

# BOSE-EINSTEIN CONDENSATION OF PARTICLES WITH SPIN

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One of possible ground states and low-lying collective modes of Bose-Einstein condensate (BEC) of atoms with arbitrary spin in a magnetic field is studied using Bogoliubov's model for weakly interacting Bose gas. The equation for the vectorial order parameter, valid at temperatures  $T \rightarrow 0$ , is derived and its specific solution is found. This solution corresponds to the formation of BEC of atoms with a definite spin projection onto magnetic field. We obtain also the necessary condition for thermodynamic stability of such a condensate and the explicit expressions for low-lying collective modes and magnetization.

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## 1. INTRODUCTION

After the first remarkable experiments concerning the observation of BEC in dilute gases of alkali atoms such as  $^{87}\text{Rb}$  [1],  $^{23}\text{Na}$  [2], and  $^7\text{Li}$  [3] the interest to this phenomenon has revived [4,5]. Later, BEC has been also obtained in other atomic species: atomic hydrogen [6], metastable  $^4\text{He}$  [7], and  $^{41}\text{K}$  [8]. The experimental observation of BEC has become possible due to development of laser cooling and trapping techniques [9]. The carried out experiments have proved many predictions of the micro-scopic theory for weakly interacting Bose gas, which originates from the pioneering work of Bogoliubov [10]. Bogoliubov's theory has become almost the first theory in which it was necessary to move essentially from the methods of standard perturbative approach while describing the interaction effects. However, this theory, in its original formulation, did not take into account the internal degrees of freedom of atoms. The spin degrees of freedom have been taken into account for a weakly interacting Bose gas (spinor BEC) in [10-19].

The realization of optical trapping of atomic condensate [20] has stimulated theoretical interest to spinor BEC. Bose condensation in a weakly interacting gas of bosonic atoms has been theoretically studied by many authors both for spin-1 [12-17] and spin-2 [18,19] bosons. These investigations are based on the effective interaction Hamiltonians of two bosons in which the interaction is characterized by a definite number of interaction constants s-wave scattering lengths. The number of scattering lengths is determined by the total spin of two interacting bosons taking into account the symmetry properties of their wave function. For example, in case of spin-1 atoms the interaction Hamiltonian contains two interaction constants [12-17], in case of spin-2 atoms there are three interaction constants [18,19]. Thus, as the spin value of the atoms grows, the number of constants which characterize the interaction of two bosons is increased under phenomenological description. Note that in the mentioned effective Hamiltonians it is difficult to interpret the physical nature of the isolated term of non-relativistic interaction not associated with neither potential nor spin-exchange interactions (see e.g. [18]).

In this paper we study a weakly interacting Bose gas

of particles with arbitrary integer spin  $S$  in a magnetic field (see also [11]). We start from the interaction Hamiltonian for two spin- $S$  bosons. This Hamiltonian is specified by two functions, which describe the potential and spin-exchange interactions of spin- $S$  atoms. According to general rules of quantum mechanics we pass from the pairwise interaction of two bosons to the standard expression for binary interaction of arbitrary number of bosons in the second quantization representation. By solving the problem of multichannel scattering for the considered Hamiltonian we could find, in principle, all scattering lengths in terms of the functions characterizing the potential and spin-exchange interactions. Thereby, it would be possible to obtain the Hamiltonians analogous to the above mentioned effective interaction Hamiltonians (see e.g. [18]). However, the Hamiltonian of the present paper gives a possibility to restrict ourselves by two interaction constants even in the case of arbitrary spin while studying the ground state, stability, and excitations in a weakly interacting gas in the presence of BEC.

## 2. SEPARATION OF A CONDENSATE

To describe the system with a spontaneously broken symmetry we address to the method of quasiaverages [21,22]. According to this method the Gibbs statistical operator is modified so that it possesses the symmetry of the degenerate state. This modification is usually done by introducing the infinitesimal "source"  $v\hat{F}$  ( $v \rightarrow 0$ ), which has the symmetry of phase under consideration into the Gibbs exponent. Then the average value of any physical quantity  $\hat{A}$  is defined as

$$\langle \hat{A} \rangle = \lim_{v \rightarrow 0} \lim_{V \rightarrow \infty} \text{Tr} \hat{w}_v \hat{A}, \quad (1)$$

where the Gibbs statistical operator  $\hat{w}_v$  has the form

$$\hat{w}_v = \exp\left(\Omega_v - \beta(\hat{H} - \mu\hat{N} + v\hat{F})\right). \quad (2)$$

Here  $\beta = 1/T$ ,  $\mu$  are the reciprocal temperature and chemical potential respectively, and  $\hat{H}$ ,  $\hat{N}$  are the system Hamiltonian and the particle number operator. The thermodynamic potential  $\Omega_v$  being a function of thermodynamic variables  $\beta, \mu$  is found from the

normalization condition  $\text{Tr } \hat{w}_v = 1$ . Notice that the limits in (1) are not permutable.

