THE SHEARING MODES APPROACH TO THE THEORY OF THE DIOCOTRON INSTABILITY OF THE CYLINDRICAL ELECTRON LAYER

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The temporal evolution of the linear diocotron instability of the cylindrical annular plasma column, which is driven by the shear of the equilibrium velocity of pure electron non-neutral plasma in crossed external magnetic and own electric fields, is investigated by using the extension of shearing modes methodology onto the cylindrical geometry. That approach does not use any spectral transforms in time and gives the solution of the initial value problems for any desired time. The evolution process leads toward the convergence to the phase-locking configuration of the mutually growing eigen and forced modes.
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1. BASIC EQUATIONS OF THE NON-MODAL APPROACH

The diocotron instability [1], is the electrostatic instability of the low-density non-neutral plasmas in magnetic field. It is driven by the shear of the equilibrium velocity of non-neutral plasma in crossed external magnetic and own electric fields. In recent years, the investigations of this instability are going far beyond traditional studies of plasma stability in Malmberg-Penning traps. The understanding the physics of this instability is important for the development of a new type of beam collimator system in high-energy colliders, which utilizes pulsed hollow electron beam to kick halo particles transversely while leaving the beam core unperturbed [2]. The diocotron instability is considered [3] as a promising mechanism leading to highly unstable flows in the pulsar inner magnetosphere.

In this paper we develop the theory of the diocotron instability of the cylindrical annular plasma column by extending the shearing modes methodology [4] onto cylindrical geometry. We consider the most simple model of the confined electron plasma as an infinitely long along the magnetic field hollow annulus with step-function electron density profile, which, nevertheless, requires the development of the shearing mode approach [4] to the rotating cylindrical plasma with a radially inhomogeneous angular velocity. The basic equation in that model is the drift-Poisson equation for the perturbed electrostatic potential ϕ

$$\left(\frac{\partial}{\partial t}+\Omega(r)\frac{\partial}{\partial\theta}\right)\nabla^2\phi(r,\theta,t)$$
$$=\frac{\omega_{pe}^2}{\omega_{ce}}\frac{\partial\phi}{\partial\theta}\left(\delta(r-b)-\delta(r-d)\right), \quad (1)$$

where the angular velocity $\Omega(r)$ is equal to

$$\Omega(r)=\frac{\omega_{pe}^2}{2\omega_{ce}}\left(1-\frac{b^2}{r^2}\right). \quad (2)$$

The boundary conditions for potential ϕ are the continuity of the potential across the edges $r=b$ and

$r=d$, i.e. $\phi(r=b-\varepsilon,\theta,t)=\phi(r=b+\varepsilon,\theta,t)$ with $\varepsilon\to 0$ and the same condition at $r=d$, the zero magnitude of the potential on the conducted boundary $r=R$ and the conditions on the jump of the $d\phi/dr$ at $r=b$ and $r=d$,

$$\frac{\partial}{\partial t}\left[\frac{\partial\phi}{\partial r}\bigg|_{r=b+\varepsilon}-\frac{\partial\phi}{\partial r}\bigg|_{r=b-\varepsilon}\right]=\frac{\omega_{pe}^2}{\omega_{ce}}\frac{1}{b}\frac{\partial\phi(b,\theta,t)}{\partial\theta}$$

$$\left(\frac{\partial}{\partial t}+\Omega(d)\frac{\partial}{\partial\theta}\right)\left[\frac{\partial\phi}{\partial r}\bigg|_{r=d+\varepsilon}-\frac{\partial\phi}{\partial r}\bigg|_{r=d-\varepsilon}\right] \quad (3)$$

$$=-\frac{\omega_{pe}^2}{\omega_{ce}}\frac{1}{d}\frac{\partial\phi(d,\theta,t)}{\partial\theta}.$$

We describe two areas: the electron layer, $b,, r,, d$, and vacuum in the rest of space. Eq. (1) in the vacuum has a form

$$\frac{\partial}{\partial t}\nabla^2\phi=0. \quad (4)$$

The solutions to Eq.(4) for the separate Fourier harmonics $\phi(r,l,t)$, determined as

$$\phi(r,\theta,t)=\sum_{l=-\infty}^{\infty}\phi(r,l,t)\exp(il\theta), \quad (5)$$

are

$$\phi(r,l,t)=C_1(l,t)r^l \quad\text{for}\quad 0<r<b,$$

$$\phi(r,l,t)=C_2(l,t)r^{-l}\left(1-\frac{r^{2l}}{R^{2l}}\right)\text{ for } d<r,, R. \quad (6)$$

