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Some Considerations on Failure of Solids and Liquids

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О некоторых общих закономерностях разрушения твердых и жидких тел

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Процесс разрушения как твердых, так и жидких тел в сплошной среде можно рассматривать как единое физическое явление. Математически описать его можно при выполнении некоторого предельного условия, в котором обычно учитывается напряженное состояние материала. Анализ классических теорий разрушения твердых тел (с точки зрения пластичности и механики разрушения) и современной теории разрушения жидких тел (явление кавитации с учетом максимального значения главного напряжения) позволил обнаружить общие закономерности в условиях их разрушения. В рамках общего подхода к разрушению тел предложено использовать для описания предельного условия разрушения такой параметр поля напряжений как трехосность напряженного состояния. Показано, что функции коллапса для твердых и жидких тел в сплошной среде имеют подобный вид.

Ключевые слова: разрушение, излом, кавитация, трехосность напряженного состояния, твердые тела, жидкости.

Nomenclature

a – characteristic dimension of a defect

B – material's mechanical parameter

$\mathbf{D}[\mathbf{u}]$ – symmetric part of the gradient of the velocity field $\mathbf{u}$

E, G' – Young's and shear moduli, respectively

g – acceleration of gravity

$G, G_{\mathrm{I}c}$ – fracture energy and critical fracture energy under Mode I, respectively

I_1 – first invariant of the stress tensor

J_2 – second invariant of the deviatoric stress tensor

K_{I} – stress intensity factor (SIF)

$K_{\mathrm{I}c}$ – fracture toughness (plane strain condition)

p – hydrostatic pressure in a liquid under static condition

$p_c(T)$ – liquid's cavitation threshold pressure

$\mathbf{S}$, s_{ij}	–	deviatoric stress tensor
S_{ij}	–	principal stresses of the deviatoric stress tensor $\mathbf{S}$ $(i,\ j = 1,\ 2,\ 3)$
t, T	–	stress triaxiality and temperature, respectively
$\mathbf{u} = \{u \quad v \quad w\}$	–	velocity field in the x_1, x_2, x_3 coordinate system
Y	–	dimensionless SIF or geometric correction factor
μ	–	liquid dynamic viscosity
κ	–	hydrostatic stress component of the stress tensor
α, β	–	parameter's of the failure criterion adopted for solids
ρ	–	mass density
σ_s, τ_s	–	normal and shear stress, respectively
σ	–	stress tensor
σ_i	–	principal stresses of the stress tensor $(i = 1,\ 2,\ 3)$
σ_0	–	remote applied stress
$\tau[\mathbf{u}]$	–	shear stress tensor in a moving liquid

Introduction. Failure in continuum media, solids or liquids, can be regarded as a single physical phenomenon which is determined by the fulfilment of a limit condition which generally involves the stress field in the medium under study. Failure conditions can be easily and conveniently written in terms of the stress invariants and the deviatoric stress invariants, and this approach is used in the following.

By considering the classical theory of plasticity or fracture mechanics concepts in the context of failure of solids and a recent theory on the cavitation phenomenon in the context of liquids, a unified discussion is herein presented, showing as the collapse functions present a similar form for both classes of continuum materials.

Furthermore, by introducing a stress field parameter (the stress triaxiality), the limit conditions of collapse for both classes of materials can be simply represented through the same equation by only setting the involved material's parameters appropriately. The values assumed by such parameters can give us useful information about the kind of expected failure.

Local collapse of continuum media, for both solids (mainly brittle solids) and liquids, frequently appears as a loss of continuity of the material. The above phenomenon occurs through the formation of cracks or high strain deformation due to plastic flow in solids, or as the appearance of voids or bubbles (due to the so-called cavitation phenomenon) in liquids.

Several studies have been carried out on collapse of solids. The classical plasticity has been described by a well established theory [1, 2] (up to recent studies on crystal plasticity [3, 4]), as well as fracture mechanics that dates back to the pioneering researches carried out by Griffith [5, 6] and to the rigorous mathematical approach performed by Westergaard [7], up to subsequent works by Liebowitz [8] and Sih [9] and the recent studies of fracture at microscopic and nanoscale level [10, 11]. Liquid collapse, also known as liquid cavitation, has also been studied by several authors [12–18]. By considering a stress-based approach to the mentioned failure phenomena in solids and liquids [4, 12], similar governing equations can be found.