Consider a gas of condensed bosonic atoms with spin  $S$ . The origination of a condensate is accompanied by the gauge symmetry breaking and, therefore, in order to remove this kind of degeneracy we should choose the "source"  $v\hat{F}$  in (2) such that  $[\hat{w}_v, \hat{N}] \neq 0$  ( $\hat{N}$  is the generator of phase transformation),

$$v\hat{F} = v_\alpha \int d^3x (\hat{\psi}_\alpha(\mathbf{x}) + \hat{\psi}_\alpha^\dagger(\mathbf{x})),$$

where  $\hat{\psi}_\alpha^\dagger(\mathbf{x})$ ,  $\hat{\psi}_\alpha(\mathbf{x})$  are the creation and annihilation operators with index  $\alpha$  taking  $2S+1$  values (the summation over the repeated indices is assumed). Then  $\langle \hat{\psi}_\alpha(\mathbf{x}) \rangle = V^{-1/2} \langle \hat{a}_{0\alpha} \rangle \sim 1$  that corresponds to the formation of a condensate of atoms with momenta  $\mathbf{p} = 0$ . The quantity  $\Psi_\alpha = V^{-1/2} \langle \hat{a}_{0\alpha} \rangle$  represents the order parameter usually called the condensate wave function ( $V$  is the volume of the system).

Note that the method of quasiaverages and the spatial correlation decay principle allow to justify the replacement of creation and annihilation operators with  $\mathbf{p} = 0$  by  $c$ -numbers,  $\hat{a}_{0\alpha}, \hat{a}_{0\alpha}^\dagger \rightarrow \sqrt{V}\Psi_\alpha, \sqrt{V}\Psi_\alpha^*$  [21-22] (the condensate separation procedure).

The basic statement of the method of quasiaverages applied to the description of BEC consists in the following [21-22]: the Gibbs statistical operator is replaced by

$$\hat{w}(\Psi) = \exp(\Omega(\Psi) - \beta(\hat{H}(\Psi) - \mu\hat{N}(\Psi))), \quad (3)$$

where  $\Psi \equiv \{\Psi_\alpha, \Psi_\alpha^*\}$  is found from the following equation:

$$\frac{\partial \Omega(\Psi)}{\partial \Psi} = 0. \quad (4)$$

### 3. THE GROUND STATE OF SPIN- $S$ BEC

In this Section we study the ground state properties of spin- $S$  BEC in a magnetic field. In doing so, we start from the Hamiltonian  $\hat{\mathcal{H}} = \hat{H} - \mu\hat{N}$ , which determines the Gibbs statistical operator (3). This Hamiltonian is given by  $\hat{\mathcal{H}} = \hat{\mathcal{H}}_0 + \hat{\mathcal{H}}_p + \hat{\mathcal{H}}_e$ , where

$$\begin{aligned} \hat{\mathcal{H}}_0 &= \sum_1 \hat{a}_{1\alpha}^\dagger [(\varepsilon_1 - \mu)\delta_{\alpha\beta} - \mathbf{h}\mathbf{S}_{\alpha\beta}] \hat{a}_{1\beta}, \quad \varepsilon_p = \frac{p^2}{2M}, \\ \hat{\mathcal{H}}_p &= \frac{1}{2V} \sum_{1234} U(1-3)\delta_{1+2,3+4} \hat{a}_{1\alpha}^\dagger \hat{a}_{2\beta}^\dagger \hat{a}_{3\alpha} \hat{a}_{4\beta}, \quad (5) \\ \hat{\mathcal{H}}_e &= \frac{1}{2V} \sum_{1234} J(1-3)\delta_{1+2,3+4} \hat{a}_{1\alpha}^\dagger \hat{a}_{2\beta}^\dagger \mathbf{S}_{\alpha\gamma} \mathbf{S}_{\beta\delta} \hat{a}_{3\gamma} \hat{a}_{4\delta}, \end{aligned}$$

where  $\mathbf{S}_{\alpha\beta}$  are the spin matrices,  $U(\mathbf{p}_{13})$ ,  $J(\mathbf{p}_{13})$  ( $\mathbf{p}_{13} = \mathbf{p}_1 - \mathbf{p}_3$ ) are the Fourier transforms of the amplitudes of potential and spin-exchange interactions respectively, and  $\mathbf{h} = g\mathbf{H}/S$ , where  $g$  is the Bohr magneton, and  $\mathbf{H}$  is an external magnetic field. For our next calculations it is convenient to introduce the so-