In electron layer, the right hand side of Eq. (1) is equal to zero, except the edges at $r=b$, and $r=d$ i.e.

$$\left(\frac{\partial}{\partial t}+\Omega(r)\frac{\partial}{\partial\theta}\right)\nabla^2\phi(r,\theta,t)=0. \quad (7)$$

Instead of application of the commonly used spectral transform in time, here we use other approach, which gives easy and transparent treating of the problem considered. That approach is grounded on the transformation of Eq. (7) to the sheared coordinates $t=t$,

$$r=\hat{r},\ \ \theta=t\Omega(r)+\hat{\theta}, \quad (8)$$

where the sheared coordinate $\hat{\theta} = \theta - t\Omega(r)$ is the characteristic for Eq.(7). In these coordinates, we have $\partial/\partial t + \Omega(r)\partial/\partial\theta = \partial/\partial t$ and Eq.(7) is integrated easily over time. That gives for the Fourier harmonic $\phi(\hat{r}, l, t)$ of the potential, determined as $\phi(\hat{r}, \hat{\theta}, t) = \sum_{l=-\infty}^{\infty} \phi(\hat{r}, l, t) \exp(il\hat{\theta})$, the equation

$$\frac{\partial^2 \phi(\hat{r}, l, t)}{\partial \hat{r}^2} + \left(\frac{1}{\hat{r}} - 2ilt\frac{d\Omega(\hat{r})}{d\hat{r}}\right)\frac{\partial \phi(\hat{r}, l, t)}{\partial \hat{r}}$$
$$-\left[\frac{l^2}{\hat{r}^2} + ilt\frac{d^2\Omega(\hat{r})}{d\hat{r}^2} + ilt\frac{1}{\hat{r}}\frac{d\Omega(\hat{r})}{d\hat{r}}\right] \quad (9)$$
$$+l^2t^2\left(\frac{d\Omega(\hat{r})}{d\hat{r}}\right)^2\Bigg]\phi(\hat{r}, l, t) = 4\pi e n_1(\hat{r}, l, t = t_0)$$

and brings into the further consideration the initial perturbation $n_1(\hat{r}, l, t_0)$ of the electron density in electron layer. The general solution to Eq.(9) is obtained straightforwardly and is equal to

$$\phi(\hat{r}, \theta, t) = \sum_{l=1}^{\infty} e^{il(\hat{\theta} + t\Omega(\hat{r}))}[(C_3(l, t)$$
$$+\frac{2\pi e}{l}\int_b^{\hat{r}} d\hat{r}_1 \hat{r}_1^{1-l} n_1(\hat{r}_1, l, t_0) e^{-ilt\Omega(\hat{r}_1)})\hat{r}^l + \quad (10)$$
$$\left(C_4(l, t) + \frac{2\pi e}{l}\int_{\hat{r}}^d d\hat{r}_1 \hat{r}_1^{1+l} n_1(\hat{r}_1, l, t_0) e^{-ilt\Omega(\hat{r}_1)}\right)\hat{r}^{-l}\right].$$

That solution is valid for any time. It does not contain any singularities, which are inherent for the solutions obtained with spectral transforms in time and compose serious obstacles for the determining the explicit time dependence for the potential for the finite time.

2. MODAL DIOCOTRON INSTABILITY

If we suppose that any initial perturbation in layer is absent, i.e. $n_1(\hat{r}, l, t_0) = 0$, the solution (10) in layer $b \leq r \leq d$ reduces to a form

$$\phi(\hat{r}, \theta, t) = \sum_{l=1}^{\infty} e^{il(\hat{\theta} + t\Omega(\hat{r}))}\left(C_3(l, t)\hat{r}^l + C_4(l, t)\hat{r}^{-l}\right) \quad (11)$$

which describes only the surface waves, which form the discrete spectrum of perturbations. The condition of the perturbed potential continuity on the boundaries $r = b$ and $r = d$ couples the coefficients $C_1(l, t)$, $C_2(l, t)$ of Eq.(6) with $C_3(l, t)$, $C_4(l, t)$, and gives the following presentation for the potential in the vacuum regions:

$$\phi(r, \theta, t) = \sum_{l=1}^{\infty} e^{il\theta} r^l$$
$$\times (C_3(l, t) + C_4(l, t)b^{-2l}), \quad 0 < r < b, \quad (12)$$

$$\phi(r, \theta, t) = \sum_{l=1}^{\infty} e^{il\theta} r^{-l} \left(R^{2l} - r^{2l}\right)\left(R^{2l} - d^{2l}\right)^{-1}$$
$$\times (C_3(l, t)d^{2l} + C_4(l, t)), \quad d < r, R. \quad (13)$$