1. **Failure in Solids: Plastic Collapse and Fracture Collapse**. Failure in solids can be regarded as an irreversible phenomenon which involves a great rearrangement of the solid structure accompanied by energy dissipation. Such an irreversible modification of the material structure can appear as a plastic flow due to the sliding – at the microscopic scale – of the lattice structure of the material, characterized by high plastic strain level (typical of the so-called ductile materials) or as a loss of continuity, due to a large strain localization (as usually occurs in brittle materials).

Fracture in solids is a common phenomenon which usually takes place due to the growth and coalescence of microcracks or voids, produced by the stressed state in the material, up to the formation of a macrodefect which can be mathematically represented as a strain localization in a narrow band which identifies the fracture position.

Typical approaches to the fracture phenomenon are the well-known SIF criterion {descending from the classical Westergaard solution in linear elastic fracture mechanics (LEFM) [7]} and the energy criterion introduced by Griffith [5, 6].

The critical SIF criterion for catastrophic fracture under pure Mode I can be written as follows:

$$K_{\mathrm{I}} = Y\sigma_0\sqrt{\pi a} \geq K_{\mathrm{I}c}, \tag{1}$$

where Y is the so-called geometric correction factor (or dimensionless SIF), σ_0 is the applied remote stress, a is a characteristic size of the existing crack, and $K_{\mathrm{I}c}$ is the fracture toughness of the material under given environmental conditions.

As mentioned above, the fracture phenomenon can also be analyzed through an energy-based approach. For example, by considering a tensile loading of infinite plate containing a straight crack with length $2a$ [in such a case: $Y=1$ in Eq. (1)], the energy criterion for collapse states that [5, 6]:

$$G = \pi a\sigma_0^2 / E \geq G_{\mathrm{I}c}, \tag{2}$$

where G is the fracture energy, i.e., the energy per unit crack surface, available (in the mechanical system) for crack extension.

The above two approaches to fracture can be linked by the classical relationships:

$$G_{\mathrm{I}c} = K_{\mathrm{I}c}^2 / E \qquad \text{(plane stress)}, \qquad G_{\mathrm{I}c} = (1-\nu^2) K_{\mathrm{I}c}^2 / E \qquad \text{(plane strain)}. \tag{3}$$

Such fracture mechanics approaches assume that fracture process occurs starting from a pre-existing crack or defect having a characteristic size a, which instantaneously grows when the critical condition [either (1) or (2)] is fulfilled. In other words, in the (theoretical) case of absence of any defect, fracture mechanics approach leads to the erroneous conclusion that no failure occurs, irrespective of the stress level in the body.

In macroscopically non-damaged solids, the failure process can be regarded as a breaking or a decohesion phenomenon produced by a sufficiently high stress level, which must be greater than the tensile strength, f_t, of the material.

The collapse condition in a material under uniform tension, σ_0, can be generally stated as follows:

$$\sigma_1 \geq \min\left(f_t, \sqrt{\frac{EG_{\mathrm{I}c}}{\pi a}}\right) \quad \text{or} \quad \sigma_1 \geq \min\left(f_t, \frac{K_{\mathrm{I}c}}{Y\sqrt{\pi a}}\right), \tag{4}$$

where the maximum principal stress $\sigma_1 = \sigma_0$ is used.

From such an equation, a characteristic flaw dimension a_0, below which collapse is strength-like and above which collapse is fracture-like, can be identified:

$$a_0 \geq \frac{EG_{\mathrm{I}c}}{\pi f_t^2} \qquad \text{(fracture-like collapse)}, \tag{5a}$$

$$a_0 < \frac{EG_{\mathrm{I}c}}{\pi f_t^2} \qquad \text{(plastic-like collapse)}. \tag{5b}$$

In the context of plasticity, a more general and common failure criterion, which has a wide applicability for several materials, is the so-called Drucker criterion [2], usually expressed by means of the stress invariants. In the present paper, the criterion is written as follows:

$$I_1 + \xi\sqrt{|J_2|} - k \geq 0 \quad \text{with} \quad k = -3\sigma_p, \quad \xi = \sqrt{3}\,\frac{k - f_t}{f_t} = -\sqrt{3}\left(\frac{3+c}{c}\right), \tag{6}$$

where the material parameters ξ and k can be determined by considering two different stressed states at failure: a hydrostatic failure compressive stress field σ_p, and a uniform tensile failure stress field f_t. The dimensionless ratio c in Eq. (6) is defined as $c = f_t/\sigma_p$. For compressive hydrostatic-insensitive materials, the parameter ξ is approximately equal to $-3\sqrt{3}$ and k is equal to $-3\sigma_p$, and the criterion (6) becomes $I_1 - 3\sqrt{3}\sqrt{|J_2|} \geq -3\sigma_p$ or equivalently:

$$-\kappa + \sqrt{3}\sqrt{|J_2|} \leq \sigma_p, \tag{7}$$

where κ is the mean stress or hydrostatic pressure ($\kappa = tr\,\sigma/3 = \sigma_{ii}/3$).