called ladder operators  $\hat{S}_\pm = \hat{S}_x \pm i\hat{S}_y$ . Then their non-zero matrix elements in the representation where  $\hat{S}_z$  is a diagonal matrix,  $\langle \alpha | \hat{S}_z | \alpha' \rangle = \alpha\delta_{\alpha\alpha'}$ , have the form

$$\begin{aligned} \langle \alpha + 1 | \hat{S}_+ | \alpha \rangle &= \sqrt{S(S+1) - \alpha(\alpha+1)}, \\ \langle \alpha - 1 | \hat{S}_- | \alpha \rangle &= \sqrt{S(S+1) - \alpha(\alpha-1)}. \end{aligned} \quad (6)$$

Now we separate the  $\mathbf{p} = 0$  components  $\hat{a}_{0\alpha}$  in the Hamiltonian (the replacement of  $\hat{a}_{0\alpha}$  by  $c$ -numbers,  $\hat{a}_{0\alpha} \rightarrow \sqrt{V}\Psi_\alpha$ ) and keep the terms only up to second order in  $\hat{a}_{\mathbf{p}\alpha}$ . We omit the higher order terms, since they should be taken into account only when examining the interaction between quasiparticles, which we will introduce in the next section. As a result the Hamiltonian takes the form  $\hat{\mathcal{H}} \approx \hat{\mathcal{H}}^{(0)} + \hat{\mathcal{H}}^{(2)}$ . The explicit expression for  $\hat{\mathcal{H}}^{(0)}$ , which contains only  $c$ -numbers  $\Psi_\alpha$  reads

$$\begin{aligned} \frac{\hat{\mathcal{H}}^{(0)}}{V} &= \frac{U(0)}{2} (\Psi^* \Psi)^2 + \frac{J(0)}{2} (\Psi^* \hat{S} \Psi)^2 \\ &\quad - \mathbf{h} \Psi^* \hat{S} \Psi - \mu \Psi^* \Psi, \end{aligned} \quad (7)$$

where

$$\Psi^* \Psi = \Psi_\alpha^* \Psi_\alpha, \quad \Psi^* \hat{S} \Psi = \Psi_\alpha^* \hat{S}_{\alpha\beta} \Psi_\beta. \quad (8)$$

The explicit form for  $\hat{\mathcal{H}}^{(2)}$  will be written in the next section.

Next, using the normalization condition  $\text{Tr } \hat{w} = 1$ , we immediately find the density of thermodynamic potential  $\omega = \Omega T / V$  in the main approximation ( $T \rightarrow 0$ ) of the model for weakly interacting Bose gas,  $\omega = \frac{U(0)}{2} (\Psi^* \Psi)^2 + \frac{J(0)}{2} (\Psi^* \hat{S} \Psi)^2 - \mathbf{h} \Psi^* \hat{S} \Psi - \mu \Psi^* \Psi$ .

Therefore, Eq.(5) for  $\Psi_\alpha$  takes the form

$$\begin{aligned} \mu \Psi_\alpha - U(0)(\Psi^* \Psi) \Psi_\alpha \\ - J(0)(\Psi^* \hat{S} \Psi) \mathbf{S}_{\alpha\beta} \Psi_\beta + \mathbf{h} \mathbf{S}_{\alpha\beta} \Psi_\beta = 0. \end{aligned} \quad (9)$$

If to introduce the normalized spin functions  $\zeta_\alpha$ ,  $\Psi_\alpha = \sqrt{n} \zeta_\alpha$ , where  $n = \Psi_\alpha^* \Psi_\alpha$  is the condensate density and  $\zeta_\alpha^* \zeta_\alpha = 1$ , then the latter equation is written as

$$\begin{aligned} \mu \zeta_\alpha - n U(0) \zeta_\alpha \\ - n J(0) (\zeta^* \hat{S} \zeta) \mathbf{S}_{\alpha\beta} \zeta_\beta + \mathbf{h} \mathbf{S}_{\alpha\beta} \zeta_\beta = 0. \end{aligned} \quad (10)$$

