DISPERSION PROPERTIES OF ARBITRARY CROSS-SECTION PLASMA-FILLED WAVEGUIDE IN A FINITE MAGNETIC FIELD

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The study presents a universal approach to solving a boundary value problem for waveguide of arbitrary cross-section with longitudinally magnetized plasma filling. The approach utilizes representation of longitudinal fields by infinite scalar series. The dispersion equation of magnetized plasma-filled waveguide of arbitrary cross-section is obtained. As examples, numerical results for three types of plasma-filled waveguides (circular, coaxial and rectangular) are presented. Their dispersion properties, as well as convergence of numerical results, are investigated numerically. The validity of the results is verified by comparison with known data.

PACS: 52.35.Hr

INTRODUCTION

The use of plasma-filled waveguides for charged particle acceleration, generation of high-power electromagnetic radiation and transport of charged-particle beams requires knowledge of eigenfrequencies and eigenfields of such structures. Calculations of the eigenfrequency spectrum of plasma-filled waveguide with an arbitrary cross-section, embedded in an external axial finite magnetic field, are associated with significant mathematical and computational difficulties.

First, longitudinal components of the electric and magnetic fields are coupled due to plasma gyrotropy caused by magnetization [1]. Second, boundary conditions at the metal walls of the arbitrary cross-section gyrotropic waveguide restrict the use of the method of variable separation [2]. As a result this method disallows to determine eigenfrequencies and eigenfields for waveguides with arbitrary cross-sections filled with plasma embedded in finite external magnetic field. Up to now either axisymmetric magnetized plasma-filled waveguides (cylindrical [1, 3-5] and coaxial [6, 7]) or arbitrary cross-section waveguides filled with plasma embedded in infinite magnetic field [8] have been considered. The lack of generalization due to specific forms of waveguides or infinite value of external magnetic field motivates the present study.

Thus, the paper is devoted to electromagnetic analysis of longitudinally magnetized plasma-filled waveguide of arbitrary constant cross-section. The analysis utilizes a new universal approach to solving a waveguide problem. It is based on field expansion into series of eigenfunctions of well-known Neumann and Dirichlet problems for Helmholtz equation in a two-dimensional domain. The domain coincides with waveguide cross-section. The dispersion equation is obtained from the existence condition of non-zero solutions for the set of linear algebraic equations for the unknown expansion coefficients. It was numerically tested for different waveguide cross-sections. Details of the developed approach and some numerical examples are addressed below.

1. DESCRIPTION OF THE APPROACH

Let us consider metallic waveguide of arbitrary constant cross-section S. The waveguide is uniformly filled with magnetized plasma. The plasma is considered to be cold, collisionless; plasma ions are immobile.

We introduce unit vectors $\mathbf{s}$ and $\mathbf{n}$ with respect to cross-section contour C. Both unit vectors belong to the plane of cross-section. Vector $\mathbf{n}$ is the outward normal to C (i.e. directed into metal). Vector $\mathbf{s}$ is directed along the cross-section contour. Vectors $\mathbf{n}$, $\mathbf{s}$ and the waveguide axis z form a right-hand triple.

By assuming that the electromagnetic fields and the velocities of plasma electrons have the form $A(\mathbf{r},t)=A(\mathbf{r}_\perp)\exp(-i\omega t+ik_z z)$, we obtain

$$E_s=\frac{1}{\zeta^4}\left\{ik\chi^2\left[\frac{\partial B_z'}{\partial n}+ig\frac{\partial B_z'}{\partial s}\right]-ik_z\frac{\zeta^4}{\chi^2}\frac{\partial E_z}{\partial s}\right\}, \quad (1)$$

$$E_n=\frac{1}{\zeta^4}\left\{-ik\chi^2\left[\frac{\partial B_z'}{\partial s}-ig\frac{\partial B_z'}{\partial n}\right]-ik_z\frac{\zeta^4}{\chi^2}\frac{\partial E_z}{\partial n}\right\}, \quad (2)$$