Among the various parameters to identify the stressed state ‘quality’ in a point of a continuum, the stress triaxiality factor [19–23] can be used:

$$t = \frac{1}{3}\,tr\,\sigma\Big/\sqrt{|J_2|} = \kappa\Big/\sqrt{|J_2|} = \frac{I_1}{3\sqrt{|J_2|}}, \tag{8}$$

i.e., the ratio between the mean hydrostatic stress and a quantity that measures the root of the distortional energy (equivalent to the von Mises stress). From the above definition, it can be deduced that $t \to \infty$ for purely hydrostatic stressed states, $t = 0$ for purely shear stressed states, and $t = 1/\sqrt{3}$ for an uniaxial 3D stressed state. By using such a definition, the condition (4) can be written as

$$\sqrt{|J_2|}(1+t) \ge \min\left(f_t, \sqrt{\frac{EG_{\mathrm{Ic}}}{\pi a}}\right) \quad \text{or} \quad \sqrt{|J_2|}(1+t) \ge \min\left(f_t, \frac{K_{\mathrm{Ic}}}{Y\sqrt{\pi a}}\right), \tag{9}$$

where the maximum principal stress is written as $\sigma_1 = \sqrt{|J_2|}(1-t)$, as is discussed below.

On the other hand, by considering the Druker plastic collapse criterion, we get:

$$\sqrt{|J_2|}\left(\frac{\xi}{3} + t\right) \ge \frac{k}{3} \tag{10}$$

and, in the particular case expressed by Eq. (7), the criterion becomes:

$$\sqrt{|J_2|}(-\sqrt{3} + t) \ge \sigma_p. \tag{11}$$

The above failure expressions (9) and (11) can be summarized in the following general equation:

$$\sqrt{|J_2|}(\pm t + \beta) \ge B, \tag{12}$$

where β and B are material constants as is discussed below.

It can be observed [25–28] that the stress triaxiality can provide a simple parameter to identify the type of collapse: stress triaxiality lower than zero ($t = 0$ corresponds to a pure shear situation) produces a shear-type plastic flow collapse, which can be identified as a special case of fracture process, while positive triaxiality produces a void formation-type fracture process (fracture along a direction normal to the principal stress) [27, 28]. Experimental observations [21] have shown that fracture never occurs for $t \le -1/3$.

2. **"Fracture" in Liquids: The Cavitation Phenomenon**. Generally speaking, liquid failure can be identified by the appearance of voids or bubbles in the liquid with loss of continuity and, consequently, the well-known corrosion in solid parts which are in contact which such broken zones produced by the so-called cavitation phenomenon.

The knowledge of the stressed state in a liquid is fundamental to predict the possibility of liquid breaking. The stressed state $\sigma = \sigma_{ij}$ can be generally written as

$$\sigma = -p \cdot \mathbf{1} + \tau[\mathbf{u}], \tag{13}$$

where p is a scalar that represents the hydrostatic pressure (a positive value of p indicates hydrostatic compression), $\mathbf{1}$ is the identity tensor, and τ is the shearing stress tensor which can be determined by an appropriate constitutive law depending on the liquid under consideration.

Generally, the shearing stress tensor τ can be written as an appropriate function of the velocity. As an example, a simple linear constitutive equation can be assumed for a Newtonian liquid:

$$\tau[\mathbf{u}] = 2\mu \cdot \mathbf{D}[\mathbf{u}], \tag{14}$$

where μ is the dynamic viscosity, and $\mathbf{D}[\mathbf{u}]$ is the symmetric part of the gradient of the velocity vector u_i, $\mathbf{D}[\mathbf{u}] = D_{ij} = 1/2(\Delta\mathbf{u} + \Delta^T\mathbf{u}) = 1/2(u_{i,j} + u_{j,i})$, $i = 1, 2, 3$. According to the above hypothesis, the stress tensor can be rewritten as

$$\sigma_{ij} = -p\delta_{ij} + \mu(u_{i,j} + u_{j,i}), \tag{15}$$

where the symbol δ_{ij} denotes the Kroneker delta function.