Its solution  $\zeta_\alpha^{(m)}$ , which is an eigenfunction of  $\hat{S}_z$ ,  $(\hat{S}_z)_{\alpha\beta} \zeta_\beta^{(m)} = m \zeta_\alpha^{(m)}$ , is of the form

$$\zeta_\alpha^{(m)} = \delta_{\alpha m}. \quad (11)$$

Assuming the vector  $\mathbf{h}$  directed along  $z$ -axis,  $\mathbf{h} = (0, 0, h)$  and taking into account that  $\hat{S}_z$  is a diagonal matrix, whereas  $\hat{S}_\pm$  have no diagonal matrix elements in the considered representation of spin matrices, one finds from Eq. (3.8)

$$n = \frac{\mu + mh}{U(0) + m^2 J(0)}. \quad (12)$$

Formulae (11), (12) result in the following expression for the density of equilibrium thermodynamic potential:

$$\omega = -\frac{1}{2} \frac{(\mu + mh)^2}{U(0) + m^2 J(0)}. \quad (13)$$

We are now in a position to study the stability of the possible ground states (11). In the considered approximation the thermodynamic potential of normal state is zero (the order parameter  $\Psi_\alpha$  vanishes). Therefore, for stability of the studied ground state the density of thermodynamic potential must be negative,  $\omega < 0$  and, consequently, according to (13), we can write the necessary condition of thermodynamic stability,

$$U(0) + m^2 J(0) > 0. \quad (14)$$

Let us find now such spin projections  $m$ , which correspond to minimum of the potential (13). For simplicity, we study the case of  $h = 0$  (or sufficiently weak  $h$ ). Then

$$\omega = -\frac{\mu^2}{2(U(0) + m^2 J(0))} < 0. \quad (15)$$

As it can be easily seen that in contrast to the usual Bogoliubov theory, where  $U(0) > 0$  (the necessary condition of stability), the case  $U(0) < 0$  is also permissible. Therefore, we have the following three situations:

1)  $U(0) > 0$ ,  $J(0) > 0$ . In this case the requirement (14) is automatically fulfilled. The density of thermodynamic potential (15) has a minimum at  $m = 0$ , in which  $\omega = -\mu^2 / 2U(0)$ . We call this case as antiferromagnetic ordering.

2)  $U(0) > 0$ ,  $J(0) < 0$  but such that the requirement (14) is fulfilled. The minimum of  $\omega$  (see (15)) is reached for  $m_{\min} = \pm(m_c - 1)$ , where

$$m_c = [-U(0)/J(0)]^{1/2}, \quad m_c \leq S + 1 \quad (16)$$

(the square brackets denote an integer part). This case corresponds to ferromagnetic ordering.

3)  $U(0) < 0$ ,  $J(0) > 0$  but again such that  $U(0) + m^2 J(0) > 0$ . Here the minimum of  $\omega$  is given by the spin projections  $m_{\min} = \pm(m_c + 1)$ , where  $m_c$

is also defined by (16) but with  $U(0) < 0$ ,  $J(0) > 0$ . This case also corresponds to ferromagnetic ordering.

#### 4. LOW-LYING COLLECTIVE MODES

Here we shall obtain the excitation spectra for spin- $S$  BEC employing the well-known diagonalization procedure (Bogoliubov's  $u-v$  transformations [10]) for the Hamiltonian quadratic in creation and annihilation operators.

The part of the spin-exchange interaction Hamiltonian (see (5)), which is quadratic in  $\hat{a}_{\mathbf{p}\alpha}$  ( $\mathbf{p} \neq 0$ ) has the form

$$\begin{aligned} \hat{\mathcal{H}}_e^{(2)} = & J(0) \Psi^* \hat{\mathbf{S}} \Psi \sum_{\mathbf{p}} \hat{a}_{\mathbf{p}}^+ \hat{\mathbf{S}} \hat{a}_{\mathbf{p}} + \\ & + \frac{1}{2} \sum_{\mathbf{p}} J(\mathbf{p}) \left[ (\hat{a}_{\mathbf{p}}^+ \hat{\mathbf{S}} \Psi) (\hat{a}_{-\mathbf{p}}^+ \hat{\mathbf{S}} \Psi) + (\hat{a}_{\mathbf{p}}^+ \hat{\mathbf{S}} \Psi) (\Psi^* \hat{\mathbf{S}} \hat{a}_{\mathbf{p}}) + \text{h.c.} \right] \end{aligned} \quad (17)$$

where we have used the notations (8). Taking into account that

$$\hat{S}_x = \frac{1}{2} (\hat{S}_+ + \hat{S}_-), \quad \hat{S}_y = -\frac{i}{2} (\hat{S}_+ - \hat{S}_-)$$

and employing (6) for nonzero matrix elements of  $\hat{S}_\pm$  as well as the explicit expression for the condensate wave function  $\Psi_\alpha^{(m)} = \sqrt{n} \delta_{\alpha m}$ , one finds

