

# WAKE FORCE EXCITED BY ULTRARELATIVISTIC ELECTRON BUNCH IN RECTANGULAR WAVEGUIDE WITH PERIODIC PERTURBED WALLS

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A perturbation method for calculation of wakefields in a periodic waveguides with periodic perturbed walls is developed. The analytical formulas for a transverse component of an alternating wake force were derived.

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## 1. INTRODUCTION

The wakefield (WF) induced by a relativistic charge particle bunch in a periodic corrugated waveguide and the corresponding wake force can be expanded into Floquet series in spatial harmonics. The spatial harmonics synchronous with the bunch usually are of interest due to their constant action on the particles that results in the well-known beam loading and beam breakup instability effects in the rf structures.

However, the alternating transverse wake force which consists of the nonsynchronous harmonics can give rise to undulating off-axis particles that should result in no less important phenomena such as the wakefield undulator radiation (WFUR) [1-5], and perhaps the pondermotive focusing. A similar focusing, but in external fields, has been experimentally demonstrated in Ref. [6,7]. Therefore there exists an interest in developing of methods of calculation of the alternating wake forces induced by a relativistic charge bunch in the periodic rf structures.

Applying the method of field decomposition in the eigen-modes in partially intersecting regions, the analytical formula of alternating wake force induced by a bunch train of relativistic charged particles in the lowest pass-band of the axial symmetric dick loaded waveguide has been derived in Ref. [8]. This results in the possibility to perform under-estimations of WFUR photon fluxes from the periodic waveguide. Authors [9, 10] have developed a perturbation method on calculating wakefields, which takes into account all modes induced by electron bunch in axially symmetrical waveguides with periodic perturbed walls. However, the classical round structures with sub-millimeter sizes, which need for generation of hard WFUR [4,5], are difficult to construct and be very expensive. So, in connection with development of the micromechanic technology of fabrication of planar sub-millimeter structures, the analytical methods on calculating wakefields in the rectangular periodic waveguides are relevant topic.

The goal of this paper consists in generalization of the perturbation method [9,10] and expanding it on rectangular waveguides with periodic perturbed walls.

## 2. PROBLEM STATEMENT

Let a bunch of  $N$  ultrarelativistic electrons moves along a planar waveguide with weakly-corrugated metallic both upper and lower surfaces. The wakefield and the corresponding wake force, which acts on the bunch electrons, have to be found.

In order to apply the method of Green's functions for arbitrary distribution of charge in the bunch, we will find the wakefields excited by a point bunch, the current density of which may be written as

$$j_x = 0, \quad j_y = 0, \quad j_z = eN\delta(x-x_0)\delta(y-y_0)\delta(t-z/v), \quad (1)$$

where  $e$  is the electron charge,  $x, y$  are transverse coordinates,  $z$  is a longitudinal coordinate,  $x_0$  and  $y_0$  are the transverse coordinates of the point bunch,  $v$  is velocity of the electrons,  $t$  is a time.

Consider both symmetrical and asymmetrical planar periodic waveguides sketched in Figs.1, 2.

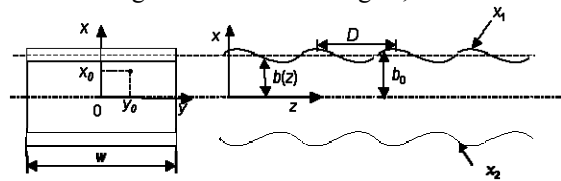


Fig.1. A rectangular waveguide symmetric with the plane  $yz$ ;  $x_{1,2} = \pm b(z)$  are the surface contours,  $D$  is the period

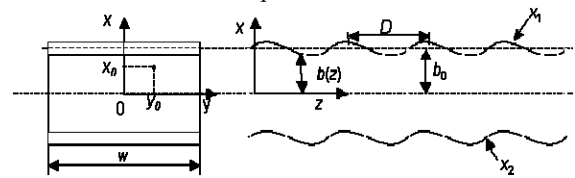


Fig.2. A rectangular waveguide asymmetric with the plane  $yz$ ;  $x_1 = b(z)$ ,  $x_2 = -b(z + D/2)$  are the surface contours

The periodic shape of the corrugated surface can be represented by a Fourier series expansion

$$b(z) = b_0 \left[ 1 + \varepsilon \sum_{p=-\infty}^{\infty} C_p e^{i \frac{2\pi p}{D} z} \right], \quad C_0 = 0, \quad (2)$$

where  $\varepsilon$  is a small parameter ( $0 < \varepsilon \ll 1$ ),  $b_0$  is the average half-distance between the upper and lower surfaces.