As is well-known, the following relationship holds for an incompressible liquid: $\mathrm{div}\, u = tr\,\mathbf{D}[\mathbf{u}] = u_{i,i} = 0$ (mass conservation equation). From the previous equation, it can be stated that $tr\,\tau = tr(2\mu\,\mathbf{D}[\mathbf{u}]) = 0$. In this case, the mean hydrostatic pressure becomes:

$$p = -\frac{1}{3} tr\ \sigma = -\frac{1}{3}\sigma_{ii}. \tag{16}$$

The above equation is true for a liquid in a static condition for which the hydrostatic pressure p represents the average of the principal stresses.

In a flowing liquid, the average of the principal stresses has not significance [16, 18], and the hydrostatic pressure p also depends on the stresses given by (14). In such a general case, $p = -1/3\, tr\ (\sigma - \tau)$ since $tr\ \tau \neq 0$, as occurs for a constitutive relationship different from the classical constitutive relationship (which holds for the so-called non-Newtonian liquids) expressed by Eq. (14). In this last case, Eq. (13) can be rewritten as

$$\sigma = -p \cdot \mathbf{1} + \tau = -\kappa \cdot \mathbf{1} + S, \tag{17}$$

where $p = -1/3\, tr(\sigma - \tau)$, $\kappa = -1/3\, tr\,\sigma = p - 1/3\, tr\,\tau$, and $\mathbf{S}$ is the deviatoric stress tensor for which $tr\,\mathbf{S} = s_{ij} = 0$ holds (the scalars and are assumed positive for hydrostatic compression states).

The breaking phenomenon in a liquid, commonly known as cavitation, takes place by the formation of bubbles inside the liquid and is classically identified by the following condition [18]:

$$p < p_c(T) \qquad \text{(cavitation)}, \tag{18a}$$

$$p > p_c(T) \qquad \text{(no cavitation)}, \tag{18b}$$

where $p_c(T)$ is the cavitation threshold (negative) pressure of the considered liquid at a given temperature T. The above condition can be rewritten for liquids in motion, by considering the principal stresses instead of the average pressure p which can loose significance in such a case [18]. The previous hypothesis requires the determination of the maximum (principal) stresses and of the corresponding directions in a point of the liquid. Principal stresses in the coordinate system coincident with the principal stress directions can be determined by considering that the stress tensor has a diagonal form in such a coordinate system:

$$\operatorname{diag}\sigma = \operatorname{diag}(-\kappa \cdot \mathbf{1} + S) = -\kappa \cdot \mathbf{1} + \operatorname{diag}\mathbf{S} \tag{19a}$$

or

$$\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} = -\kappa \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \operatorname{diag}\mathbf{S}. \tag{19b}$$

In two dimensions, the above relationship becomes:

$$\begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} = -\kappa \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} S_{11} & 0 \\ 0 & S_{22} \end{bmatrix} = -\kappa \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} S_{11} & 0 \\ 0 & -S_{11} \end{bmatrix} \tag{20}$$

$$\text{since} \qquad S_{11} + S_{22} = 0,$$

where the principal stresses of the deviatoric tensor $\mathbf{S}$, S_{11}, S_{22}, are equal to $S_{11} = (s_{11}^2 + s_{12}^2)^{1/2}$, $S_{22} = -(s_{11}^2 + s_{12}^2)^{1/2}$ (since $s_{11} = -s_{22}$), and the term $s_{ij} = \sigma_{ij} + \delta_{ij}\kappa$ is the generic component of $\mathbf{S}$. Since the principal stresses of the deviatoric tensor $\mathbf{S}$ are:

$$S_{11} = \sigma_1 - \frac{1}{2}(\sigma_1 + \sigma_2), \qquad S_{22} = \sigma_2 - \frac{1}{2}(\sigma_1 + \sigma_2), \tag{21}$$

the maximum difference between the maximum (σ_1) and the minimum (σ_2) principal stress is equal to $\sigma_1 - \sigma_2 = S_{11} - S_{22} = 2S_{11}$.

The condition of rupture (18), formulated in terms of the maximum principal stress, becomes:

$$\sigma_1 \geq |p_c| \tag{22a}$$

or

$$\sigma_1 = S_{11} - \kappa = (s_{11}^2 + s_{12}^2) - \kappa = \mu[u_{1,1}^2 + (u_{1,2} + u_{2,1})^2]^{1/2} - \kappa \geq |p_c|. \tag{22b}$$

When the above condition is fulfilled, a slit vacuum cavity initially opens in a direction perpendicular to the maximum tension axis; subsequently the flow vorticity can rotate the major axis of the ellipse, and vapour fills the created cavity [18]. If also $\sigma_2 = S_{22} - \kappa \geq |p_c|$ then the cavity opens also along the other principal direction.