$$\begin{aligned} (\Psi^* \hat{\mathbf{S}} \Psi) (\hat{a}_{\mathbf{p}}^+ \hat{\mathbf{S}} \hat{a}_{\mathbf{p}}) &= nm \sum_{\alpha} \alpha \hat{a}_{\mathbf{p}\alpha}^+ \hat{a}_{\mathbf{p}\alpha}, \\ (\hat{a}_{\mathbf{p}}^+ \hat{\mathbf{S}} \Psi) (\hat{a}_{-\mathbf{p}}^+ \hat{\mathbf{S}} \Psi) &= nm^2 \hat{a}_{\mathbf{p}m}^+ \hat{a}_{-\mathbf{p}m}^+ + \\ &+ \frac{n}{2} S_m S_{-m} \left[ \hat{a}_{\mathbf{p}m-1}^+ \hat{a}_{-\mathbf{p}m+1}^+ + \hat{a}_{\mathbf{p}m+1}^+ \hat{a}_{-\mathbf{p}m-1}^+ \right], \\ (\hat{a}_{\mathbf{p}}^+ \hat{\mathbf{S}} \Psi) (\Psi^* \hat{\mathbf{S}} \hat{a}_{\mathbf{p}}) &= nm^2 \hat{a}_{\mathbf{p}m}^+ \hat{a}_{\mathbf{p}m} + \\ &+ \frac{n}{2} \left[ S_{-m}^2 \hat{a}_{\mathbf{p}m-1}^+ \hat{a}_{\mathbf{p}m-1} + S_m^2 \hat{a}_{\mathbf{p}m+1}^+ \hat{a}_{\mathbf{p}m+1} \right], \end{aligned}$$

where

$$S_m = \sqrt{S(S+1) - m(m+1)}.$$

As a result the "Hamiltonian" (17) takes the form

$$\begin{aligned} \hat{\mathcal{H}}_e^{(2)} = & J(0) nm \sum_{\mathbf{p}} \left[ (m-1) \hat{a}_{\mathbf{p}m-1}^+ \hat{a}_{\mathbf{p}m-1} + m \hat{a}_{\mathbf{p}m}^+ \hat{a}_{\mathbf{p}m} + (m+1) \hat{a}_{\mathbf{p}m+1}^+ \hat{a}_{\mathbf{p}m+1} \right] + \\ & + J(0) nm \sum_{\mathbf{p}, \alpha} \alpha \hat{a}_{\mathbf{p}\alpha}^+ \hat{a}_{\mathbf{p}\alpha} + \frac{n}{2} \sum_{\mathbf{p}} J(\mathbf{p}) m^2 \left[ \hat{a}_{\mathbf{p}m}^+ \hat{a}_{-\mathbf{p}m}^+ + 2 \hat{a}_{\mathbf{p}m}^+ \hat{a}_{\mathbf{p}m} + \hat{a}_{-\mathbf{p}m} \hat{a}_{\mathbf{p}m} \right] + \\ & + \frac{n}{2} \sum_{\mathbf{p}} J(\mathbf{p}) \left[ S_m S_{-m} \left( \hat{a}_{\mathbf{p}m-1}^+ \hat{a}_{-\mathbf{p}m+1}^+ + \hat{a}_{-\mathbf{p}m+1} \hat{a}_{\mathbf{p}m-1} \right) + S_{-m}^2 \hat{a}_{\mathbf{p}m-1}^+ \hat{a}_{\mathbf{p}m-1} + S_m^2 \hat{a}_{\mathbf{p}m+1}^+ \hat{a}_{\mathbf{p}m+1} \right] \end{aligned}$$

Here the summation index  $\alpha$  in the second term takes all the values of spin projections except  $m-1$ ,  $m$ , and  $m+1$ . We have separated these three spin projections and written them as the first term of the above "Hamiltonian". In a similar manner assuming the vector  $\mathbf{h}$  directed along  $z$ -axis,  $\mathbf{h} = (0, 0, h)$ , and eliminating the chemical potential  $\mu$  by using (12), we can find the explicit expressions for  $\hat{\mathcal{H}}_0^{(2)}$ ,  $\hat{\mathcal{H}}_{\mathbf{p}}^{(2)}$  and thereby the total Hamiltonian quadratic in creation and annihilation operators:

$$\hat{\mathcal{H}}^{(2)} = \hat{\mathcal{H}}_\alpha^{(2)} + \hat{\mathcal{H}}^{(2)}(m) + \hat{\mathcal{H}}^{(2)}(m-1, m+1), \quad (18)$$