where $B_z'=B_z+i\,k_z/k\,gE_z$, $k=\omega/c$, $\chi^2=k_z^2-\varepsilon_1k^2$, $g=\varepsilon_2k^2/\chi^2$, $\zeta^4=\chi^4\left(1-g^2\right)$, $\varepsilon_1=1-\omega_p^2/\left(\omega^2-\omega_H^2\right)$, $\varepsilon_2=-\omega_p^2\omega_H/\left(\omega(\omega^2-\omega_H^2)\right)$, $\varepsilon_3=1-\omega_p^2/\omega^2$, ω_p and ω_H are the Langmuir and cyclotron frequencies of plasma electrons, c is the speed of light in vacuum, E_z and B_z are the longitudinal field components related to each other by the equations

$$\left(\Delta_\perp+a\right)B_z'=bE_z, \quad (3)$$

$$\left(\Delta_\perp+c\right)E_z=dB_z'. \quad (4)$$

Here $\Delta_\perp$ is the transverse part of Laplace operator, $a=-\zeta^4/\chi^2$, $b=-i\varepsilon_2kk_z\zeta^4/\chi^4$, $d=i\varepsilon_2kk_z/\varepsilon_1$, $c=-\left(\varepsilon_3\chi^4+\varepsilon_2^2k^2k_z^2\right)/\left(\varepsilon_1\chi^2\right)$.

The equations (3) and (4) should be supplemented by the following boundary conditions:

$$E_z\big|_C=0,\ \partial B_z'/\partial n\big|_C=-ig\,\partial B_z'/\partial s\big|_C, \quad (5)$$

which reduce the tangential electric-field components at the waveguide wall to zero.

Let us seek the solution of the waveguide problem (3) - (5) in the form:

$$B_z'=\sum_k a_k\psi_k\left(\mathbf{r}_\perp\right)+\sum_k b_k\varphi_k\left(\mathbf{r}_\perp\right), \quad (6)$$

$$E_z=\sum_k c_k\varphi_k\left(\mathbf{r}_\perp\right), \quad (7)$$

where expansion coefficients a_k, b_k and c_k are unknown constants. Here we introduce two complete countable sets of functions

$$\left(\Delta_\perp + k_{1i}^2\right)\varphi_i = 0\,, \qquad \varphi_i\big|_C = 0\,, \tag{8}$$

$$\left(\Delta_\perp + k_{2i}^2\right)\psi_i = 0\,, \qquad \partial\psi_i/\partial n\big|_C = 0\,. \tag{9}$$

They are the eigenfunctions of the well-known self-adjoint Neumann and Dirichlet problems for Helmholtz equation in a two-dimensional domain. Whatever the contour shape is, these functions are orthogonal

$$\int\limits_S \varphi_i^*\varphi_k dS = k_{1i}^{-2}\delta_{ik}\,,\ \int\limits_S \psi_i^*\psi_k dS = k_{2i}^{-2}\delta_{ik}\,. \tag{10}$$

Multiply (3) and the complex conjugated (9) by ψ_i^* and B_z', respectively. Then subtract the results and integrate the obtained expression over S. Using the orthogonality condition (10), the second Green's formula and the boundary conditions on C, one finally comes to

$$\sum_k A_{ik}^{(1)}a_k + \sum_k B_{ik}^{(1)}b_k + \sum_k C_{ik}^{(1)}c_k = 0\,, \tag{11}$$

where $B_{ik}^{(1)} = (a - k_{2i}^2)\Phi_{ik}$, $C_{ik}^{(1)} = -b\Phi_{ik}$, $A_{ik}^{(1)} = \delta_{ki}\,(a - k_{2k}^2)/k_{2k}^2 - igT_{ik}$, $T_{ik} = \int_C ds\psi_i^*\,\partial\psi_k/\partial s$, $\Phi_{ik} \equiv \int\limits_S dS\psi_i^*\varphi_k = \left(k_{2i}^2 - k_{1k}^2\right)^{-1}\int\limits_C ds\psi_i^*\,\partial\varphi_k/\partial n$.