The above criterion must be corrected in presence of impurities (such as bubbles, etc.) that facilitate the cavitation phenomenon. If a defect having a characteristic dimension a, is supposed to exist in a point of the liquid, Eq. (22) can be written as follows [16]:

$$\sigma_1 \geq \left| p_c(a, T) \right| = \left| K_c(T) \right| / \sqrt{a} < \left| p_c(T) \right|, \tag{23}$$

where $K_c(T)$ is a material parameter for a given temperature T.

Similar to the case of collapse in solids, the cavitation phenomenon, in general, takes place when the following condition is fulfilled [16]:

$$\sigma_1 \geq \min\left(\left| p_c(T) \right|, \frac{\left| K_c(T) \right|}{\sqrt{a}} \right) = \frac{\left| K_c(T) \right|}{\sqrt{a}}. \tag{24}$$

Defining the stress triaxiality factor similar to the case of solids (see Section 1):

$$t = \kappa / \sqrt{\left| J_2 \right|} \qquad \text{with} \qquad J_2 = s_{11} s_{22} - s_{12}^2, \tag{25}$$

the maximum principal stress can be rewritten as follows:

$$\sigma_1 = (s_{11}^2 + s_{12}^2)^{1/2} - \kappa = (-s_{11} s_{22} + s_{12}^2)^{1/2} - \kappa = \sqrt{\left| J_2 \right|} - \kappa. \tag{26}$$

Then, the cavitation phenomenon takes place when

$$\sigma_1 = \sqrt{\left| J_2 \right|}(1 - t) \geq \left| p_c \right|. \tag{27}$$

As can be also observed under pure shear ($t = 0$), liquid rupture can occur when $\sqrt{\left| J_2 \right|} \geq \left| p_c \right|$. If a high level of hydrostatic compressive pressure exists ($t >> 0$, since $\kappa > 0$ for hydrostatic compression), $\sigma_{11} = \sqrt{\left| J_2 \right|}(1 - t) < 0$ and no cavitation occurs.

Some examples of fracture collapse in solids and fluids are given in Fig. 1.

a *b* *c*

Fig. 1. Fracture collapse in a ductule (*a*) or brittle (*b*) solid material and rupture (cavitation) in a liquid (*c*).

3. **Discussion**. From the previous Sections, we can deduce that the failure condition in flawed solids [see Eqs. (9) and (11)] and that in liquids [see Eq. (27)] present a similar form which can be synthesized as follows:

$$\sigma_1 = \sqrt{|J_2|}(1+t) \geq \min(C_1, C_2), \tag{28a}$$

where

$$C_1, C_2 = \begin{cases} f_t, \sqrt{\dfrac{EG_{\mathrm{I}c}}{\pi a}} = \dfrac{K_{\mathrm{I}c}}{Y\sqrt{\pi a}} & \text{for solids,} \\ |p_c(T)|, \dfrac{|K_c(T)|}{\sqrt{a}} & \text{for liquids,} \end{cases} \tag{28b}$$

where the same definition of κ is used for both solids and liquids, i.e., $\kappa = 1/3\, tr\, \sigma$ (instead of $\kappa = -1/3\, tr\, \sigma$ as usually made for liquids). Equation (29) is valid for both solids and liquids once the constants C_1 and C_2 are appropriately defined.

In the above general relationship, a difference in the meaning of the stress σ_1 must be considered: the failure in liquids takes place in the point where the principal stress attains the maximum value (the imperfections are assumed to be equally distributed in the liquid domain), while the stress in solids must be interpreted as the reference stress (applied to the structure) that produces the critical condition of the SIF.

By considering the previous Sections, it can also be noted that the failure conditions in unflawed solids [see Eqs. (9) and (10)] and liquids [see Eq. (27)] have a similar form which can be synthesized in this way:

$$\sqrt{|J_2|}(\beta + \alpha t) \geq B, \tag{29}$$

where the material constants α, β, and B depend on the mechanical material behavior and the presence of defects. The values assumed by the three parameters involved in Eq. (29) are summarized in Table 1.

Table 1

Parameters Involved in the General Failure Function, $\sqrt{|J_2|}(\alpha t + \beta) \geq B$, for Both Unflawed and Flawed Solids and Liquids

Material	Mechanical behavior and defects	Failure parameters		
		α	β	B
Solids	Unflawed brittle	1	1	f_t
	Unflawed ductile	1	$\xi/3$	$k/3$
	Flawed	1	1	$\sqrt{EG_{\mathrm{I}c}/(\pi a)}$
Liquids	Unflawed	-1	1	$\lvert p_c(T)\rvert$
	Flawed	-1	1	$\lvert p_c(a,T)\rvert = \lvert K_c(T)\rvert/\sqrt{a}$

In order to compare the failure phenomenon occurrence in unflawed solids and liquids, two 2D examples related to collapse in simple situations are discussed below.