where

$$\begin{aligned} \hat{\mathcal{H}}_\alpha^{(2)} &= \sum_{\mathbf{p}, \alpha} \left[ \varepsilon_{\mathbf{p}} - J(0) nm (m - \alpha) + h(m - \alpha) \right] \hat{a}_{\mathbf{p}\alpha}^+ \hat{a}_{\mathbf{p}\alpha}, \\ \hat{\mathcal{H}}^{(2)}(m) &= \sum_{\mathbf{p}} \left[ \varepsilon_{\mathbf{p}} + g_m(\mathbf{p}) \right] \hat{a}_{\mathbf{p}m}^+ \hat{a}_{\mathbf{p}m} + \\ &+ \frac{1}{2} \sum_{\mathbf{p}} g_m(\mathbf{p}) \left[ \hat{a}_{\mathbf{p}m}^+ \hat{a}_{-\mathbf{p}m}^+ + \hat{a}_{\mathbf{p}m} \hat{a}_{-\mathbf{p}m} \right], \end{aligned} \quad (19)$$

$$\begin{aligned} \hat{\mathcal{H}}^{(2)}(m-1, m+1) = & \sum_{\mathbf{p}} [\varepsilon_p - h + \beta_m(\mathbf{p})] \hat{a}_{\mathbf{p}m+1}^+ \hat{a}_{\mathbf{p}m+1} \\ & + \sum_{\mathbf{p}} [\varepsilon_p + h + \beta_{-m}(\mathbf{p})] \hat{a}_{\mathbf{p}m-1}^+ \hat{a}_{\mathbf{p}m-1} \\ & + \sum_{\mathbf{p}} \alpha_m(\mathbf{p}) [a_{\mathbf{p}m-1}^+ a_{-\mathbf{p}m+1}^+ + a_{\mathbf{p}m-1} a_{-\mathbf{p}m+1}]. \end{aligned}$$

The introduced quantities  $\alpha_m(\mathbf{p})$ ,  $\beta_m(\mathbf{p})$ ,  $g_m(\mathbf{p})$  are given by

$$\begin{aligned} \alpha_m(\mathbf{p}) &= \frac{n}{2} J(\mathbf{p}) S_m S_{-m}, \\ \beta_m(\mathbf{p}) &= \frac{n}{2} J(\mathbf{p}) S_m^2 + nJ(0)m, \\ g_m(\mathbf{p}) &= n(U(\mathbf{p}) + m^2 J(\mathbf{p})). \end{aligned} \quad (20)$$

Now we are in a position to carry out the diagonalization of the total Hamiltonian (18). In this connection let us note that the "Hamiltonians" (19) contain the creation and annihilation operators with not overlapping sets of indices  $\alpha$ ,  $m-1$ ,  $m$ ,  $m+1$  ( $\alpha \neq m \pm 1, m$ ).

Therefore, we can perform their diagonalization independently. The evidence of this statement is also associated with the fact that (18) can be considered as the Hamiltonian of the system consisting of four kinds ( $\alpha, m, m \pm 1$ ) of noninteracting particles.

The "Hamiltonian"  $\hat{\mathcal{H}}_{\alpha}^{(2)}$  has already a diagonal form with the following spectrum:

$$\omega_{m,\alpha}(\mathbf{p}) = \varepsilon_p - J(0)nm(m-\alpha) + h(m-\alpha). \quad (21)$$

To carry out the diagonalization of  $\hat{\mathcal{H}}^{(2)}(m-1, m+1)$  we introduce the creation and annihilation operators  $\hat{b}_{\mathbf{p}m+\sigma}$  ( $\sigma = \pm 1$ ),

$$\begin{aligned} \hat{a}_{\mathbf{p}m+\sigma} &= u_{m,\sigma}(\mathbf{p}) \hat{b}_{\mathbf{p}m+\sigma} + v_{m,\sigma}(\mathbf{p}) \hat{b}_{-\mathbf{p}m-\sigma}^+, \\ \hat{a}_{\mathbf{p}m+\sigma}^+ &= u_{m,\sigma}^*(\mathbf{p}) \hat{b}_{\mathbf{p}m+\sigma}^+ + v_{m,\sigma}^*(\mathbf{p}) \hat{b}_{-\mathbf{p}m-\sigma} \end{aligned}$$

in terms of which this "Hamiltonian" has the diagonal form,

$$H^{(2)}(m+\sigma) = \sum_{\mathbf{p},\sigma} \omega_{m,\sigma}(\mathbf{p}) b_{\mathbf{p}m+\sigma}^+ b_{\mathbf{p}m+\sigma} + E_0,$$

where  $\omega_{m,\sigma}(\mathbf{p})$  and  $E_0$  are the excitation spectra and the ground state energy respectively. In order that the introduced operators meet the canonical commutation relations, the functions  $u_{m,\sigma}(\mathbf{p})$ ,  $v_{m,\sigma}(\mathbf{p})$  must obey the relationships:

$$\begin{aligned} |u_{m,\sigma}(\mathbf{p})|^2 - |v_{m,\sigma}(\mathbf{p})|^2 &= 1, \\ u_{m,\sigma}(\mathbf{p})v_{m,-\sigma}(-\mathbf{p}) - v_{m,\sigma}(\mathbf{p})u_{m,-\sigma}(-\mathbf{p}) &= 0. \end{aligned}$$