The same procedure applied to equations (3), (4) and to the complex conjugated equation (8) leads to the following:

$$\sum_k A_{ik}^{(2)}a_k + \sum_k B_{ik}^{(2)}b_k + \sum_k C_{ik}^{(2)}c_k = 0\,, \tag{12}$$

$$\sum_k A_{ik}^{(3)}a_k + \sum_k B_{ik}^{(3)}b_k + \sum_k C_{ik}^{(3)}c_k = 0\,, \tag{13}$$

where $A_{ik}^{(2)} = (a - k_{2k}^2)\Phi_{ki}$, $B_{ik}^{(2)} = \delta_{ki}\,(a - k_{1k}^2)/k_{1k}^2$, $C_{ik}^{(2)} = -\delta_{ki}\,b/k_{1k}^2$, $A_{ik}^{(3)} = d\Phi_{ki}$, $B_{ik}^{(3)} = \delta_{ki}\,d/k_{1k}^2$, $C_{ik}^{(3)} = -\delta_{ki}\left(c - k_{1k}^2\right)/k_{1k}^2$.

Set of linear homogeneous equations (11) - (13) have nontrivial solution only if the determinant of the set is equal to zero.

$$D\left(\omega, k_z\right) \equiv \det\begin{pmatrix} A_{ik}^{(1)} & B_{ik}^{(1)} & C_{ik}^{(1)} \\ A_{ik}^{(2)} & B_{ik}^{(2)} & C_{ik}^{(2)} \\ A_{ik}^{(3)} & B_{ik}^{(3)} & C_{ik}^{(3)} \end{pmatrix} = 0\,. \tag{14}$$

Equation (14) is the unified dispersion relation for magnetized plasma-filled waveguides of arbitrary cross-section. The shape of cross-section determines the eigenvalues of the problems (8), (9) and also their eigenfuctions presented in integrands of T_{ik} and Φ_{ik}.

Below we apply (14) to analysis of dispersion properties of cylindrical, coaxial and rectangular waveguides filled with magnetized plasma. Solutions of the boundary value problems (8), (9) in these cases have rather simple form (see, for example [9]).

2. NUMERICAL EXAMPLES

2.1. CYLINDRICAL WAVEGUIDE

Let us consider cylindrical waveguide of radius R_0 filled with magnetized plasma. For circular cross-section the eigenfunctions of the problems (8), (9) in polar coordinates $\mathbf{r}_\perp = \{r, \varphi\}$ have the form:

$$\psi_i = A_i J_l(k_{2i}r)\exp\left(il\varphi\right), \tag{15}$$

$$\varphi_i = B_i J_l(k_{1i}r)\exp\left(il\varphi\right), \tag{16}$$

where $k_{1i} = \mu_{l,i}/R_0$, $k_{2i} = \mu_{l,i}'/R_0$, $\mu_{l,i}$ and $\mu_{l,i}'$ are the zeros of Bessel function $J_l(x)$ and its derivative, constants A_i and B_i are determined by the conditions (10).

In axisymmetric case basis functions are characterized by the pair indexes (azimuthal l and radial i). Basis functions with different azimuthal indexes are orthogonal and total fields with different azimuthal indexes can be considered independently. So, above we assume (for simplicity) that the total indexes of basis functions coincide with their radial indexes.

Using (15), (16), we come to

$$T_{ik} = \frac{-2il}{\sqrt{\mu_{li}'^2 - l^2}\sqrt{\mu_{lk}'^2 - l^2}}\,, \tag{17}$$

$$\Phi_{ik} = \frac{2R_0^2}{\sqrt{\mu_{li}'^2 - l^2}\left(\mu_{li}'^2 - \mu_{lk}^2\right)}\,. \tag{18}$$

Substitution (17) and (18) into (14) gives the dispersion equation of cylindrical waveguide filled with magnetized plasma. This equation was solved numerically (Fig. 1,a). The results were compared with solutions of exact dispersion equation [3]. The convergence of numerical results turns out to be rather good. It was found that three terms ($N = 3$) of series (11) - (13) suffice to achieve good agreement with [3] both in frequencies and fields (Figs. 1,b,c) even for the highly gyrotropic plasma.