3.1. ***Collapse of a Solid Undefined 2D Strip under Shear Load***. As an example of application of the collapse functions determined above, an undefined 2D strip in plane stress condition, under self weight and an applied shear load, is considered in the present Section (Fig. 2). The stressed state in the material can be approximately defined by the following equations:

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} = -\kappa \cdot \mathbf{1} + \mathbf{S} =$$

$$= -\frac{\rho g}{2} \begin{bmatrix} h(1+e) - x_2(2e+1) & 0 \\ 0 & h(1+e) - x_2(2e+1) \end{bmatrix} +$$

$$+ \begin{bmatrix} \frac{\rho g}{2}[h(1-e) + x_2(2e+1)] & G'\frac{d}{h} \\ G'\frac{d}{h} & -\frac{\rho g}{2}[h(1-e) + x_2(2e+1)] \end{bmatrix}, \qquad (30)$$

where ρ, g, and G' are solid density, acceleration of gravity and shear modulus, respectively. Furthermore, d is the displacement applied to the top edge of the strip, and $e = \sigma_x(0)/\sigma_y(0)$ is the ratio between the horizontal stress and the vertical stress at the bottom of the strip.

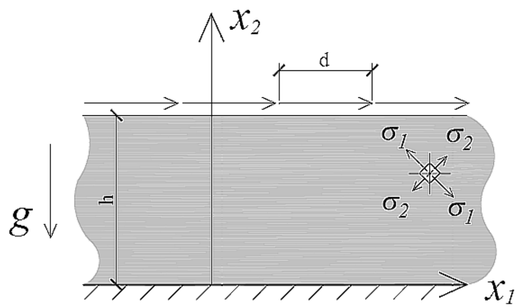


Fig. 2. 2D infinite strip under self weight and shear load.

In the principal stress coordinate system, the stress tensor becomes:

$$\text{diag}\,\sigma = \frac{\rho g}{2} \begin{bmatrix} h(1+e) - x_2(2e+1) & 0 \\ 0 & h(1+e) - x_2(2e+1) \end{bmatrix} + \begin{bmatrix} \sqrt{|J_2|} & 0 \\ 0 & -\sqrt{|J_2|} \end{bmatrix}, \qquad (31)$$

where $\sqrt{|J_2|} = \sqrt{\frac{\rho^2 g^2}{4}[h(1-e) + x_2(2e-1)]^2 + \frac{G'^2 d^2}{h^2}}$.

It can be observed that, at the top free surface, we have:

$$\sigma = \frac{\rho g h e}{2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \begin{bmatrix}\frac{\rho g h e}{2} & G'\frac{d}{h}\\ G'\frac{d}{h} & -\frac{\rho g h e}{2}\end{bmatrix}, \tag{32}$$

$$\operatorname{diag}\sigma(x_2 = h) = \frac{\rho g h e}{2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \sqrt{\frac{\rho^2 g^2 h^2 e^2}{4} + \frac{G'^2 d^2}{h^2}}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix},$$

and

$$\sqrt{|J_2|} = \sqrt{\frac{\rho^2 g^2 h^2 e^2}{4} + \frac{G'^2 d^2}{h^2}}.$$

By introducing the stress triaxiality t (which is a function of x_2 only):

$$t(x_2) = \frac{\kappa}{\sqrt{|J_2|}} = \frac{-\frac{\rho g}{2} h(1+e) - x_2(2e+1)}{\sqrt{\frac{\rho^2 g^2}{4}[h(1-e) + x_2(2e-1)]^2 + \frac{G'^2 d^2}{h^2}}}, \tag{33}$$

the maximum principal stress can be expressed as follows:

$$\sigma_1(x_2) = \kappa + \sqrt{|J_2|} = \sqrt{|J_2|}(1 + t(x_2)) = \frac{\rho g(h - x_2)}{2} +$$

$$+\sqrt{\frac{\rho^2 g^2 (h - x_2)^2}{4} + \frac{G'^2 d^2}{h^2}}, \tag{34}$$

and the failure conditions can be written as

$$\sigma_1(x_2) = \sqrt{|J_2|}(1 + t(x_2)) \ge f_t,$$

$$\sqrt{|J_2|}\left(\frac{\xi}{3} + t\right) \ge \frac{k}{3}, \tag{35}$$

for an undamaged brittle or ductile material, respectively, or in dimensionless form:

$$F_1 = \frac{\sigma_1(x_2)}{f_t} - 1 = \sqrt{|J_2|}\frac{(1 + t(x_2))}{f_t} - 1 \ge 0, \qquad F_2 = \frac{\sqrt{|J_2|}}{k/3}\left(\frac{\xi}{3} + t\right) - 1 \ge 0. \tag{36}$$