We apply the boundary conditions (3) to (12) - (13), and obtain the system of equations for $C_3(l, t)$ and $C_4(l, t)$, i.e.

$$\frac{\partial C_3(l, t)}{\partial t} + il\Omega(d)\left(C_3(l, t) + \frac{C_4(l, t)}{R^{2l}}\right)$$
$$= i\frac{\omega_{pe}^2}{2\omega_{ce}}\left[C_3(l, t)\left(1 - \frac{d^{2l}}{R^{2l}}\left(1 - \frac{b^{2l}}{d^{2l}}\right)\right) + \frac{C_4(l, t)}{d^{2l}}\right], \quad (14)$$
$$\frac{\partial C_4}{\partial t} = -i\frac{\omega_{pe}^2}{2\omega_{ce}}\left(C_3(l, t)b^{2l} + C_4(l, t)\right).$$

The solution to Eqs.(14) has a modal form,

$$C_3(l, t) = c_1(l)e^{-i\omega(l)t + \gamma(l)t} + c_2(l)e^{-i\omega(l)t - \gamma(l)t},$$
$$C_4(l, t) = c_1(l)a_1 e^{-i\omega(l)t + \gamma(l)t} + c_2(l)a_2 e^{-i\omega(l)t - \gamma(l)t}, \quad (15)$$
where

$$a_{1,2} = b^{2l}\left(\frac{2\omega_{ce}}{\omega_{pe}^2}(\omega(l) \pm i\gamma(l)) - 1\right)^{-1},$$

and

$$\omega(l) = \frac{\omega_{pe}^2}{4\omega_{ce}}\left[l\left(1 - \frac{b^2}{d^2}\right) + \frac{d^{2l}}{R^{2l}}\left(1 - \frac{b^{2l}}{d^{2l}}\right)\right], \quad (16)$$

$$\gamma(l) = \frac{\omega_{pe}^2}{4\omega_{ce}}\left\{4\frac{b^{2l}}{d^{2l}}\left[1 - l\left(1 - \frac{b^2}{d^2}\right)\frac{d^{2l}}{R^{2l}}\right]\right.$$
$$\left. - \left[2 - l\left(1 - \frac{b^2}{d^2}\right) - \frac{d^{2l}}{R^{2l}}\left(1 - \frac{b^{2l}}{d^{2l}}\right)\right]^2\right\}^{1/2}, \quad (17)$$

which define the known frequency and growth rate for diocotron instability in cylindrical annular plasma column [1] with conducted boundary. It follows from (17), that instability is absent for $l = 0$ and $l = 1$. and exists when

$$4\frac{b^{2l}}{d^{2l}}\left[1 - l\left(1 - \frac{b^2}{d^2}\right)\frac{d^{2l}}{R^{2l}}\right]$$
$$> \left[2 - l\left(1 - \frac{b^2}{d^2}\right) - \frac{d^{2l}}{R^{2l}}\left(1 - \frac{b^{2l}}{d^{2l}}\right)\right]^2. \quad (18)$$

3. MODAL DIOCOTRON INSTABILITY INTERPRETED IN TERMS OF EDGE WAVES INTERACTION

The application of the transformation to shearing coordinates (1) opens the way to effective analysis of the diocotron instability in terms of edge waves interaction [5], applied for the diocotron instability in plane geometry in Ref. [4]. Writing the functions $C_3(l, t)$ and $C_4(l, t)$ in the complex form [4],

$$C_3(l, t) = Q_3(l, t)e^{i\varepsilon_3(l, t)}, C_4(l, t) = Q_4(l, t)e^{i\varepsilon_4(l, t)}, (19)$$

the edge perturbation of the potential can be regarded as two edge waves with amplitudes $Q_3(l, t)$ and $Q_4(l, t)$ and phases $\varepsilon_3(l, t)$ and $\varepsilon_4(l, t)$. By substituting Eqs. (19) into Eqs. (14) and separating the real and imaginary parts at $r = b$ and $r = d$, we obtain, that amplitudes $Q_3(l, t)$ and $Q_4(l, t)$, and the relative phase

