NUMERICAL ALGORITHM BASED ON THE PDE METHOD FOR SOLUTION OF THE FOKKER PLANCK EQUATION

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(Received September 7, 2011)

This paper discus a fast and accurate algorithm for numerical solution of Fokker-Planck equation (FPE) based on the PDE (Partial Differential Equation) method. The PDE concepts and methods largely are used in computer simulation of fluid-dynamical systems. This method can be used for studying of stochastic beam dynamics in one dimensional phase space in the storage ring. The performances of the PDE-method are calculated using the stochastic cooling process in the CR storage ring (FAIR, Germany).

PACS: 29.20.D, 29.27.Fh, 29.85.-c

1. INTRODUCTION

The large variety of physical phenomena uses FPE. Formally, the FPE equilibrium solution can be easily determined, while its time evolution can be analytically obtained only in few particular cases. Numerical solution of the FPE in general is interested for a number of stochastic physics problems. In this paper, we describe an algorithm for a numerical solution of the Fokker-Planck equation. This problem, especially if variable coefficients are included in the model, is computationally very expensive. The solution of FPE often takes a long time even with today's high-speed computers. The algorithm described here makes the calculation of the cumulative distribution functions for predicted process durations computationally much less expensive. This improvement is achieved by solving the Kolmogorov backward equation numerically instead of employing the previously used closed form solution. Additionally, the algorithm can determine the optimum fit for one of the model parameters (the starting point z) directly, thereby reducing the dimension of the parameter search space by one. The resulting method is shown to be notably faster than the standard (closed-form solution) method for parameter estimation.

The FPE is a second order partial differential equation; it can be put in the form

$$\frac{\partial \Psi(t,z)}{\partial t} = \left(\frac{\partial}{\partial z} F(t,z) - \frac{1}{2} \frac{\partial^2}{\partial z^2} D(t,z) \right) \Psi(t,z)\,, \tag{1}$$

where $F(t,z)$ and $D(t,z)$ are known functions which may depend, in principle, on time, and $\Psi(t,z)$ represents the unknown solution. It can be easily shown [1] that this solution corresponds to the z-coordinate probability distribution of a mass less particle whose dynamics is described by the Langevin equation. The algorithm described in this article involves numerical solution of the parabolic PDE (1). Solving PDEs numerically is a well-established topic both in mathematics and in applied areas. There exists a vast body of literature and there are also many 'black box' PDE solvers available (e.g. the PDE toolbox of Matlab or Mathematic packages). Nevertheless, in order to keep the text as self-contained as possible, we give here a rough sketch of the numerical method. For a more detailed description we refer to the following references [2-5].

2. PDE ALGORITHM FOR FPE

The key to the numerical solution of PDEs such as eq. (1) by using finite difference methods is to discretise 'space' z and time t: instead of the full z-interval $[0,\, a]$ we only consider the discrete set of z-values $\{0,\, \Delta z,\, 2\Delta z, ...N\Delta z\}$ where $\Delta z = a/N$ for some positive integer N and instead of the t-interval [0,1] we only consider the t -values $\{0,\, \Delta t,\, 2\Delta t, ...\}$. The algorithm computes values ψ_{ij} which approximate the true solution Ψ by $\psi_{ij} = \Psi(i\Delta t,\, j\Delta z)$ for $i = 0,\, 1,\, 2,\, ...$ and $j = 0,\, 1, ...,\, N$. The accuracy of this approximation depends on the step sizes Δt and Δz. The algorithm works by considering a grid row with fixed t at a time, starting with an approximation of the initial condition: $\psi_{00},\, \psi_{0N}$. Then, in each step, the algorithm uses the approximation $\psi_{i0},\, \psi_{iN}$ for time $i\Delta t$ to compute an approximation for time $(i+1)\Delta t$. To simplify the presentation we consider the function $\Psi(t,\, z)$. A simple calculation shows that ? solves with homogeneous boundary conditions $\Psi(t,\, 0) = \Psi(t,\, a) = 0$. We denote the computed solution for time $n\Delta t$ by $\Psi_n = (\psi_{n1}, ..., \psi_{iN-1})$. We do not include the outermost points ψ_{n0} and ψ_{nm} since these are always zero due to the boundary conditions. Using numerical approximations for the partial derivatives we can then write the application of