## 3. SOLUTION METHOD

### 3.1. EQUATION FOR WAKEFIELD

In periodic structures the electromagnetic fields  $\mathbf{G} = (\mathbf{E}, \mathbf{H})$  excited by a charged particle moving in longitudinal direction with velocity  $v$  may be represented by the Fourier's integral over frequencies  $\omega$  and Floquet series in spatial harmonics

$$\mathbf{G}(\mathbf{r}, t) = \int_{-\infty}^{\infty} \left\{ e^{i\omega \left( t - \frac{z}{v} \right)} \sum_{p=-\infty}^{\infty} \mathbf{G}_{\omega, p}(\mathbf{r}_{\perp}) e^{i \frac{2\pi p}{D} z} \right\} d\omega, \quad (3)$$

where  $\mathbf{r} = (x, y, z)$ ,  $\mathbf{r}_{\perp} = (x, y)$ ,  $\mathbf{G}_{\omega, p}(\mathbf{r}_{\perp})$  is the  $p^{\text{th}}$  spatial harmonic of the Fourier's component.

Using the transformation (3) and expressing the transversal field components through longitudinal, the transversal components of the wake force spatial harmonics may be found from the Maxwell equations as:

$$F_{\omega,p,x} = \frac{ie}{\alpha_p^2} \left\{ \left[ \frac{2\pi p}{D} - \frac{\omega}{v\gamma^2} \right] \frac{\partial E_{\omega,p,z}}{\partial x} - \frac{2\pi p}{D} \frac{v}{c} \frac{\partial H_{\omega,p,z}}{\partial y} \right\}, \quad (4)$$

$$F_{\omega,p,y} = \frac{ie}{\alpha_p^2} \left\{ \left[ \frac{2\pi p}{D} - \frac{\omega}{v\gamma^2} \right] \frac{\partial E_{\omega,p,z}}{\partial y} + \frac{2\pi p}{D} \frac{v}{c} \frac{\partial H_{\omega,p,z}}{\partial x} \right\}, \quad (5)$$

where we have introduced the following definitions:

$$\gamma = 1/\sqrt{1-(v/c)^2}; \quad k = \omega/c; \quad k_p = \frac{2\pi p}{D} - \frac{\omega}{v}; \quad \alpha_p^2 \equiv k^2 - k_p^2,$$

$c$  is the velocity of light.

The spatial harmonics of the longitudinal field components satisfy the transformed wave equations

$$\Delta_{\perp} E_{\omega,p,z} + \alpha_p^2 E_{\omega,p,z} = \frac{4\pi i}{v} \left( \frac{2\pi p}{D} - \frac{\omega}{v\gamma^2} \right) j_{\omega,p,z}, \quad (6)$$

$$\Delta_{\perp} H_{\omega,p,z} + \alpha_p^2 H_{\omega,p,z} = 0, \quad (7)$$

where  $\Delta_{\perp} = \partial^2/\partial x^2 + \partial^2/\partial y^2$ ;  $j_{\omega,p,z}$  is a current density spatial harmonic of the point bunch Eq.(1)

$$j_{\omega,p,z} = \frac{eN}{2\pi} \delta(x-x_0) \delta(y-y_0) \delta_{0,p}. \quad (8)$$

Here  $\delta_{0,p}$  is the Kronecher's symbol.

### 3.2. BOUNDARY CONDITIONS

The equations (6), (7) should be complemented with the boundary conditions for conducting walls:

$$E_z \left( x, y = \pm \frac{w}{2}, z \right) = H_y \left( x, y = \pm \frac{w}{2}, z \right) = 0, \quad (9)$$

$$E_{\tau} \left( x = x_{1,2}, y, z \right) = H_n \left( x = x_{1,2}, y, z \right) = 0. \quad (10)$$

For the symmetrical waveguide (Fig.3) a tangential component of the electric field  $E_{\tau}$  and a normal component of the magnetic field  $H_n$  are expressed in terms of the transverse components  $E_x, E_z$  i  $H_x, H_z$ , accordingly:

$$\pm E_x(x_{1,2}, y, z) \sin \alpha + E_z(x_{1,2}, y, z) \cos \alpha = 0, \quad (11)$$

$$\pm H_x(x_{1,2}, y, z) \cos \alpha - H_z(x_{1,2}, y, z) \sin \alpha = 0, \quad (12)$$

where  $\operatorname{tg} \alpha = db(z)/dz$ .