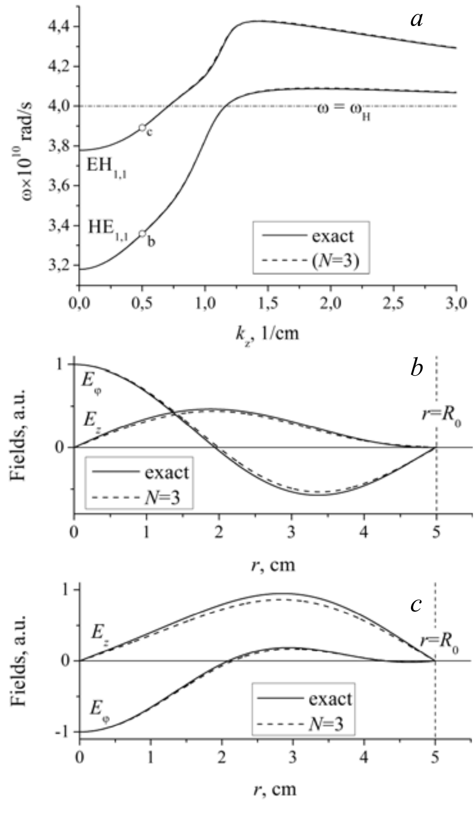

Fig. 1. Frequencies and fields of cylindrical waveguide filled with magnetized plasma. The fields correspond to the circles on dispersion curves ($\omega_p = 3\times10^{10}$ rad/sec, $\omega_H = 4\times10^{10}$ rad/sec, $R_0 = 5$ cm)

2.2. COAXIAL WAVEGUIDE

We now turn to consideration of coaxial waveguide with outer radius R_0 and inner radius R_i filled with magnetized plasma. For coaxial waveguide the boundary value problems (8), (9) have the following solutions

$$\psi_i \equiv A_i Z_l^{(2)}(k_{2i}r)\exp(il\varphi) = = A_i\left[J_l(k_{2i}r)+\alpha_i N_l(k_{2i}r)\right]\exp(il\varphi), \quad (19)$$

$$\varphi_i \equiv B_i Z_l^{(1)}(k_{1i}r)\exp(il\varphi) = = B_i\left[J_l(k_{1i}r)+\beta_i N_l(k_{1i}r)\right]\exp(il\varphi). \quad (20)$$

Here $N_l(x)$ is the Neumann function, $k_{1i}=\gamma_{l,i}/R_0$, $k_{2i}=\gamma'_{l,i}/R_0$, $\gamma_{l,i}$ and $\gamma'_{l,i}$ are the zeros of functions $Z_l^{(1)}(x/C_0)$ and $Z_l'^{(2)}(x/C_0)$, $C_0=R_0/R_i$, $\alpha_i=-J_l'(k_{2i}R_0)/N_l'(k_{2i}R_0)$, $\beta_i=-J_l(k_{1i}R_0)/N_l(k_{1i}R_0)$, constants A_i and B_i should be found from (10).

Using (19), (20), we obtain

$$T_{ik}=-2\pi i l A_i A_k\left\{Z_l^{(2)}(\gamma'_{li})Z_l^{(2)}(\gamma'_{lk})- -Z_l^{(2)}(\gamma'_{li}/C_0)Z_l^{(2)}(\gamma'_{lk}/C_0)\right\}, \quad (21)$$

$$\Phi_{ik}=\frac{2\pi R_0^2 A_i B_k}{\gamma_{li}'^2-\gamma_{lk}^2}\left\{\gamma_{lk}Z_l'^{(1)}(\gamma_{lk})Z_l^{(2)}(\gamma'_{li})- -\gamma_{lk}/C_0\, Z_l'^{(1)}(\gamma_{lk}/C_0)Z_l^{(2)}(\gamma'_{li}/C_0)\right\}. \quad (22)$$

The equation (14) in terms of (21) and (22) becomes the dispersion equation of coaxial waveguide filled with magnetized plasma. We solved this equation numerically (Fig. 2,a) for the same parameters as used in the case of cylindrical waveguide.