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the differential operator

$$L = \frac{\partial}{\partial z} F(t,z) - \frac{1}{2}\frac{\partial^2}{\partial z^2} D(t,z) \qquad (2)$$

as a matrix vector multiplication. Collecting all the terms we get

$$L\Psi(n\Delta t, n\Delta z) = L^N \Psi_n^n, \qquad n[1,...N]\,, \qquad (3)$$

where L^N is the tri-diagonal matrix given by

$$L^N = \begin{pmatrix} d_1 & p_1 & 0 & 0 \\ r_2 & d_2 & p_2 & 0 \\ 0 & r_i & d_i & p_i \\ 0 & 0 & r_N & d_N \end{pmatrix}, \qquad (4)$$

where

$$\begin{aligned} r_i &= -\frac{F_{i-1}}{2\Delta z} + \frac{D_{i-1}}{2\Delta z^2}\,, \qquad i \in [2,\, N]\,, \\ d_i &= -\frac{D_i}{\Delta z^2}\,, \qquad i \in [1,\, N]\,, \\ p_i &= \frac{F_{i+1}}{2\Delta z} + \frac{D_{i+1}}{2\Delta z^2}\,, \qquad i \in [1,\, N-1]\,. \end{aligned} \qquad (5)$$

The approximations introduced above suggest the following approximation to the PDE:

$$\frac{\Psi^{n+1} - \Psi^n}{\Delta t} = L^N \left(\theta \Psi^{n+1} + (1-\theta)\right) \Psi^n\,, \qquad (6)$$

where $\theta \in [0,1]$ is a parameter of the method. For $\theta = 0$ the derivative on the right hand side is evaluated only for the current approximation Ψ^n. For all values $\theta > 0$ the derivative is evaluated for a mixture of Ψ^n and Ψ^{n+1}. In these cases one has to solve a system of linear equations to compute Ψ^{n+1} from Ψ^n: by rearranging the terms in (6) we get

$$\left(I - \Delta t \theta L^N\right) \Psi_i^{n+1} = \left(I - \Delta t (1-\theta) L^N\right) \Psi_i^n\,, \qquad (7)$$

where I is the $N \times N$ identity matrix. As we will discuss below, the choice of the parameter θ affects the stability of the method. Common choices are $\theta = 0$ (Euler scheme), $\theta = 1/2$ (Crank Nicolson scheme) and $\theta = I$ (implicit Euler scheme).

3. SOLVER WITH PDE METHOD

By using the equations discussed previously the numerical calculation procedure is as follows. One has to construct the set of the linear algebraic equations following the equation (7). Taking into account the matrix of operators (5) the left side of the equation (7) is written as multiplication of the tri-diagonal matrix (4) on the vector Ψ_i^{n+1}. The right side of eq.(7) can be written as a vector f_i, which is obtained by multiplying matrix $\left(I - \Delta t(1-\theta)L^N\right)$ on the known Ψ_i^n. In general a tridiagonal system for N unknowns may be written as

$$\begin{pmatrix} b_1 & c_1 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 \\ 0 & a_i & b_i & c_i \\ 0 & 0 & a_N & b_N \end{pmatrix} \begin{pmatrix} \psi_1^{n+1} \\ \psi_2^{n+1} \\ \psi_i^{n+1} \\ \psi_N^{n+1} \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix}, \qquad (8)$$

where:

$$\begin{aligned} a_i &= -\frac{T}{2}\left(\frac{D_{i-1}}{\Delta z^2} + \frac{F_{i-1}}{\Delta z}\right), \qquad b_i = 1 + T\frac{D_i}{\Delta z^2}\,, \\ c_i &= -\frac{T}{2}\left(\frac{D_{i+1}}{\Delta z^2} - \frac{F_{i+1}}{\Delta z}\right), \end{aligned} \qquad (9)$$

and

$$f_i = \Psi_i^n + \frac{W}{2}\left(-\frac{F_{i+1}\Psi_{i+1}^n - F_{i-1}\Psi_{i-1}^n}{\Delta z} + \frac{D_{i+1}\Psi_{i+1}^n - 2D_i\Psi_i^n + D_{i-1}\Psi_{i-1}^n}{\Delta z^2}\right), \qquad (10)$$