In Fig. 3a, the qualitative collapse function patterns are displayed. As can be observed, the collapse for a ductile solid occurs first at the base of the strip, while the collapse for a brittle solid occurs first at the top of the strip. In Fig. 3b, the stress triaxiality factor along a vertical line of the solid strip is represented: such a parameter increases from the base to the top of the strip.

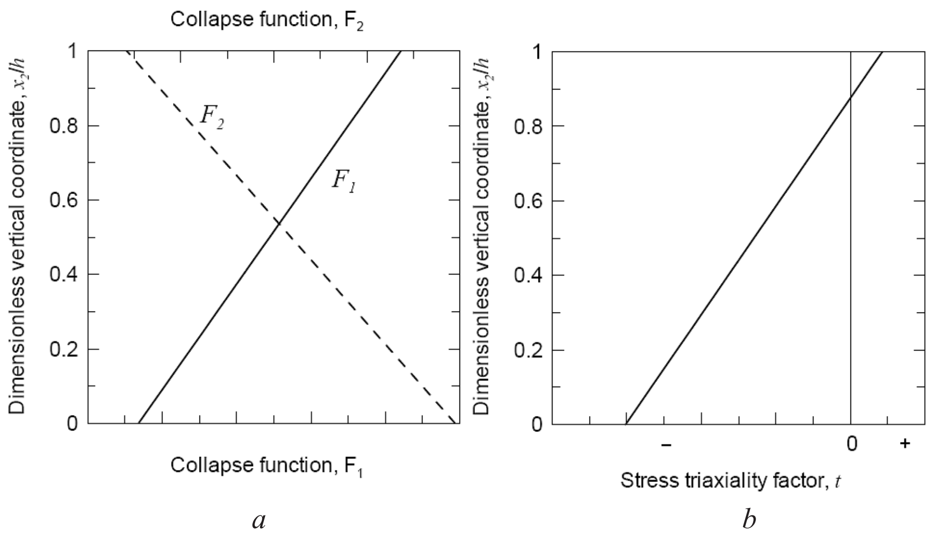


Fig. 3. Brittle or ductile solid strip under shear: (a) collapse functions; (b) stress triaxiality factor.

3.2. ***Collapse of a Liquid under Shear Flow***. Fracture in liquids can occur even under shear flow if the maximum tension stress criterion described above is adopted. By assuming a one-dimensional flow with a velocity field expressed as follows (Fig. 4):

$$\{u \quad v \quad w\} = \{Ux_2/h \quad 0 \quad 0\}, \tag{37}$$

the constitutive equation (14) becomes:

$$\tau[u] = \mu \begin{bmatrix} 0 & U/h & 0 \\ U/h & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{38}$$

The liquid is assumed to be in the gravity field (the direction of gravity is $-x_2$). Under the previous hypothesis, the stressed state in the liquid (only quantities on plane $x - y$ are considered) can be expressed as follows:

$$\sigma = -\kappa \cdot \mathbf{1} + \mathbf{S} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} = -\rho g(h - x_2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \mu \begin{bmatrix} 0 & U/h \\ U/h & 0 \end{bmatrix}, \tag{39}$$

where ρ and g are the liquid density and the acceleration of gravity, respectively. In the principal stress coordinate system, the stress tensor becomes:

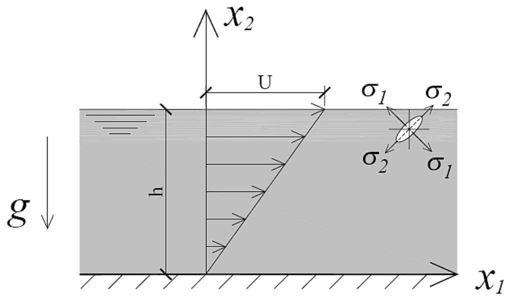


Fig. 4. Shear flow of a liquid in the gravitational field.

$$\operatorname{diag}\sigma = -\rho g(h-x_2)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \mu\begin{bmatrix}U/h & 0\\ 0 & -U/h\end{bmatrix}. \tag{40}$$