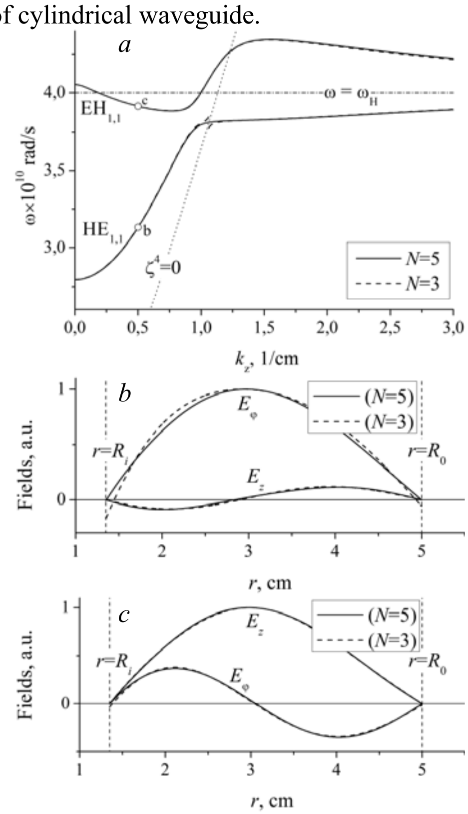

Fig. 2. Frequencies and fields of coaxial waveguide filled with magnetized plasma. The fields correspond to frequencies marked by circles ($\omega_p=3\times10^{10}$ rad/sec, $\omega_H=4\times10^{10}$ rad/sec, $R_0=5$ cm, $R_i=1.35$ cm)

Compared to this case, the number N should be increased to multiply precision of the field evaluation in the neighborhood of the waveguide boundary (Figs. 2,b,c). Moreover, when N is insufficiently high the dispersion curves can discontinue at $\zeta^4=0$ (see Fig. 2,a). As seen from (1), (2), when $\zeta^4=0$ field expressions are of indeterminate form 0/0 (otherwise fields would be nonregular). Thus, the number of terms in field expansion should be enough to avoid computational errors in this case.

2.3. RECTANGULAR WAVEGUIDE

We now proceed to study rectangular waveguide filled with magnetized plasma. The waveguide cross-section has the width a and the height b. For such a cross-section the eigenfunctions of the problems (8), (9) in rectangular coordinate system $\mathbf{r}_\perp=\{x,y\}$ have the form:

$$\psi_i\equiv\psi_{mn}=A_{mn}\cos(m\pi x/a)\cos(n\pi y/b), \quad (23)$$

$$\varphi_k\equiv\varphi_{m'n'}=B_{m'n'}\sin(m'\pi x/a)\sin(n'\pi y/b), \quad (24)$$

where m, n, m', n' are the integer numbers, $k_{1k}=\pi\sqrt{m'^2/a^2+n'^2/b^2}$, $k_{2i}=\pi\sqrt{m^2/a^2+n^2/b^2}$, A_{mn} and $B_{m'n'}$ are the normalization constants (see, (10)), $m',n'\neq0$.

Using (23), (24), we come to

$$T_{ik}\equiv T_{mn,m1n1}=A_{mn}A_{m1n1}\left(n^2m_1^2-n_1^2m^2\right)\times \times\frac{\left((-1)^{m_1-m}-1\right)}{m_1^2-m^2}\frac{\left((-1)^{n_1-n}-1\right)}{n_1^2-n^2}, \quad (25)$$

$$\Phi_{ik}\equiv\Phi_{mn,m'n'}=ab\pi^{-2}m'n'A_{mn}B_{m'n'}\times \times\frac{\left((-1)^{m'-m}-1\right)}{m'^2-m^2}\frac{\left((-1)^{n'-n}-1\right)}{n'^2-n^2}. \quad (26)$$

As evident from (25), (26), coupling coefficients $T_{mn,m'n'}$ and $\Phi_{mn,m'n'}$ differ from zero only if the differences $(m'-m)$ and $(n'-n)$ are odd simultaneously. Thus, one can take into consideration functions (23), (24) with even sum of indices (both indices are odd or even) and separately those which have odd sum of indices (one of the indices is odd and another is even).

Together with (25) and (26), the equation (14) becomes the dispersion equation of rectangular waveguide filled with magnetized plasma. This equation was solved numerically. The results are presented in Fig. 3. The figure also demonstrates applicability of our approach to analysis of plasma waves ($\omega_H<\omega_p$ in the present case) in addition to cyclotron waves presented in previous examples. We have also calculated the eigenfields of the rectangular plasma-filled waveguide. As seen from the Fig. 3, we have achieved rather good accuracy in field evaluation near the waveguide boundary with relatively small number of terms in field expansion.