(10) $W = \Delta t(1-\theta)$, $T = \Delta t\theta$. To solve a tridiagonal system of equations (8) one can use so called "the tridiagonal matrix algorithm" also known as the "Thomas algorithm" [7], which is a simplified form of Gauss elimination. By Thomas algorithm the solution can be obtained in N operations instead of N^3 required by Gaussian elimination.

4. STOCHASTIC COOLING FACILITY AND PARAMETERS

The stochastic cooling in the accelerator cycling rings is carried out by application of the following scheme (Fig.1) [8]:

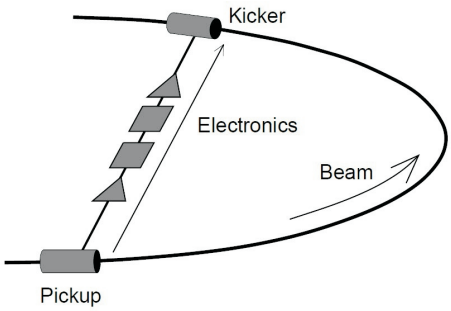

Fig.1. *The general scheme of stochastic cooling*

Pickup is the diagnostic device which measures a particle deviation. In the Kicker, the electromagnetic field impulse in the perpendicular direction to the beam is created. Pickup and Kicker are connected by the amplifier. The amplifier signal is proportional to a signal from Pickup. The stochastic cooling system is planned to be applied in the Collector Ring (CR, FAIR) [9],(Fig.2). For this matter the following scheme of stochastic cooling for antiprotons is developed.

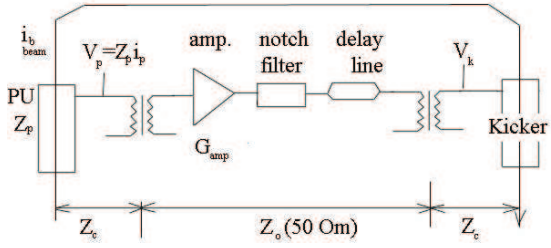

Fig.2. *Stochastic cooling scheme at the CR machine for antiprotons*

For the numerical calculations the parameters of the stochastic cooling system given in Table are used.

Main parameters of the Stochastic Cooling System at the CR machine for antiprotons

Total number of antiprotons, N_0	10^8
Beam energy, GeV	3
Frequency slip factor :	
Of the ring	-0.017
Local: from pick-up to kicker	-0.041
Revolution frequency, MHz	1.315
Bandpass (fmin, fmax), GHz	1, 2
Min/Max harmonic number	800/1600
Effective temperature for amplifier noise , K	73
Pick-up impedance, Z_p, Ω	720
Kicker-impedance, Z_c, Ω	2880
Number of pick-ups, n_p	2
Number of kickers, n_k	2
Gain	2×10^7

We are interested in the evolution of the $\Psi(t,\delta)$ distribution with a time. The initial distribution $\Psi(0,\delta)$ is a parabolic. The *rms* values δ_{rms} of $\Psi(t,\delta)$ distribution is calculated at each time step. The accuracy we defined as difference of calculated integrals of $\Psi(t,\delta)$ function at a time t_{max} and $t=0$.

$$\varepsilon = \frac{1}{N_0}\left(\int_{-\delta_{max}}^{\delta_{max}} \Psi(t_{max},\delta) - \int_{-\delta_{max}}^{\delta_{max}} \Psi(0,\delta)\right) .$$

Here N_0 is the total number of particles. In our simulation we take $t_{max} = 25\,s$. The beam equilibrium (when diffusion and cooling terms are equal) begins in $20\,s$. Obviously during further $5\,s$ one should observe stable δ_{rms}, which indicate stability of the PDE method.