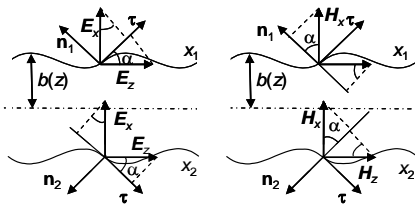


Fig.3. The boundary conditions for the symmetrical waveguide

In case of the asymmetrical structure (Fig.4) the connection between the components  $E_z$  i  $E_x, H_x$  i  $H_z$  is given by:

$$E_x(x_{1,2}, y, z) \sin \alpha + E_z(x_{1,2}, y, z) \cos \alpha = 0, \quad (13)$$

$$\pm H_x(x_{1,2}, y, z) \cos \alpha \mp H_z(x_{1,2}, y, z) \sin \alpha = 0. \quad (14)$$

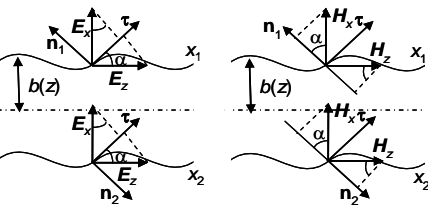


Fig.4. The boundary conditions for the asymmetrical waveguide

Expanding the fields on the surfaces  $x_{1,2}$  in terms of a Taylor's series close to the imaginary planes  $x = \pm b_0$  (see Figs.1,2) and taking into account Eq.(2), it is possible to get new boundary conditions:

– on the plane  $x = b_0$

$$E_{\omega,p,z}(b_0) = -\varepsilon \sum_{q=-\infty}^{\infty} b_0 C_{p-q} \left[ \left( 1 - \frac{2\pi(p-q)k_q}{D\alpha_q^2} \right) \frac{\partial E_{\omega,q,z}(b_0)}{\partial x} + \frac{2\pi(p-q)k}{D\alpha_q^2} \frac{\partial H_{\omega,q,z}(b_0)}{\partial y} \right] \quad (15)$$

$$-\varepsilon^2 \sum_{q=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} b_0^2 C_s C_{p-q-s} \left[ \left( \frac{1}{2} - \frac{2\pi s k_q}{D\alpha_q^2} \right) \frac{\partial^2 E_{\omega,q,z}(b_0)}{\partial x^2} + \frac{2\pi s k}{D\alpha_q^2} \frac{\partial^2 H_{\omega,q,z}(b_0)}{\partial x \partial y} \right] + o(\varepsilon^2),$$

$$\frac{\partial H_{\omega,p,z}(b_0)}{\partial x} = -\frac{k}{k_p} \frac{\partial E_{\omega,p,z}(b_0)}{\partial y} \quad (16)$$

$$-\frac{\alpha_p^2}{k_p} \varepsilon \sum_{q=-\infty}^{\infty} \frac{b_0 C_{p-q}}{\alpha_q^2} \left[ k \frac{\partial^2 E_{\omega,q,z}(b_0)}{\partial x \partial y} \right.$$

$$\left. + k_q \frac{\partial^2 H_{\omega,q,z}(b_0)}{\partial x^2} - \frac{2\pi(p-q)\alpha_q^2}{D} H_{\omega,q,z}(b_0) \right] + o(\varepsilon);$$

– on the plane  $x = -b_0$

$$E_{\omega,p,z}(-b_0) = \pm \varepsilon \sum_{q=-\infty}^{\infty} b_0 C_{p-q} \left[ \left( 1 - \frac{2\pi(p-q)k_q}{D\alpha_q^2} \right) \frac{\partial E_{\omega,q,z}^{(\pm)}(-b_0)}{\partial x} + \frac{2\pi(p-q)k}{D\alpha_q^2} \frac{\partial H_{\omega,q,z}^{(\pm)}(-b_0)}{\partial y} \right] \quad (17)$$

$$-\varepsilon^2 \sum_{q=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} b_0^2 C_s C_{p-q-s} \left[ \left( \frac{1}{2} - \frac{2\pi s k_q}{D\alpha_q^2} \right) \frac{\partial^2 E_{\omega,q,z}^{(\pm)}(-b_0)}{\partial x^2} + \frac{2\pi s k}{D\alpha_q^2} \frac{\partial^2 H_{\omega,q,z}^{(\pm)}(-b_0)}{\partial x \partial y} \right] + o(\varepsilon^2),$$

$$\frac{\partial H_{\omega,p,z}(-b_0)}{\partial x} = -\frac{k}{k_p} \frac{\partial E_{\omega,p,z}(-b_0)}{\partial y} \quad (18)$$

$$\pm \frac{\alpha_p^2}{k_p} \varepsilon \sum_{q=-\infty}^{\infty} \frac{b_0 C_{p-q}}{\alpha_q^2} \left[ k \frac{\partial^2 E_{\omega,q,z}(-b_0)}{\partial x \partial y} \right.$$

$$\left. + k_q \frac{\partial^2 H_{\omega,q,z}(-b_0)}{\partial x^2} - \frac{2\pi(p-q)\alpha_q^2}{D} H_{\omega,q,z}(-b_0) \right] + o(\varepsilon).$$

Here and later on, the upper sign “+” corresponds to the symmetrical structure, the lower sign “-” corresponds to asymmetrical one.