Simple mathematical manipulations (see e.g. [22]) result to the following expression for low-lying collective modes:

$$\omega_{m,\sigma}(\mathbf{p}) = nm\sigma \left( J(0) - \frac{1}{2} J(\mathbf{p}) \right) - \sigma h \pm \quad (22)$$

$$\pm \left[ \varepsilon_p^2 + \varepsilon_p n J(\mathbf{p}) (S(S+1) - m^2) + \left( \frac{nJ(\mathbf{p})m}{2} \right)^2 \right]^{\frac{1}{2}},$$

moreover,

$$\begin{aligned} u_{m,\sigma}(\mathbf{p}) &= \frac{\alpha_m(\mathbf{p})}{\sqrt{\alpha_m^2(\mathbf{p}) - (\omega_{m,\sigma}(\mathbf{p}) - \gamma_{m,\sigma})^2}}, \\ v_{m,\sigma}(\mathbf{p}) &= \frac{\omega_{m,\sigma}(\mathbf{p}) - \gamma_{m,\sigma}}{\sqrt{\alpha_m^2(\mathbf{p}) - (\omega_{m,\sigma}(\mathbf{p}) - \gamma_{m,\sigma})^2}}, \end{aligned}$$

where

$$\gamma_{m,\sigma} = \varepsilon_p + \beta_{m\sigma}(\mathbf{p}) - \sigma h \quad (23)$$

(here  $\beta_{m\sigma}(\mathbf{p})$  depends on the product  $m\sigma$ ). In fact, the functions  $u_{m,\sigma}(\mathbf{p})$ ,  $v_{m,\sigma}(\mathbf{p})$  do not depend on  $\sigma$  because, as it can be easily shown, the quantity  $\omega_{m,\sigma}(\mathbf{p}) - \gamma_{m,\sigma}$  is independent of  $\sigma$ . The sign plus before the square root in (22) corresponds (for  $\sigma = 1$ ) to the wave, which propagates in one direction, whereas the sign minus corresponds (for  $\sigma = -1$ ) to the wave propagating in opposite direction. Notice that the obtained spectra as well as (21) contain only the amplitude of spin-exchange interaction and does not depend on the amplitude of potential interaction.

When  $m = 0$  (the antiferromagnetic ordering), the excitation spectrum (22) takes the form

$$\omega(\mathbf{p}) = \sqrt{\varepsilon_p^2 + \varepsilon_p n J(\mathbf{p}) S(S+1)} \pm h.$$

In this case for  $h = 0$  and  $\mathbf{p} \rightarrow 0$  we have

$$\omega(p) = cp, \quad c = \sqrt{\frac{n}{2M} J(0) S(S+1)}.$$

In ferromagnetic case (when  $m = S$ ) the excitation spectrum is of the form

$$\omega(\mathbf{p}) = \varepsilon_p + \frac{nJ(\mathbf{p})S}{2} (1 - \sigma) + nJ(0)S\sigma - \sigma h.$$

In a similar manner we can perform the diagonalization of  $\hat{\mathcal{H}}^{(2)}(m)$  and thereby to obtain another spectrum of excitations. This spectrum depends on the amplitudes of spin-exchange and potential interactions,

$$\omega_m(\mathbf{p}) = \sqrt{\varepsilon_p^2 + 2\varepsilon_p n (U(\mathbf{p}) + m^2 J(\mathbf{p}))}. \quad (24)$$

The corresponding functions  $u_{m\mathbf{p}}$  and  $v_{m\mathbf{p}}$  can be easily found and have the form,

$$u_m(\mathbf{p}) = \frac{\varepsilon_p + \omega_m(\mathbf{p})}{2\sqrt{\varepsilon_p \omega_m(\mathbf{p})}}, \quad v_m(\mathbf{p}) = \frac{\omega_m(\mathbf{p}) - \varepsilon_p}{2\sqrt{\varepsilon_p \omega_m(\mathbf{p})}}.$$

When  $J(\mathbf{p}) = 0$  the excitation spectrum (24) coincides with those, obtained Bogoliubov [10]. At small  $\mathbf{p}$  the spectrum has the phonon behavior,

$$\omega_m(\mathbf{p}) = cp, \quad c = \sqrt{\frac{n}{M} (U(0) + m^2 J(0))},$$

In this formula and in (24) we have chosen the arithmetic value of the square root.