$\varepsilon = \varepsilon_3 - \varepsilon_4$ of the edge diocotron waves evolve according to equations

$$\frac{dQ_3}{dt} = \frac{\omega_{pe}^2}{2\omega_{ce}} d^{-2l} Q_4 \sin\varepsilon \left[1 - l\left(1 - \frac{b^2}{d^2}\right)\frac{d^{2l}}{R^{2l}} \right],$$

$$\frac{dQ_4}{dt} = \frac{\omega_{pe}^2}{2\omega_{ce}} b^{2l} Q_3 \sin\varepsilon, \tag{20}$$

and

$$\frac{d\varepsilon}{dt} = \Gamma\left(\cos\varepsilon + \beta(t)\right) \tag{21}$$

where

$$\Gamma = \frac{\omega_{pe}^2}{2\omega_{ce}} \left[\frac{Q_3}{Q_4} + \frac{Q_4}{Q_3}\frac{b^{2l}}{d^{2l}} \left[1 - l\left(1 - \frac{b^2}{d^2}\right)\frac{d^{2l}}{R^{2l}} \right] \right], \tag{22}$$

and

$$\beta(t) = \left(2 - l\left(1 - \frac{b^2}{d^2}\right) - \frac{d^{2l}}{R^{2l}}\left(1 - \frac{b^{2l}}{d^{2l}}\right) \right)$$
$$\times \left\{ \frac{Q_3}{Q_4} + \frac{Q_4}{Q_3}\frac{b^{2l}}{d^{2l}}\left[1 - l\left(1 - \frac{b^2}{d^2}\right)\frac{d^{2l}}{R^{2l}} \right] \right\}^{-1}, \tag{23}$$

From Eqs. (20) one can obtain the integral,

$$Q_3^2 = Q_4^2 \frac{b^{2l}}{d^{2l}}\left[1 - l\left(1 - \frac{b^2}{d^2}\right)\frac{d^{2l}}{R^{2l}} \right] + C. \tag{24}$$

Due to the exponential growth of amplitudes Q_3, Q_4 with time from infinitesimal beginnings, the amplitudes become

$$Q_3^2 \approx Q_4^2 \frac{b^{2l}}{d^{2l}}\left[1 - l\left(1 - \frac{b^2}{d^2}\right)\frac{d^{2l}}{R^{2l}} \right] ? \ C; \tag{25}$$

then $\beta(t)$ and Γ approaches the values

$$\beta_0 = \left(1 - \frac{l}{2}\left(1 - \frac{b^2}{d^2}\right) - \frac{d^{2l}}{2R^{2l}}\left(1 - \frac{b^{2l}}{d^{2l}}\right) \right)$$
$$\times \left(\frac{b}{d} \right)^{-l}\left[1 - l\left(1 - \frac{b^2}{d^2}\right)\frac{d^{2l}}{R^{2l}} \right]^{-1/2}, \tag{26}$$

$$\Gamma_0 = \frac{\omega_{pe}^2}{\omega_{ce}}\left(\frac{b}{d} \right)^l\left[1 - l\left(1 - \frac{b^2}{d^2}\right)\frac{d^{2l}}{R^{2l}} \right]^{1/2}. $$

At condition (18), under which the diocotron instability develops, β_0 is less than unity and therefore, the stationary (or fixed) points of the equation (21), where $d\varepsilon/dt = 0$, exist and are determined by the equation $\cos\varepsilon + \beta_0 = 0$. The solutions of this equation are two sets of stationary points: stable (or attractors) at

$$\varepsilon_k = \left(\pi - \cos^{-1}\beta_0\right) + 2k\pi, \tag{27}$$

and unstable at

$$\varepsilon_k = -\left(\pi - \cos^{-1}\beta_0\right) + 2k\pi. \tag{28}$$

The solution of the equation $d\varepsilon/dt = \Gamma_0\left(\cos\varepsilon + \beta_0\right)$ with initial condition $\varepsilon = \varepsilon_0$ at $t = t_0 = 0$, has a simple form

$$\tan\frac{\varepsilon}{2} = -\sqrt{\frac{1+\beta_0}{1-\beta_0}}\left(\frac{1 + Ae^{\Gamma_0 t\sqrt{1-\beta_0^2}}}{1 - Ae^{\Gamma_0 t\sqrt{1-\beta_0^2}}} \right), \tag{29}$$

where

$$A = \frac{(1-\beta_0)\tan\frac{\varepsilon_0}{2} + \sqrt{1-\beta_0^2}}{(1-\beta_0)\tan\frac{\varepsilon_0}{2} - \sqrt{1-\beta_0^2}}. \tag{30}$$

As it follows from Eq.(29), the initial perturbations with an arbitrary value of the initial phase of each wave, will evolve with time to the ultimate value ε_* of relative phase,

$$\cos\varepsilon_* = \beta_0, \tag{31}$$

which does not depend on the initial data.