### 3.3. ONE-DIMENSIONAL EQUATIONS

From the boundary conditions (9) it follows that the longitudinal components of the fields may be written in form of Fourier series with the period  $2w$

$$E_{\omega,p,z}(x, y) = \sum_{m=1}^{\infty} E_{\omega,m,p,z}(x) \sin \left[ \frac{\pi m}{w} \left( y + \frac{w}{2} \right) \right], \quad (19)$$

$$H_{\omega,p,z}(x, y) = \sum_{m=1}^{\infty} H_{\omega,m,p,z}(x) \cos \left[ \frac{\pi m}{w} \left( y + \frac{w}{2} \right) \right]. \quad (20)$$

Thus Eqs.(6), (7) may be substituted by the one-dimension ones:

$$\frac{\partial^2 E_{\omega,m,p,z}(x)}{\partial x^2} + g_{m,p}^2 E_{\omega,m,p,z}(x) = -\frac{ia_m \omega}{\gamma^2} \delta(x-x_0) \delta_{0,p}, \quad (21)$$

$$\frac{\partial^2 H_{\omega,m,p,z}(x)}{\partial x^2} + g_{m,p}^2 H_{\omega,m,p,z}(x) = 0, \quad (22)$$

$$a_m \equiv \frac{4eN}{v^2 w} \sin \left[ \frac{\pi m}{w} \left( y_0 + \frac{w}{2} \right) \right], \quad g_{m,p}^2 \equiv \alpha_p^2 - \left( \frac{\pi m}{w} \right)^2.$$

For the boundary conditions (15)-(18) are the expansion in terms of the small parameter  $\varepsilon$ , we will solve Eqs.(21), (22) by a successive approximation method expanding the fields in terms of the powers of  $\varepsilon$ :

$$E_{\omega,m,p,z} = E_{\omega,m,p,z}^{(0)} + E_{\omega,m,p,z}^{(1)} + E_{\omega,m,p,z}^{(2)} + \dots, \quad (23)$$

$$H_{\omega,m,p,z} = H_{\omega,m,p,z}^{(0)} + H_{\omega,m,p,z}^{(1)} + H_{\omega,m,p,z}^{(2)} + \dots$$

where  $E_{\omega,m,p,z}^{(n)}, H_{\omega,m,p,z}^{(n)} \propto \varepsilon^n$ .

## 4. SOLUTION

### 4.1. ZERO ORDER WAKEFIELD

Leaving the zero order terms in both Eqs.(21)-(22) and Eqs.(15)-(18), we can obtain

$$\frac{\partial^2 E_{\omega,m,0,z}^{(0)}(x)}{\partial x^2} + g_{m,0}^2 E_{\omega,m,0,z}^{(0)}(x) = -\frac{ia_m}{\gamma^2} \delta(x - x_0), \quad (24)$$

$$E_{\omega,m,0,z}^{(0)}(\pm b_0) = 0, \quad (25)$$

$$\frac{\partial^2 H_{\omega,m,0,z}^{(0)}(x)}{\partial x^2} + g_{m,0}^2 H_{\omega,m,0,z}^{(0)}(x) = 0, \quad (26)$$

$$\frac{\partial H_{\omega,m,0,z}^{(0)}(\pm b_0)}{\partial x} = 0. \quad (27)$$

Omitting suitable calculations, recovering the time dependence of the fields by the inverse Fourier transform

$$\int_{-\infty}^{\infty} \mathbf{F}_{\omega,p}(\mathbf{r}_\perp) e^{i\omega\tau} d\omega = 2\pi \sum_n \text{Res}[\mathbf{F}_{\omega,p}(\mathbf{r}_\perp)]_n e^{i\omega\tau} \quad (28)$$

(where  $\tau = t - z/v$ ), we obtain for the zero order only the synchronous spatial harmonic of the wake field:

$$E_{0,z}^{(0)}(x, y, \tau) = -\frac{8\pi eN}{b_0 w} \sum_{m=1}^{\infty} \sin \left[ \frac{\pi m}{w} \left( y + \frac{w}{2} \right) \right] \quad (29)$$

$$\times \sin \left[ \frac{\pi m}{w} \left( y_0 + \frac{w}{2} \right) \right] \sum_{s=1}^{\infty} (-1)^s e^{-v\gamma \sqrt{\left( \frac{\pi m}{w} \right)^2 + \left( \frac{\pi s}{2b_0} \right)^2} |\tau|}$$

$$\times \begin{cases} \sin \left[ \frac{\pi s}{2b_0} (x_0 + b_0) \right] \sin \left[ \frac{\pi s}{2b_0} (x - b_0) \right], & x_0 \leq x \leq b_0 \\ \sin \left[ \frac{\pi s}{2b_0} (x_0 - b_0) \right] \sin \left[ \frac{\pi s}{2b_0} (x + b_0) \right], & -b_0 \leq x \leq x_0 \end{cases}$$

$$H_{0,z}^{(0)}(x, y, \tau) = 0. \quad (30)$$

From Eq.(29) follows the well-known result [10], that the zero order wakefield fall off exponentially in the distance  $v|\tau| \sim b_0/\gamma$ , that points out Coulomb's nature of this fields.