5. COEFFICIENTS F AND D WITH THE OPTION OF CONSIDERING FEEDBACK THROUGH THE BEAM. SIMULATIONS FOR CR, ANTIPROTON BEAM COOLING

Prior to the calculation of the precision and stability of the FPE numerical solutions given in previous sections the coefficients F and D must be defined. These coefficients characterise an electronic system of stochastic cooling. Their optimization is a subject of many works. For our simulations the simple formulae given in refs.[8, 9] are considered. For these formulae the parameters of the stochastic cooling system, which is planned to be used in the CR are used. The parameters necessary here for coefficients are summarized in Table. The coherent coefficient F depending on the momentum $\delta = dp/p$ is written by

$$F(\delta) = 2ef_0^2\sqrt{n_p n_k Z^k Z_p}\sum_{n=n1}^{n2} Re\left\{\frac{G(\delta,\,n)}{1-S(\delta,\,n)}\right\}\,eV/s\,. \tag{11}$$

e - electron charge, Z_k, Z_p - electrical impedances coupled to the beam in the circuit convention [8], n_p, n_k - number of the pick-up and kicker units, f_0 - revolution frequency, G - voltage gain including the filter and phase shifts in the pick-ups and kickers. The gain is reduced by factor $(1-S)$ due to the beam-feedback effect. The function $S(\delta,\,n)$ is so-called open-loop gain. Summing is done over all harmonics from n1 to $n2$. The incoherent coefficient D given in ref. [8] consist of two noise power components. One is the amplifier noise, and the other the beam noise. The D is written by

$$D(\delta) = 2\left[kTf_0^2 n_k Z_k \sum_{n=n1}^{n2}\frac{|G(\delta,\,n)|^2}{|1-S(\delta,\,n)|^2 P(n)^2}\right.$$
$$\left.\frac{2\pi e^2 f_0^2 n_p n_k Z_p Z_k \Psi(\delta)}{|K|}\sum_{n=n1}^{n2}\frac{|G(\delta,\,n)|^2}{n|1-S(\delta,\,n)|^2}\right]\,, \tag{12}$$

k is Boltzmann's constant, T is the temperature of the electronic devise (K), Z_p and Z_k are the impedances of a pick-up and a kicker, e - electron charge. For the numerical calculations the parameters of the stochastic cooling system given in Table are used. This system is planned to be applied in the Collector Ring (CR) [9].

Fig.3 shows the rms momentum spread during the cooling process. It is confirmed that the numerical solvers, obtained by PDE method, quite well represent the result received by CERN code [8,10]. Fig.4 demonstrates the particle number during cooling process calculated by PDE method and CERN code.

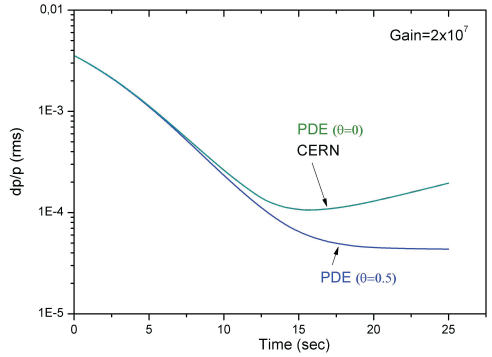

Fig.3. *Rms energy spread calculates by PDE method and CERN code*

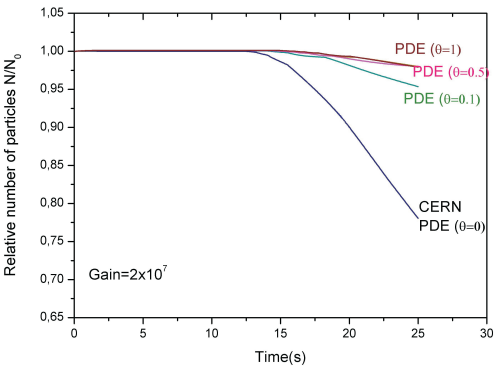

Fig.4. *Relative number of particles obtained by PDE method with different parameter θ and CERN code*

One can see that solutions obtained by PDE method with parameter $\theta > 0$ are more stable for the equilibrium state.