This solution of the initial value problem reveals the linear stage of the instability development as a process of the formation of the phase locked configuration. Figure illustrates such configurations for separate modes $l = 5$ with relative phase ε_*, determined by Eq.(31). On Figure, we use the values of the parameter b/d, for which the growth rate $\gamma(l)$ attains the maximal values. These values are $b/d = 0.83$ with $\varepsilon_* = 2.17$ rad for $l = 5$. The time of the developing of such configuration is comparable with the inverse growth rate time of the diocotron instability.
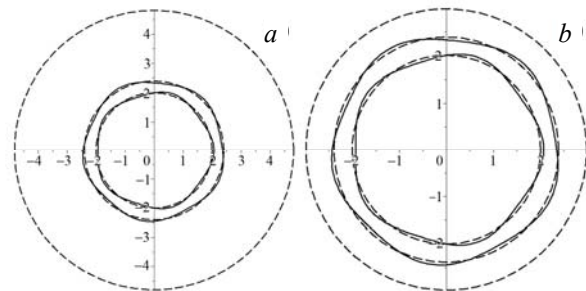
Phase-locked configuration for azimuthal wave number $l = 5$ (a) with $b/d = 0.838$, $d/R = 0.8$, $\varepsilon_* = 2.136$ rad, and (b) with $b/d = 0.8316$, $d/R = 0.5$, $\varepsilon_* = 2.352$ rad

4. NON-MODAL ANALYSIS OF THE DIOCOTRON INSTABILITY

Now we obtain the complete solution of the boundary and initial value problems, determined by the condition of the continuity of the potential ϕ and by Eqs. (3) with accounting for the initial perturbation of the electron density. The condition of the potential continuity at the inner and outer surfaces of the electron cylinder gives the connection formulae for the functions $C_1(l,t)$, $C_2(l,t)$ and $C_3(l,t)$, $C_3(l,t)$, and as a result, presentation of solutions (6) through the functions $C_3(l,t)$ and $C_4(l,t)$ of the solution (11). We obtain for the vacuum region, $0 < r < b$,

$$\phi(\hat{r},\hat{\theta},t) = \sum_{l=1}^{\infty} e^{il\left(\hat{\theta}+t\Omega(\hat{r})\right)} \hat{r}^l \left(C_3(l,t) + C_4(l,t)b^{-2l} \right.$$
$$\left. + \frac{2\pi e}{l}b^{-2l}\int_b^d d\hat{r}_1 \hat{r}_1^{1+l} n_1(\hat{r}_1,l,t_0)e^{-ilt\Omega(\hat{r}_1)} \right),$$

and for region $r > d$

$$\phi\left(\hat{r},\hat{\theta},t\right)=\sum_{l=1}^{\infty}e^{il\left(\hat{\theta}+t\Omega(\hat{r})\right)}\hat{r}^{-l}\left(C_3\left(l,t\right)d^{2l}+C_4\left(l,t\right)\right.$$
$$\left.+\frac{2\pi e}{l}d^{2l}\int_b^d d\hat{r}_1\hat{r}_1^{1-l}n_1\left(\hat{r}_1,l,t_0\right)e^{-ilt\Omega(\hat{r}_1)}\right).$$

The application of the conditions (3) to the above solutions gives the inhomogeneous equations for $C_3\left(l,t\right)$ and $C_4\left(l,t\right)$,

$$\frac{\partial C_3\left(l,t\right)}{\partial t}+il\Omega(d)\left(C_3\left(l,t\right)+\frac{C_4\left(l,t\right)}{R^{2l}}\right)$$
$$=i\frac{\omega_{pe}^2}{2\omega_{ce}}\left[C_3\left(l,t\right)\left(1-\frac{d^{2l}}{R^{2l}}\left(1-\frac{b^{2l}}{d^{2l}}\right)\right)+\frac{C_4\left(l,t\right)}{d^{2l}}\right]+f_1\left(l,t\right),$$
$$\frac{\partial C_4}{\partial t}=-i\frac{\omega_{pe}^2}{2\omega_{ce}}\left(C_3\left(l,t\right)b^{2l}+C_4\left(l,t\right)\right)+f_2\left(l,t\right), \quad (32)$$

where functions $f_{1,2}$, determines the effect of the initial perturbations of the electron density introduced by solution (10) into the boundary conditions (3), and are equal to

$$f_1\left(l,t\right)=i\frac{\omega_{pe}^2}{2\omega_{ce}}\frac{2\pi e}{l}\int_b^d d\hat{r}_1 n_1\left(\hat{r}_1,l,t_0\right)e^{-ilt\Omega(\hat{r}_1)}$$
$$\times\left\{\hat{r}_1^{1-l}\left[1-\frac{d^{2l}}{R^{2l}}-l\left(\frac{b^2}{\hat{r}_1^2}-\frac{b^2}{d^2}\right)\right]\right. \quad (33)$$
$$\left.+\hat{r}_1^{1+l}\left[1+l\left(1-\frac{b^2}{\hat{r}_1^2}\right)\right]\right\},$$