### 4.2. FIRST ORDER WAKE FORCE

Leaving the first order infinitesimals in Eqs. (21)-(22), we have

$$\frac{\partial^2 E_{\omega,m,p,z}^{(1)}(x)}{\partial x^2} + g_{m,p}^2 E_{\omega,m,p,z}^{(1)}(x) = 0, \quad (31)$$

$$\frac{\partial^2 H_{\omega,m,p,z}^{(1)}(x)}{\partial x^2} + g_{m,p}^2 H_{\omega,m,p,z}^{(1)}(x) = 0. \quad (32)$$

These equations should be complemented with the suitable boundary conditions:

– on the plane  $x=b_0$

$$E_{\omega,m,p,z}^{(1)}(b_0) = -b_0 \varepsilon C_p \left( 1 - \frac{2\pi p v \gamma^2}{D \omega} \right) \frac{\partial E_{\omega,m,0,z}^{(0)}(b_0)}{\partial x}, \quad (33)$$

$$\frac{\partial H_{\omega,m,p,z}^{(1)}(b_0)}{\partial x} = -\frac{k}{k_p} \frac{\pi m}{w} \left[ E_{\omega,m,p,z}^{(1)}(b_0) + b_0 \varepsilon C_p \frac{\alpha_p^2}{\alpha_0^2} \frac{\partial E_{\omega,m,0,z}^{(0)}(b_0)}{\partial x} \right]; \quad (34)$$

– on the plane  $x=-b_0$

$$E_{\omega,m,p,z}^{(1)}(-b_0) = \pm b_0 \varepsilon C_p \left( 1 - \frac{2\pi p v \gamma^2}{D \omega} \right) \frac{\partial E_{\omega,m,0,z}^{(0)}(-b_0)}{\partial x}, \quad (35)$$

$$\frac{\partial H_{\omega,m,p,z}^{(1)}(-b_0)}{\partial x} = -\frac{k}{k_p} \frac{\pi m}{w} \left[ E_{\omega,m,p,z}^{(1)}(-b_0) \mp b_0 \varepsilon C_p \frac{\alpha_p^2}{\alpha_0^2} \frac{\partial E_{\omega,m,0,z}^{(0)}(-b_0)}{\partial x} \right]. \quad (36)$$

Inserting the found  $E_{\omega,p,z}^{(1)}(x, y)$ ,  $H_{\omega,p,z}^{(1)}(x, y)$  into Eqs.(19), (20), taking into account Eqs. (4), (5), and applying the transform (28), we can obtain the time dependence of the first order  $p^{\text{th}}$  spatial harmonic of the wake force transverse components:

$$F_{p,x}^{(1)}(\mathbf{r}_\perp, \tau) = i \frac{16\pi^2 e^2 N \varepsilon p C_p}{w D} \sum_{m=1}^{\infty} \sin \left[ \frac{\pi m}{w} \left( y + \frac{w}{2} \right) \right] \quad (37)$$

$$\times \sum_{n=0}^{\infty} \frac{(-1)^n}{1 + \delta_{0,n}} X_{m,n}(x) e^{i\omega_{m,p,n}\tau},$$

where  $X_{m,n}(x) = \frac{\sin \left[ \frac{\pi m}{w} \left( y_0 + \frac{w}{2} \right) \right]}{\sinh \left( \frac{2\pi m b_0}{w} \right)} \left\{ \text{sh} \left[ \frac{\pi m}{w} (x_0 + b_0) \right] \cos \left[ \frac{\pi n}{2b_0} (x + b_0) \right] \pm \text{sh} \left[ \frac{\pi m}{w} (x_0 - b_0) \right] \cos \left[ \frac{\pi n}{2b_0} (x - b_0) \right] \right\}$

$$F_{p,y}^{(1)}(\mathbf{r}_\perp, \tau) = i \frac{2\pi^2 e^2 N \varepsilon C_p D}{\gamma^2 b_0 w^2 p} \sum_{m=1}^{\infty} m \cos \left[ \frac{\pi m}{w} \left( y + \frac{w}{2} \right) \right], \quad (38)$$

$$\times \sum_{n=1}^{\infty} (-1)^n n Y_{m,n}(x) e^{i\omega_{m,p,n}\tau}$$

where  $Y_{m,n}(x) = \frac{\sin \left[ \frac{\pi m}{w} \left( y_0 + \frac{w}{2} \right) \right]}{\text{sh} \left( \frac{2\pi m b_0}{w} \right)} \left\{ \text{sh} \left[ \frac{\pi m}{w} (x_0 + b_0) \right] \sin \left[ \frac{\pi n}{2b_0} (x + b_0) \right] \pm \text{sh} \left[ \frac{\pi m}{w} (x_0 - b_0) \right] \sin \left[ \frac{\pi n}{2b_0} (x - b_0) \right] \right\},$

