$$\mu(0)>\mu_1, \qquad (4)$$

where $\mu_1<1$ is the value of the magnetic permeability at a limiting frequency (in the optical frequency range) above which the permeability loses sense at all (see [2], para 79). The magnitude of $\mu(0)$ for paramagnetic media is $\mu(0)>1$, while for diamagnetic media $\mu(0)<1$, however the inequality Eq.(4) is always satisfied.

BASIC PART

D.A. Kirzhnitz noted in 1976 [1] that the actual response functions are not the dielectric and magnetic permittivities but rather the functions

$$F_1(\omega,k)=\frac{1}{\varepsilon^l(\omega,k)} \quad \text{and}$$
$$F_2(\omega,k)=\frac{1}{k^2c^2-\omega^2\varepsilon^{tr}(\omega,k)}, \qquad (5)$$

written in the $\vec{E},\vec{B},\vec{D}-$ representation with allowance for spatial dispersion of the medium [4]. Taking into account the known relationship of the $\vec{E},\vec{B},\vec{D}$ and $\vec{E},\vec{B},\vec{D},\vec{H}$ electrodynamics representations [4], *viz.* $\varepsilon^l(\omega,k)=\varepsilon(\omega,k)$ and

$$1-\frac{1}{\mu(\omega,k)}=\frac{\omega^2}{k^2c^2}\left\{\varepsilon^{tr}(\omega,k)-\varepsilon^l(\omega,k)\right\}, \qquad (6)$$

we are able to re-write the functions Eq.(5) in a form convenient for the analysis of magnetic media,

$$F_1(\omega,k)=\frac{1}{\varepsilon(\omega,k)} \quad \text{and}$$

$$F_2(\omega,k)=\frac{1}{k^2c^2/\mu(\omega,k)-\omega^2\varepsilon(\omega,k)}.$$

Specifically, it is the functions of Eq.(5) and Eq.(5,a) that are analytical in the upper half-plane of the complex frequency plane.

The longitudinal dielectric permittivity $\varepsilon^l(\omega,k)$ was given a detailed analysis in [1], proceeding from the Kramers-Kronig relation,

$$\mathrm{Re}\frac{1}{\varepsilon^l(0,k)}-1=$$
$$=\frac{2}{\pi}\int_0^\infty\frac{d\omega'}{\omega'}\mathrm{Im}\frac{1}{\varepsilon^l(\omega',k)}<0,\quad \mathrm{Re}\frac{1}{\varepsilon^l(0,k)}<1. \qquad (7)$$

A fact of significant importance, specially underlined in paper [1], is that the inequalities Eq.(7) permit existence of equilibrium media not only with positive, but also with negative values of $\varepsilon^l(0,k)$, (unlike Eq.(1)). According to Kirzhnitz [1], this provides an explanation to the existence of superconductors as equilibrium media.

It should be noted, however, that the inequalities Eq.(7) are valid, both for non-conducting and for conducting media, solely in the limit $\omega / k \to 0$. It is known [4] that the point $\omega = 0, k = 0$ is an essential singularity for the dielectric permittivity functions. At $\omega / k \to 0$ the function $\varepsilon^l(\omega,k)$ remains finite for non-conducting, as well for conducting media. Evidently, this point is essentially singular for the response functions Eq.(5), too. This is something to be taken in consideration in the analysis of the Kramers-Kronig relations for the response functions. In particular, in the limiting case $k/\omega \to 0$, where $\varepsilon^l(\omega,0) = \varepsilon^{tr}(\omega,0) = \varepsilon(\omega)$, ε^l behaves as $\varepsilon^l(\omega) \approx 4\pi i / \omega$ in the low frequency range. This pole needs being taken in consideration in the analysis of the Kramers-Kronig relations, exactly which will be done below.

3. First, we will analyze the response function $F_2(\omega,k)$ which involves the transverse dielectric permittivity. As far as we know, the Kramers-Kronig relation for this function was seriously discussed, with respect to the magnetic permeability $\mu(\omega,k)$, in the monograph [5] alone (earlier results of paper [6] were not sufficiently grounded, as noted in [1]). The results obtained in [5] can be expressed as the inequalities

$$\mathrm{Re}\frac{\mu(0,k)}{c^2k^2} = \frac{2}{\pi}\int_0^\infty \frac{d\omega'}{\omega'}\mathrm{Im}\frac{1}{\omega^2\varepsilon^l(\omega',k) - c^2k^2/\mu(\omega',k)} > 0. \quad \mathrm{Re}\,\mu(0,k) > 0 \tag{8}$$

The static magnetic permeability differs from unity only for magnetic (and hence, quantum-physical) media which the inequalities Eq.(8) relate to. Note now that in the case of magnetic media the low-frequency limit reads as

$$\varepsilon(\omega,k)) = 1 - \frac{\alpha}{\omega^2} + i\begin{cases} 4\pi\sigma(k)/\omega \\ \gamma(k)\omega \end{cases}, \tag{9}$$

The upper line in Eq.(9) refers to conducting magnetic media and the lower one to non-conducting. It follows from Eqs.(8) - (9) that

$$1 - \frac{1}{\mu(0,k)} = -\frac{\alpha}{k^2c^2}. \tag{10}$$

For paramagnetic materials (*i.e.* $\mu(0,k) > 1$) we have $\alpha < 0$, while for the diamagnetic (including superconductors), $\alpha > 0$.