$$f_2\left(l,t\right)=i\frac{\omega_{pe}^2}{2\omega_{ce}}\frac{2\pi e}{l}\int_b^d d\hat{r}_1\hat{r}_1^{1+l}n_1\left(\hat{r}_1,l,t_0\right)$$
$$\times e^{-ilt\Omega(\hat{r}_1)}\left(1+l\left(1-\frac{b^2}{\hat{r}_1^2}\right)\right). \quad (34)$$

The system (32) - (34) compose second initial value problem in the investigation of the stability cylindrical annular plasma column, the solution of which gives complete linear description of the temporal evolution of the diocotron instability. The solutions to system (32) for $C_3\left(l,t\right)$ and $C_4\left(l,t\right)$ with $l\geq 2$ are obtained straightforwardly and are given by

$$C_3\left(l,t\right)=c_1\left(l\right)e^{-i\omega(l)t+\gamma(l)t}$$
$$+c_2\left(l\right)e^{-i\omega(l)t-\gamma(l)t}+\hat{C}_3\left(l,t\right), \quad (35)$$

$$C_4\left(l,t\right)=c_1\left(l\right)a_1 e^{-i\omega(l)t+\gamma(l)t}$$
$$+c_2\left(l\right)a_2 e^{-i\omega(l)t-\gamma(l)t}+\hat{C}_4\left(l,t\right), \quad (36)$$

where

$$a_{1,2}=\left(d^{2l}1\pm i\frac{2\omega_{ce}}{\omega_{pe}^2}\gamma\left(l\right)\right)\left(l\frac{d^{2l}}{R^{2l}}\left(1-\frac{b^2}{d^2}\right)-1\right)^{-1}.$$

The first two terms in Eqs. (35), (36) describe the modal temporal evolution with growth rate $\gamma\left(l\right)$ (16), of the initial perturbations of the electrostatic potential on the boundary surfaces at $r=b$ and $r=d$, which are determined by constants c_1 and c_2. The functions $\hat{C}_3\left(l,t\right)$ and $\hat{C}_4\left(l,t\right)$ are

$$\hat{C}_3\left(l,t\right)=-\frac{i\omega_{pe}^2 d^{-2l}}{4\omega_{ce}\gamma\left(l\right)}\left(1-l\frac{d^{2l}}{R^{2l}}\left(1-\frac{b^2}{d^2}\right)\right)$$
$$\times\left\{\int_{t_0}^t dt_1 e^{-i\omega(l)\left(t-t_1\right)}\left[\left(f_2\left(l,t_1\right)-a_1 f_1\left(l,t_1\right)\right)e^{-\gamma(l)\left(t-t_1\right)}\right.\right. \quad (37)$$
$$\left.\left.+\left(a_2 f_1\left(l,t_1\right)-f_2\left(l,t_1\right)\right)e^{\gamma(l)\left(t-t_1\right)}\right],$$

$$\hat{C}_4\left(l,t\right)=-\frac{i\omega_{pe}^2 d^{-2l}}{4\omega_{ce}\gamma\left(l\right)}\left(1-l\frac{d^{2l}}{R^{2l}}\left(1-\frac{b^2}{d^2}\right)\right)$$
$$\times\left\{\int_{t_0}^t dt_1 e^{-i\omega(l)\left(t-t_1\right)}\left[a_2\left(f_2\left(l,t_1\right)-a_1 f_1\left(l,t_1\right)\right)e^{-\gamma(l)\left(t-t_1\right)}\right.\right.$$
$$\left.\left.+a_1\left(a_2 f_1\left(l,t_1\right)-f_2\left(l,t_1\right)\right)e^{\gamma(l)\left(t-t_1\right)}\right]\right\}. \quad (38)$$

The functions $\hat{C}_3\left(l,t\right)$, $\hat{C}_4\left(l,t\right)$ for any values of l introduce the non-modal modification of the modal evolution, that arises from the initial perturbations of the electron density, which are sheared due to the rotation of electron column with inhomogeneous angular velocity $\Omega(\hat{r})$. Now, the solution for the electrostatic potential in region $b<r<d$ may be presented in a simple form,

$$\phi\left(\hat{r},\hat{\theta},t\right)=\phi_{(0)}\left(\hat{r},\hat{\theta},t\right)+\phi_{(1)}\left(\hat{r},\hat{\theta},t\right)+\phi_{(2)}\left(\hat{r},\hat{\theta},t\right). (39)$$