$\omega_{m,p,n}$  are the frequencies of Čerenkov's waves

$$\omega_{m,p,n} = \frac{vD}{4\pi p} \left[ \left( \frac{\pi m}{w} \right)^2 + \left( \frac{2\pi p}{D} \right)^2 + \left( \frac{\pi n}{2b_0} \right)^2 \right], \quad (39)$$

at  $\gamma \rightarrow \infty$ ;  $g_{m,0} = \pm i |\pi m/w|$ .

It should be noted that there is essential anisotropy of transverse components of the alternative wake forces. Thus, the  $y$ -component  $F_{p,y} \sim 1/\gamma^2$  is strongly depressed comparatively with  $F_{p,x}$ , because the electric  $y$ -component of Lorentz's force is compensated by the magnetic one with an accuracy  $1/\gamma^2$ .

### 4.3. SECOND ORDER WAKE FORCE

Electron energy losses connected with wakfield excitation are defined the synchronous harmonic longitudinal component of electric field which appear in the

series (23) for terms of the second order of smallness. To find this field one needs to solve the equation

$$\frac{\partial^2 E_{\omega,m,0,z}^{(2)}(x)}{\partial x^2} + g_{m,0}^2 E_{\omega,m,0,z}^{(2)}(x) = 0 \quad (40)$$

with the boundary conditions on the plane  $x=b_0$

$$E_{\omega,m,0,z}^{(2)}(b_0) = -\varepsilon \sum_{q=-\infty}^{\infty} b_0 C_{-q} \left( \frac{2\pi q}{D} \right) \times \frac{(\gamma^2 - 1) - g_{m,q}^2}{[1 - 2\pi q v \gamma^2 / (D\omega)] g_{m,q}^2} \frac{\partial E_{\omega,m,q,z}^{(1)}(b_0)}{\partial x} \quad (41)$$

and on the plane  $x=-b_0$

$$E_{\omega,m,0,z}^{(2)}(-b_0) = \pm \varepsilon \sum_{q=-\infty}^{\infty} b_0 C_{-q} \left( \frac{2\pi q}{D} \right) \times \frac{(\gamma^2 - 1) - g_{m,q}^2}{[1 - 2\pi q v \gamma^2 / (D\omega)] g_{m,q}^2} \frac{\partial E_{\omega,m,q,z}^{(1)}(-b_0)}{\partial x}. \quad (42)$$

Next, we find the time dependence of the synchronous (zero) spatial harmonic of the longitudinal component of the electrical field in the case  $\gamma \rightarrow \infty$ :

$$F_{0,z}^{(2)}(x,y,\tau) = -\frac{8\pi^2 e^2 N \varepsilon^2 b_0}{w D v} \sum_{m=1}^{\infty} \sin \left[ \frac{\pi m}{w} \left( y + \frac{w}{2} \right) \right] \times \sum_{q=1}^{\infty} q |2C_q|^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{1 + \delta_{0,n}} Z_{m,n}(x) \omega_{m,q,n} \cos(\omega_{m,q,n} \tau), \quad (43)$$

where

$$Z_{m,n}(x) = \frac{\sin \left[ \frac{\pi m}{w} \left( y_0 + \frac{w}{2} \right) \right]}{\text{sh}^2 \left( \frac{2\pi m b_0}{w} \right)} \times \left\{ \left\{ (-1)^n \text{sh} \left[ \frac{\pi m}{w} (x_0 + b_0) \right] \pm \text{sh} \left[ \frac{\pi m}{w} (x_0 - b_0) \right] \right\} \text{sh} \left[ \frac{\pi m}{w} (x + b_0) \right] + \left\{ (-1)^n \text{sh} \left[ \frac{\pi m}{w} (x_0 - b_0) \right] \pm \text{sh} \left[ \frac{\pi m}{w} (x_0 + b_0) \right] \right\} \text{sh} \left[ \frac{\pi m}{w} (x - b_0) \right] \right\}.$$