Therefore, at the free surface we have:

$$\operatorname{diag}\sigma(x_2 = h) = \mu\begin{bmatrix}U/h & 0\\ 0 & U/h\end{bmatrix}.$$

By introducing the stress triaxiality t (which is a function of x_2 only):

$$t(x_2) = \frac{\kappa}{\sqrt{|J_2|}} = \frac{\rho g(h-x_2)h}{\mu U}, \tag{41}$$

the maximum principal stress can be expressed in terms of such a parameter as

$$\sigma_1(x_2) = \sqrt{|J_2|}(1 - t(x_2)) = \frac{\mu U}{h} - \rho g(h - x_2). \tag{42}$$

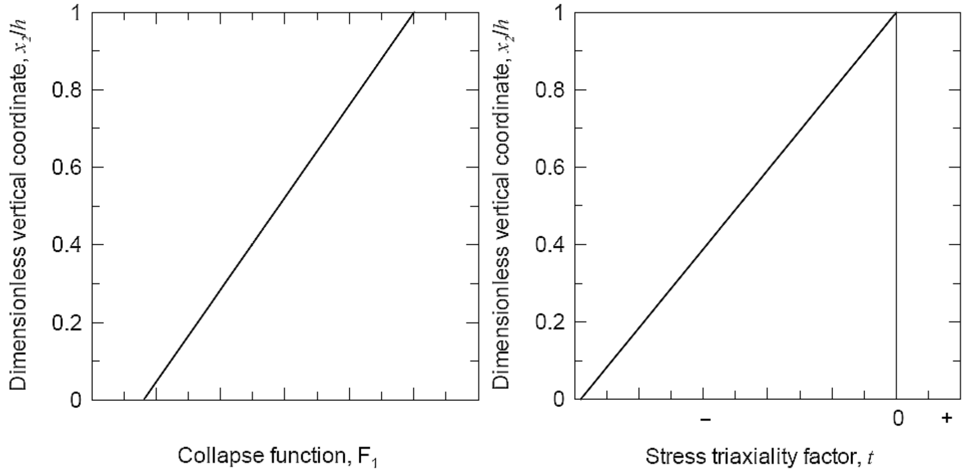
The condition of liquid fracture (or cavitation) can be finally written as follows:

$$\sigma_1(x_2) = \sqrt{|J_2|}(1 - t(x_2)) = \mu U - \rho g(h - x_2) \ge |p_c|, \tag{43}$$

or in dimensionless form:

$$F_1 = \frac{\sigma_1(x_2)}{|p_c|} - 1 = \frac{\mu}{|p_c|}U - \frac{\rho g h}{|p_c|}\frac{(h-x_2)}{h} - 1 \ge 0. \tag{44}$$

In Fig. 5, the qualitative collapse function pattern is displayed. As can be observed, since $(h - x_2) > 0,$ the cavitation phenomenon takes place first at the top free surface of the liquid located at $x_2 = h.$ In Fig. 5b, the stress triaxiality factor along a vertical line of the liquid is represented: it increases from the base to the top of the liquid domain.



a *b*

Fig. 5. Liquid under shear flow: (a) collapse function along a vertical line of the liquid; (b) stress triaxiality factor.

Conclusions. In the present paper, the failure phenomena in continuum media, solids or liquids, have been regarded as a single physical phenomenon which is determined by the fulfilment of a limiting condition.

By considering the classical plasticity or fracture mechanics theory to describe the failure of solids and the theory of the cavitation phenomenon to describe the failure of liquids, a unified description has been proposed, showing that collapse functions present a similar form for both classes of continuum materials.

The stress triaxiality, as a stress field parameter, allows us to express the limit conditions of collapse for both classes of materials through a single equation, i.e., the structure of the governing failure equation in both solids or liquids failure is the same. By appropriately setting the involved material parameters, the failure function can be explicitly written. The values of such parameters provide useful information about the type of expected failure.

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Резюме

Процес руйнування твердих і рідких тіл у суцільному середовищі можна розглядати як єдине фізичне явище. Математично описати його можна при виконанні деякої граничної умови, в якій зазвичай враховується напружений стан матеріалу. Аналіз класичних теорій руйнування твердих тіл (із точки зору пластичності і механіки руйнування) і сучасної теорії руйнування рідких тіл (явище кавітації з урахуванням максимального значення головного напруження) дозволив виявити загальні закономірності в умовах їх руйнування. У рамках загального підходу до руйнування тіл запропоновано використову-

вати для описування граничної умови руйнування такий параметр поля напружень, як тривісний напружений стан. Показано, що функції коллапса для твердих і рідких тіл у суцільному середовищі подібні.

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