6. CONCLUTIONS

To study the beam dynamic in the storage ring, where the stochastic cooling is used, the so-called PDE-method for solution of the Fokker-Planck equation is proposed. In the paper the stability and convergence of the method are studied. As an example, here the results of simulation by this method are given for the stochastic cooling process at the CR storage ring. The numerical results on FPE solution with PDE method are compared with CERN numerical methods. This work in the near future will be extended to higher dimensions for studying the dynamics of particles in storage rings. Acknowledgements: This research has been supported by the Frankfurt University (Germany) under grant of the HIC for FAIR.

References

1. A. Voss and J. Voss // *Journal of Mathematical Psychology.* 2008, v.52, p.1-9.

2. G.W. Bluman, G.J. Reid // *Journal of Mathematical Analysis and applications.* 1989, v.144, p.565.

3. V. Palleschi, et.al. // *Physics Letters A.* 1990, v.146, N7, p.8.

4. Winfried Auzinger. *Numerik partieller Differential-gleichungen.* Eine: "Einfurung", 2003.

5. Rolf Rannacher. *Numerische Mathematik (Numerik gewohnlicher Differentialgleichungen).* Vorlesungs-skriptum. Universitet Geidelberg: SS, 2007.

6. T. Kikuchi , S. Kawata, T. Katayama. *Numerical Solver with CIP method for Fokker-Planck Equation of Stochastic cooling.* PAC07, Albuquerque, New Mexica, USA: "THPAN048". 2007, p.3336-3338.

7. S.D. Conte and C. de'Boor. *Elementary Numerical Analysis.* New York: "McGraw-Hill", 1972.

8. D. Mohl, G. Petrucci, L. Thorndahl, S. van der Meer. Physics and technique of stochastic cooling // *McGraw-Hill,Physics Reports (Review Section of Physics Letters).* 1980, v.58, N.2, p.73-119.

9. A. Dolinskii, F. Nolden, M. Steck. Lattice Considerations for the Collector and Accumulator Ring of the FAIR Project // *Proceedings of the "COOL'07 conference"*, 2007, Bad Kreuznach, September, 2007, p.106-109.

10. L. Thorndal, D. Mohl. *CERN code for calculation of stochastic cooling.* Private communications.

ЧИСЛЕННЫЙ АЛГОРИТМ, БАЗИРУЮЩИЙСЯ НА PDE МЕТОДЕ РЕШЕНИЯ УРАВНЕНИЯ ФОККЕРА-ПЛАНКА

М. Долинская

Рассмотрено быстрый и точный алгоритм численного решения уравнения Фоккера-Планка, который базируется на так-называемом PDE методе. Эта концепция и методология преимущественно использовалась для компьютерных симуляций жидкостно-динамических систем. Представленный метод применен для изучения динамики стохастического пучка в одномерном фазовом пространстве на накопительном кольце. Исследуются возможности метода для изучения процессов стохастического охлаждения на накопительному кольце CR (FAIR, Germany).

ЧИСЕЛЬНИЙ АЛГОРИТМ, ЩО БАЗУЄТЬСЯ НА PDE МЕТОДІ ДЛЯ РОЗВ'ЯЗАННЯ РІВНЯННЯ ФОККЕРА-ПЛАНКА

М. Долінська

Розглянуто швидкий та точний алгоритм чисельного розв'язання рівняння Фоккера-Планка, що базується на так-званому PDE методі. Ця концепція та методологія переважно використовувалася для комп'ютерних симуляцій рідинно-динамічних систем. Представлений метод застосовано для вивчення динамики стохастичного пучка в одноміроному фазовому просторі на накопичувальному кільці. Досліджуються можливості PDE метода для вивчання процесів стохастичного охолодження на накопичувальному кільці CR (FAIR, Germany).