The magnetization is defined as  $\mathbf{M} = g \text{Tr} \hat{w}(\Psi) \hat{\mathbf{S}}$ ,

where  $g$  is the Bohr magneton,  $\hat{\mathbf{S}}$  is the spin operator in the second quantization representation and the Gibbs statistical operator  $\hat{w}(\Psi)$  is given by (3). Up to the second order in  $\hat{a}_{\mathbf{p}\alpha}$  this magnetization is of the form

$$\begin{aligned} \mathbf{M} &= gVm + gm \sum_{\mathbf{p} \neq 0} (|u_m(\mathbf{p})|^2 + |v_m(\mathbf{p})|^2) f_m(\mathbf{p}) \\ &+ gm \sum_{\mathbf{p} \neq 0} (|u_{m,1}(\mathbf{p})|^2 + |v_{m,1}(\mathbf{p})|^2) (f_{m+1}(\mathbf{p}) + f_{m-1}(\mathbf{p})) \end{aligned}$$

$$+ g \sum_{\mathbf{p} \neq 0} (f_{m+1}(\mathbf{p}) + f_{m-1}(\mathbf{p})) + g \sum_{\mathbf{p} \neq 0} \sum_{\alpha \neq m, m \pm 1} \alpha f_{\alpha}(\mathbf{p}),$$

where we have taken into account the fact that the functions  $u_{m,\sigma}(\mathbf{p})$ ,  $v_{m,\sigma}(\mathbf{p})$  do not depend on  $\sigma$ ,  $v_{m-1}(\mathbf{p}) = v_{m+1}(\mathbf{p})$ ,  $u_{m-1}(\mathbf{p}) = u_{m+1}(\mathbf{p})$ . The functions  $f_m(\mathbf{p})$ ,  $f_{m+1}(\mathbf{p})$ ,  $f_{m-1}(\mathbf{p})$ ,  $f_{\alpha}(\mathbf{p})$  represent the boson distribution functions of quasiparticles with chemical potential  $\mu = 0$  and excitation spectra  $\omega_m(\mathbf{p})$ ,  $\omega_{m,1}(\mathbf{p})$ ,  $\omega_{m,-1}(\mathbf{p})$ ,  $\omega_{m,\alpha}(\mathbf{p})$  respectively (see (24), (22), (21)).

In conclusion, we have studied BEC of atoms with arbitrary spin in a magnetic field on the basis of the model for weakly interacting Bose gas. We have derived the equation, which determines the ground state of spin- $S$  BEC at  $T \rightarrow 0$  and found its specific solution. This solution corresponds to the formation of BEC of spin- $S$  atoms with a definite spin projection  $m$  that also holds for an ideal Bose gas [11]. The explicit expression for thermodynamic potential as a function of chemical potential and spin projection has been obtained. It generalizes the thermodynamic potential for weakly interacting Bose gas to the case when both potential and spin-exchange interactions act between bosons. The thermodynamic stability of the obtained ground state has been studied and the spin projections which give a minimum of thermodynamic potential have been found. These projections are determined by the integral part of the ratio of the potential and spin-exchange interaction amplitudes. The expressions for low-lying collective modes corresponding to the ground state (11) as well as the magnetization have been obtained. Notice that Eq. (10) for the order parameter has also other solutions different from (11). The goal of our present research is to seek such solutions.

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## БОЗЕ-ЭЙНШТЕЙНОВСКАЯ КОНДЕНСАЦИЯ ЧАСТИЦ СО СПИНОМ

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На основе модели Боголюбова слабовзаимодействующего бозе-газа изучены одно из возможных основных состояний и низколежащие коллективные возбуждения бозе-эйштейновского конденсата (БЭК) атомов с произвольным целым спином в магнитном поле. Получено уравнение для векторного параметра порядка, справедливое при температурах  $T \rightarrow 0$ , и найдено его частное решение. Это решение соответствует образованию БЭК атомов с определенной проекцией спина на направление магнитного поля. Найдены также необходимое условие термодинамической устойчивости такого конденсата и выражения для спектров элементарных возбуждений и намагниченности.

## БОЗЕ-ЭЙНШТЕЙНІВСЬКА КОНДЕНСАЦІЯ ЧАСТИНОК ЗІ СПІНОМ

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На основі моделі Боголюбова слабо неідеального бозе-газу вивчено один із можливих основних станів і колективні збудження бозе-эйштейнівського конденсату (БЕК) атомів із довільним цілим спіном у магнітному полі. Отримано рівняння для векторного параметра порядку, справедливе при температурах  $T \rightarrow 0$ , та знайдено його частковий розв'язок. Цей розв'язок відповідає утворенню БЕК атомів із визначеною проекцією спину на магнітне поле. Одержано також необхідну умову термодинамічної стійкості такого конденсату та вирази для спектрів елементарних збуджень і намагніченості.