Now we aim at writing a Kramers-Kronig relation for the function

$$\frac{k^2c^2}{\omega^2}\left\{\frac{1}{k^2c^2 - \omega^2\varepsilon^{tr}(\omega,k)} - \frac{1}{k^2c^2}\right\} = \frac{\varepsilon^{tr}(\omega,k)}{k^2c^2 - \omega^2\varepsilon^{tr}(\omega,k)}. \tag{11}$$

The real part of this relation, taken in the d.c. limit for a classic medium (characte-rized by a sole first order pole in $\varepsilon^{tr}(\omega,k)$), can be represented as

$$\frac{\mathrm{Re}\,\varepsilon^{tr}(0,k)k^2c^2 - (4\pi\sigma)^2}{c^4k^4} = = \frac{2}{\pi}\int_0^\infty \frac{d\omega'}{\omega'}\left\{\frac{k^2c^2\left[\mathrm{Im}\,\varepsilon^{tr}(\omega',k)\right]}{\left|k^2c^2 - \omega'^2\varepsilon^{tr}(\omega',k)\right|^2} - \frac{4\pi\sigma}{k^2c^2\omega'}\right\} = -\beta_2^2 < 0. \tag{12}$$

When deriving this relation, we have considered the low-frequency pole in the verse dielectric permittivity, $\mathrm{Im}\,\varepsilon^{tr}(\omega,k) \to 4\pi\sigma(k)/\omega$ at $\omega \to 0$, where σ(*k*) is the d.c. conductivity of the medium. As follows from Eq.(10),

$$\varepsilon^{tr}(0,k) = (4\pi\sigma / ck)^2 - c^2k^2\beta_2^2, \tag{13}$$

i.e. the transverse permittivity $\varepsilon^{tr}(0,k)$ for conductors can assume both positive and negative values. Interestingly enough, Eq.(13) is also valid in the limit $k/\omega \to 0$ when spatial dispersion can be neglected. Hence, it should be true for the longitudinal permittivity in this limit, too. As for non-conducting ($\sigma = 0$) media, Eq.(10) gives for the real part

$$\frac{\mathrm{Re}\,\varepsilon^{tr}(0,k)}{k^2c^2} = = \frac{2k^2c^2}{\pi}\int_0^\infty \frac{d\omega'\,\mathrm{Im}\,\varepsilon^{tr}(\omega',k)}{\omega'\left|k^2c^2 - \omega'^2\varepsilon^{tr}(\omega',k)\right|^2} > 0, \ \mathrm{Re}\,\varepsilon^{tr}(0,k) > 0, \tag{14}$$

so that the inequality $\varepsilon^{tr}(0,k) > 0$ always holds.

PRINCIPAL CONCLUSIONS

1. The Kramers-Kronig relations should be applied not to dielectric permittivities but rather to response functions of a medium,

$$F_1(\omega,k) = \frac{1}{\varepsilon^l(\omega,k)}$$

and $F_2(\omega,k) = \dfrac{1}{k^2c^2 - \omega^2\varepsilon^{tr}(\omega,k)}$.

2. For the longitudinal dielectric permittivity of non-conducting media (as well as of conducting media with spatial dispersion, *i.e.* with $\omega / k << 1$) the inequality Eq. (7) is valid, namely

$$\frac{1}{\mathrm{Re}\,\varepsilon^l(0,k)} < 1.$$

3. For the transverse permittivity of classical conducting media the relation Eq.(13) is true, which also remains in force for the longitudinal dielectric permittivity under the conditions allowing the spatial dispersion to be neglected,

$$\varepsilon^{tr}(0,k) = (4\pi\sigma / ck)^2 - c^2k^2\beta_2^2.$$

4. In the case of classical non-conducting media the inequality Eq.(14) is always valid,

$$\mathrm{Re}\,\varepsilon^{tr}(0,k) > 0.$$

5. The magnetic permeability obeys the inequality Eq.(8),

$$\mu(0,k) > 0.$$

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О СООТНОШЕНИЯХ КРАМЕРСА-КРОНИГА С УЧЕТОМ ПРОСТРАНСТВЕННОЙ ДИСПЕРСИИ

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В развитие идей Д.А. Киржница [1] с помощью соотношений Крамерса-Кронига в электродинамике в $\vec{E},\vec{B},\vec{D}$ -представлении исследуются свойства продольной и поперечной диэлектрических проницаемостей для функций электродинамического отклика изотропной среды. В электродинамике в $\vec{E},\vec{B},\vec{D},\vec{H}$ -представлении исследованы также свойства магнитной проницаемости среды.

ПРО СПІВВІДНОШЕННЯ КРАМЕРСА-КРОНІГА З УРАХУВАННЯМ ПРОСТОРОВОЇ ДИСПЕРСІЇ

В.П. Макаров, А.А. Рухадзе

У розвиток ідей Д.А. Киржниця [1] за допомогою співвідношень Крамерса-Кроніга в електродинаміці в $\vec{E},\vec{B},\vec{D}$ -уявленні досліджуються властивості повздовжньої та поперечної діелектричних проникностей для функцій електродинамічного відгуку ізотропного середовища. В електродинаміці в $\vec{E},\vec{B},\vec{D},\vec{H}$ -уявленні досліджені також властивості магнітної проникності середовища.