Here $\phi_{(0)}\left(\hat{r},\hat{\theta},t\right)$ is determined by a general solution of the homogeneous system (32),

$$\phi_{(0)}\left(\hat{r},\hat{\theta},t\right)=\sum_{l=2}^{\infty}e^{-i\omega(l)t+il\left(\hat{\theta}+t\Omega(\hat{r})\right)}$$
$$\times\left[\left(c_1\left(l\right)e^{\gamma(l)t}+c_2\left(l\right)e^{-\gamma(l)t}\right)\hat{r}^l\right.$$
$$\left.+\left(a_1 c_1\left(l\right)e^{\gamma(l)t}+a_2 c_2\left(l\right)e^{-\gamma(l)t}\right)\hat{r}^{-l}\right], \quad (40)$$

where the Fourier harmonic with $l=1$, which is stable in the geometry considered, is omitted,

$$\phi_{(1)}\left(\hat{r},\hat{\theta},t\right)=\sum_{l=1}^{\infty}e^{il\left(\hat{\theta}+t\Omega(\hat{r})\right)}\left(\hat{C}_3\left(l,t\right)\hat{r}^l+\hat{C}_4\left(l,t\right)\hat{r}^{-l}\right), \quad (41)$$

$$\phi_{(2)}\left(\hat{r},\hat{\theta},t\right)=\sum_{l=1}^{\infty}e^{il\left(\hat{\theta}+t\Omega(\hat{r})\right)}\frac{2\pi e}{l}$$
$$\times\left(\int_b^{\hat{r}}d\hat{r}_1\hat{r}_1\left(\frac{\hat{r}}{\hat{r}_1}\right)^l n_1\left(\hat{r}_1,l,t_0\right)e^{-ilt\Omega(\hat{r}_1)}\right.$$
$$\left.+\int_{\hat{r}}^d d\hat{r}_1\hat{r}_1\left(\frac{\hat{r}_1}{\hat{r}}\right)^l n_1\left(\hat{r}_1,l,t_0\right)e^{-ilt\Omega(\hat{r}_1)}\right), \quad (42)$$

is formed by the initial perturbations in Eq. (9). The exact solution (41), (42) are valid for any time, at which the linear theory of the diocotron instability is valid. The integration of $\phi_{(2)}$ on time by parts displays the decay of $\phi_{(2)}\left(r,\theta,t\right)$ as t^{-1} for $t ? \left(l\Omega(d)\right)^{-1}$. The obtained asymptotics reveals that the origin of this non-modal time dependence, which is attributed usually to the continuous spectrum, is the non-modal effect of the continuous shearing of the initial disturbance of the electron density determined by $e^{-ilt\Omega(\hat{r}_1)}$ function in Eq. (42). For the better understanding the contents of

Eqs. (37), (38) it is instructive to obtain the large time, $t\,?\,\left(l\Omega(d)\right)^{-1}$, asymptotics for coefficients $\hat{C}_3(l,t)$ and $\hat{C}_4(l,t)$. The integration of (37) on time, in which only the exponentially growing terms are retained, yields $\hat{C}_3(l,t)=\hat{C}_{31}(l,t)+\hat{C}_{32}(l,t)$, where

$$
\hat{C}_{31}(l,t)=\frac{i\pi e\omega_{pe}^4 d^{-2l}}{4l\omega_{ce}^2\gamma(l)}e^{-(i\omega(l)-\gamma(l))(t-t_0)}
$$

$$
\times\int_b^d d\hat{r}_1\,\frac{n_1(\hat{r}_1,l,t_0)e^{-ilt_0\Omega(\hat{r}_1)}}{\omega(l)-l\Omega(\hat{r}_1)+i\gamma(l)}
$$

$$
\times\left[a_2\hat{r}_1^{1-l}\left(1-\frac{d^{2l}}{R^{2l}}-l\left(\frac{b^2}{\hat{r}_1^2}-\frac{b^2}{d^2}\right)\right)\right. \tag{43}
$$

$$
\left.+\hat{r}_1^{1+l}\left(1+l\left(1-\frac{b^2}{\hat{r}_1^2}\right)\right)\left(1+\frac{a_2}{R^{2l}}\right)\right]
$$

$$
\hat{C}_{32}(l,t)=\frac{\pi ed^{-2(l-1)}}{l^2 t}\frac{\omega_{pe}^2}{4\omega_{ce}\gamma(l)}\left\{n_1(d,l,t_0)e^{-ilt\Omega(d)}\frac{d^2}{b^2}\right.
$$