#### 4.4. GAUSSIAN BUNCH

The wake forces derived above for a point charge may be used as Green's functions to find the force distribution within a bunch with arbitrary charge density as

$$\mathbf{F}_p(\mathbf{r}_\perp, \tau) = \frac{v}{eN} \int_0^\infty d\tau' \iint_{S_\perp} d^2 r'_\perp \rho(\mathbf{r}'_\perp, \tau - \tau') \mathbf{F}_p(\mathbf{r}_\perp, \mathbf{r}'_\perp, \tau'). \quad (44)$$

We will consider a typical case when longitudinal bunch size is considerably more than transversal, and linear charge density is Gaussian distribution

$$\rho(x, y, \tau) = \frac{eN}{\sigma_z \sqrt{2\pi}} e^{-\frac{(v\tau)^2}{2\sigma_z^2}} \delta(x - x_0) \delta(y - y_0), \quad (45)$$

where  $\sigma_z$  is the root-mean-square bunch length.

The wake force components Eqs.(37), (38), (43), and the density Eq.(45) are substituted into the definition Eq.(44). As a result we get the wake force induced by the Gaussian electron bunch:

$$F_{p,x}^{(1)}(x, y, \tau) = i\varepsilon \frac{8\pi^2 e^2 N p C_p}{w D} \sum_{m=1}^{\infty} \sin \left[ \frac{\pi m}{w} \left( y + \frac{w}{2} \right) \right] \times \sum_{n=0}^{\infty} \frac{(-1)^n X_{m,n}(x)}{1 + \delta_{0,n}} e^{-\frac{(v\tau)^2}{2\sigma_z^2}} W \left( \frac{\omega_{m,p,n} \sigma_z}{v\sqrt{2}} - i \frac{v\tau}{\sqrt{2}\sigma} \right), \quad (46)$$

$$F_{p,y}^{(1)}(x, y, \tau) = i\varepsilon \frac{\pi^2 e^2 N D C_p}{\gamma^2 b_0 w^2 p} \sum_{m=1}^{\infty} m \cos \left[ \frac{\pi m}{w} \left( y + \frac{w}{2} \right) \right] \times \sum_{n=1}^{\infty} (-1)^n n Y_{m,n}(x) e^{-\frac{(v\tau)^2}{2\sigma_z^2}} W \left( \frac{\omega_{m,p,n} \sigma_z}{v\sqrt{2}} - i \frac{v\tau}{\sqrt{2}\sigma} \right), \quad (47)$$

$$F_{0,z}^{(2)}(x, y, \tau) = -\varepsilon^2 \frac{4\pi^2 e^2 N b_0}{w D v} \sum_{m=1}^{\infty} \sin \left[ \frac{\pi m}{w} \left( y + \frac{w}{2} \right) \right] \sum_{q=1}^{\infty} q |2C_q|^2 \times \sum_{n=0}^{\infty} \frac{(-1)^n Z_{m,n}(x)}{1 + \delta_{0,n}} \omega_{m,q,n} e^{-\frac{(v\tau)^2}{2\sigma_z^2}} \text{Re} \left\{ W \left( \frac{\omega_{m,q,n} \sigma_z}{v\sqrt{2}} - i \frac{v\tau}{\sqrt{2}\sigma} \right) \right\}, \quad (48)$$

where  $W(z) = e^{-z^2} \text{erfc}(-iz)$  [10].

## 5. EXAMPLE

The WF characteristics of the electron bunch with  $eN=1$  nC which moves along the waveguide with the shape of the upper surface  $b(z)=b_0[1+\varepsilon \cos(2\pi z/D)]$  ( $b_0=D=0.3$  mm,  $w=10b_0$ ,  $\varepsilon=0.1$ ) are represented in Figs.5-8. The structures both symmetrical and asymmetrical are considered. Figs.5,7 and Figs.6,8 correspond to the bunches both long  $\sigma_z=D$  and short  $\sigma_z=0.1D$ , accordingly. The distributions of  $|F_{1,x}|$  in  $x$ -direction (Figs.5,6) point out surface wave character of the excited wakefield. A difference in the  $|F_{1,x}|$  distributions for the symmetrical and asymmetrical waveguides is appreciable only close to the axis.  $F_{1,x}$  equals to zero on the axis of the symmetrical structure unlike the asymmetric one.