Despite the obtained results, the electromagnetic properties of rectangular waveguide filled with magnetized plasma still remain almost unexplored and require further study. At present this study is in progress.

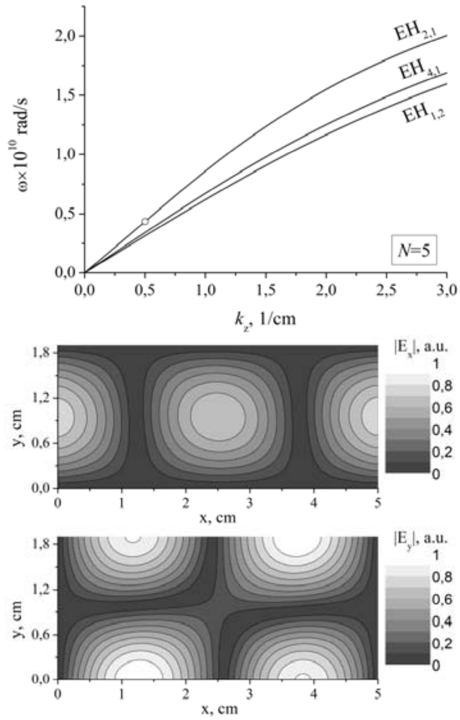

Fig. 3. Frequencies and fields of rectangular waveguide filled with magnetized plasma. The fields correspond to frequency marked by circle (ω_p=4×10^{10} rad/sec, $\omega_H = 3\times10^{10}$ rad/sec, $a = 5$ cm, $b = 1.9$ cm)

CONCLUSIONS

A unified dispersion equation for smooth metallic waveguides of arbitrary cross-section filled with longitudinally magnetized plasma is derived analytically. This equation is not solely of academic interest. It also can contribute to choice of optimal electrodynamic structure with gyrotropic filling for different applications. The dispersion equation has been successfully applied for three examples of such structure. The results were compared with known data, including those for vanishing plasma gyrotropy ($\omega_p \to 0, \omega_H \to 0,\ \omega_H \to \infty$). They show that our approach can be efficiently used in accurate evaluation of the wave frequencies and fields even in the case of essential gyrotropy of plasma filling.

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Article received 29.03.2013.

ДИСПЕРСИОННЫЕ СВОЙСТВА ПЛАЗМЕННОГО ВОЛНОВОДА ПРОИЗВОЛЬНОГО ПОПЕРЕЧНОГО СЕЧЕНИЯ В КОНЕЧНОМ МАГНИТНОМ ПОЛЕ

В.И. Щербинин, Г.И. Загинайлов, В.И. Ткаченко

Предложен новый универсальный метод решения задачи на собственные частоты и поля регулярного волновода произвольного сечения с магнитоактивной плазмой. Метод использует скалярные разложения для продольных полей волновода. Получено дисперсионное уравнение магнитоактивного плазменного волновода с произвольной формой поперечного сечения. Как пример, рассмотрены три типа (цилиндрический, коаксиальный и прямоугольный) плазменных волноводов. Их дисперсионные свойства, а также сходимость численных результатов исследованы численно. Достоверность результатов показана путем сравнения с известными данными.

ДИСПЕРСІЙНІ ВЛАСТИВОСТІ ПЛАЗМОВОГО ХВИЛЕВОДУ ІЗ ДОВІЛЬНИМ ПОПЕРЕЧНИМ ПЕРЕРІЗОМ У КІНЦЕВОМУ МАГНІТНОМУ ПОЛІ

В.І. Щербінін, Г.І. Загинайлов, В.І. Ткаченко

Було запропоновано новий універсальний метод розв'язання задачі на власні частоти та поля регулярного хвилеводу довільного перерізу із магнітоактивною плазмою. Метод використовує скалярні розкладення для поздовжніх полів хвилеводу. Було отримано дисперсійне рівняння магнітоактивного плазмового хвилеводу з довільною формою поперечного перерізу. Як приклад, було розглянуто три типа (циліндричний, коаксіальний та прямокутний) плазмових хвилеводів. Їхні дисперсійні властивості, а також збіжність чисельних результатів було досліджено чисельно. Достовірність результатів було продемонстровано шляхом порівняння із відомими даними.