$$
\times\frac{\left[a_2 d^{-l}\left(1-\frac{d^{2l}}{R^{2l}}\right)+d^l\left(1+l\left(1-\frac{b^2}{d^2}\right)\right)\left(1+\frac{a_2}{R^{2l}}\right)\right]}{\omega(l)-l\Omega(d)+i\gamma(l)}
$$

$$
-\frac{n_1(b,l,t_0)}{\omega(l)+i\gamma(l)}\frac{b^2}{d^2}\left[a_2 b^{-l}\left(1-\frac{d^{2l}}{R^{2l}}-l\left(1-\frac{b^2}{d^2}\right)\right)\right.
$$

$$
\left.\left.-b^l\left(1+\frac{a_2}{R^{2l}}\right)\right\}+O\left(t^{-2}\right). \tag{44}
$$

The same asymptotic is for

$$
\hat{C}_4(l,t)=a_1\hat{C}_3(l,t) \tag{45}
$$

experience the power-law decay with time (as t^{-1} in the non-modal parts of the perturbed potential (39) $\phi_{(2)}(r,\theta,t)$ and $\hat{C}_{32}(l,t)$). Only the exponentially growing with time function $\hat{C}_{31}(l,t)$, presented by Eq. (43), survives. Changing $\hat{\theta}+t\Omega(\hat{r})=\theta$ in Eq. (43)

with $\hat{C}_{31}(l,t)$ determined by Eq. (43), we obtain the observed in the laboratory frame the modal presentation for $\phi_{(1)}(r,\theta,t)$ as for the normal unstable diocotron wave. It follows from the last expression, that the relative phase difference ε of the edge surface waves, determines as $a_1=|a_1|e^{i\varepsilon}$ becomes constant and is equal to that corresponds to the formation of the phase locked state for the edge surface waves. This result reveals, that the accounting for the initial perturbations of the electron density does not destroy of the phase locked configuration. The performed analysis displays, that in spite of the similar spatial and temporal dependencies with normal unstable diocotron mode, solution actually is not a normal mode. As a solution of the inhomogeneous system it is a *forced* wave, which is resulted from the interaction of separate spatial mode of the electrostatic potential, formed by the rotating initial perturbation of the electron density, with the unstable modal diocotron wave.

REFERENCES

1. R.C. Davidson, Hei-Wai Chan, Chiping Chen, and S. Lund // *Rev. Modern Physics. (63)*. 1991, p. 341.
2. G. Stancari, A. Valishev, G. Annala, G. Kuznetsov, V. Shiltsev, D.A. Still, and L.G. Vorobiev // *Phys. Rev. Lett. (107)*. 2011, p. 084802.
3. J. Petry, Journal Astr. and Astrophys, *(503)*. 2009, 1. V.V. Mikhailenko, Hae June Lee, V.S. Mikhailenko // *Problems of Atomic Science and Technology. Series «Phys. Plasmas» (19)*. 2012, p. 082112.
4. N.A. Bakas, P.J. Ioannou // *Physics of Fluids (21)*. 2009, p. 024102.

ПОДХОД СДВИГОВЫХ МОД К ТЕОРИИ ДИОКОТРОННОЙ НЕУСТОЙЧИВОСТИ ЦИЛИНДРИЧЕСКОГО СЛОЯ ЭЛЕКТРОНОВ

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Временная линейная эволюция диокотронной неустойчивости цилиндрического слоя электронов, которая возбуждается широм равновесной скорости электронов в скрещенных внешнем магнитном и собственном электрическом полях, исследуется используя обобщение методологии сдвиговых мод на цилиндрическую геометрию. Этот подход не использует спектральное преобразование по времени и дает решение начальной задачи для любого времени. Эволюционный процесс приводит к образованию конфигурации с фазовой синхронизацией взаимно растущих собственных и вынужденных мод.

ПІДХІД ЗСУВНИХ МОД ДО ТЕОРІЇ ДІОКОТРОННОЇ НЕСТІЙКОСТІ ЦІЛІНДРИЧНОГО ШАРУ ЕЛЕКТРОНІВ

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Часова лінійна еволюція діокотронної нестійкості циліндричного шару електронів, яка збуджується широм рівноважної швидкості електронів у схрещених зовнішньому магнітному та власному електричному полях, досліджується використовуючи узагальнення методології зсувних мод на циліндричну геометрію. Цей підхід не використовує спектральне перетворення по часовій змінній і дає розв'язок задачі на початкові дані для любого часу. Еволюційний процес веде до утворення конфігурації з фазовою синхронізацією взаємно зростаючих власних та вимушених мод.