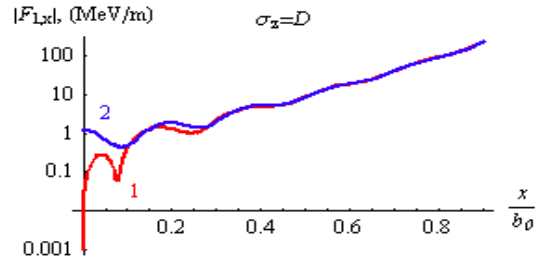


Fig. 5. The wake force transverse distribution for the long bunch. 1-symmetrical, 2-asymmetrical waveguides

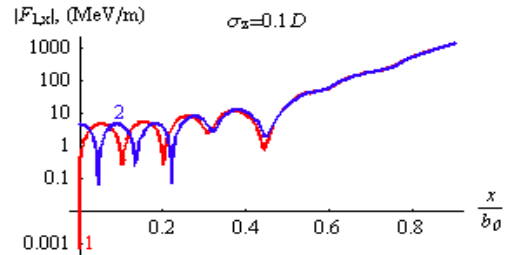


Fig. 6. The wake force transverse distribution for the short bunch. 1-symmetrical, 2-asymmetrical waveguides

In Figs.7,8 we presented the wake force distribution along the bunch. These dependences have been built at  $x_0=0.75 b_0$ ,  $y_0=0$ . It is interesting to note that in case of the long bunch ( $\sigma_z=D$ , Fig.7) the wakefield localizes within the bunch. The first half-bunch loses energy while the second one adsorbs it fully without storing the radiation into the waveguide after the bunch. In this case the wakefield "spot" moves with the bunch velocity, and the absolute value of the transverse component of

the alternating wake force  $F_{1,x}$  reaches the maximum at the maximum of the charge density where the longitudinal component of the synchronous harmonic of the electric field changes its sign. For the short bunch ( $\sigma_z=0.1D$ , Fig.8) the WF distribution is classical, a good part of the bunch loses energy to generate the wakefield (the curve 1,  $F_{0,z}$ ), and the tail particles receive both the maximal alternative transverse momentum in the non-synchronous harmonics (the curve 2,  $F_{1,x}$ ) and acceleration with a high gradient in the synchronous harmonic (the curve 1,  $F_{0,z}$ ).

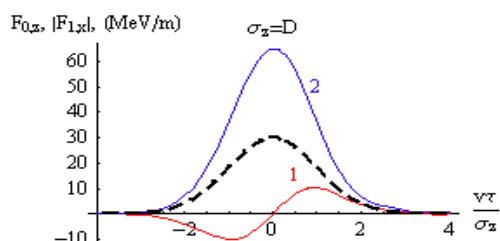


Fig.7. The WF distribution along the long bunch. 1-  $F_{0,z}$ , 2-  $|F_{1,x}|$ , the dash line is the charge distribution

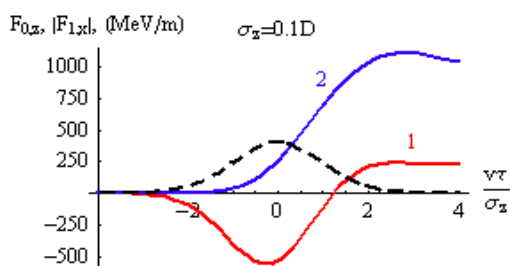


Fig.8. The WF distribution along the short bunch. 1-  $F_{0,z}$ , 2-  $|F_{1,x}|$ , the dash line is the charge distribution

## SUMMARY

The perturbation method for calculation of wakefields [9,10] have been generalized on the rectangular waveguides with periodically perturbed walls with the following main results:

In the symmetrical and asymmetrical structures the wake forces are different only at the symmetry plane.

The transverse components of the alternating wake force are strongly anisotropic.

For a long bunch the wakefield localizes within the bunch. The transverse components of the alternating

wake force are maximal at the maximum of the charge density.

For a short bunch the WF distribution is classical, the maximum of the of the alternating wake force dislocated toward the bunch tail.

The wakefields excited by an electron bunch passing through the sub-millimeter planar periodic waveguide may be used for both a high gradient acceleration and generation of the wakefield undulator radiation.

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## КИЛЬВАТЕРНАЯ СИЛА, ИНДУЦИРОВАННАЯ УЛЬТРАРЕЛЯТИВИСТСКИМ ЭЛЕКТРОННЫМ СГУСТКОМ В ПРЯМОУГОЛЬНОМ СЛАБОФРИРОВАННОМ ВОЛНОВОДЕ

Анатолій Опанасенко

Развит метод расчета кильватерных сил в слабофрированных прямоугольных волноводах. Выведены аналитические формулы для поперечных компонент знакопеременной кильватерной силы.

## КІЛЬВАТЕРНА СИЛА, ІНДУКОВАНА УЛЬТРАРЕЛЯТИВІСТСЬКИМ ЕЛЕКТРОННИМ ЗГУСТКОМ У ПРЯМОКУТНОМУ СЛАБОФРОВАНОМУ ХВИЛЕВОДІ

Анатолій Опанасенко

Розвинуто метод розрахунку кильватерних сил у слабофрованих прямокутних хвилеводах. Виведено аналітичні формули для поперечних компонент знакозмінної кильватерної сили.