THE INFLUENCE OF INSIDENT AND REFLECTED POTENTIAL WAVE ON THE POINT VORTEX EVOLUTION NEAR A SOLID SURFACE

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The influence of oscillation potential modes upon the point vortex movement near a solid wall is studied. The equations describing the point vortex movement in the given field of a potential wave in the presence of a solid wall are obtained. The character of the vortex movement is shown to change qualitatively under the influence of a potential wave. All the possible modes of vortex movement under the influence of a potential wave are analyzed.

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1. INTRODUCTION

In all the hydrodynamic spheres there are two types of extremely important objects. They are wave and vortex. Both objects are claimed to explain a lot of phenomena in hydrodynamic media. The problem is that vortexes, unlike waves, are the solutions of nonlinear hydrodynamic equations. Linear approximation for vortexes has little content. This causes a number of obstacles for their introduction and studying. The latter gets even more complicated because the fruitful idealization of nonlinear waves – „one-dimensionalization" – doesn't work for vortexes. Vortexes are inevitably many-dimensional. The closest to quasiparticles and well-studied vortexes appear in two-dimensional, ideal hydrodynamics. They are point vortexes. The point vortexes movement equations in the Hamiltonian form were obtained by Kirchhoff. The evolution of interacting point vortexes is studied rather fully. Two vortexes movement was studied as early as in the works (see, for example, [1]). Three point vortexes evolution was analyzed in detail in the papers [2,3] (see also [4,5]). Non-integrability of four or more vortexes problem, in case of general position and chaos initiation, is proved in the papers [6-9]. The studying of liquid boundaries was started rather long ago by Helmholtz. Today the properties of waves, on the one hand, and vortexes, on the other hand, have been studied rather deeply. But in hydrodynamic media waves and vortexes are usually present simultaneously. So the research on the mutual influence of the main hydrodynamic objects is of great importance. The base of such influence studies is made in Lighthill works [11,12]. They explore the potential waves generation by vortex movements. The research of the reverse influence of potential waves on the vortexes evolution has been started comparatively recently [14,13]. It has turned out that the potential waves cause the character of point vortexes evolution to change qualitatively. The example of it can be the phenomenon of vortexes (with the same vorticity sign) collapse under the influence of potential oscillations, even with small amplitude [14,13]. Such a collapse of point vortexes with shared vorticity sign is impossible without any potential oscillations.

The paper studies the influence of oscillation potential modes upon the point vortex movement near a solid wall. It is shown that the character of the vortex movements changes qualitatively. All the possible modes of vortex movement under the influence of a potential wave are analyzed.

2. VORTEX MOVEMENT EQUATIONS IN THE FIELD OF A POTENTIAL WAVE NEAR A WALL

Let us first discuss the velocity field of a potential wave in the presence of a solid wall in a compressed liquid. Let us consider the potential waves amplitudes small. In this case, there is a potential mode in the compressed liquid – sound waves (see, for example, [15]). In the presence of a wall two the most interesting stationary situations of sound wave spreading are realized. They are its spreading parallel to the wall and falling to the media boundary at some angle. In this paper we shall consider the field of falling and reflected waves from the wall as the most general case. Let the liquid occupy half-space $y > 0$, and the impermeable boundary $y = 0$. Without loss of generality, let us consider potential waves spreading in the plane (x, y).

Then the velocity field potential is determined by the falling and reflected wave from the solid boundary $y = 0$ and has the following form.

$$\varphi_s = a_0 \cos\left(y k_y\right) \cos(k_x x - \Omega t) \tag{1}$$

The angle α of the wave falling on the medium boundary $\tan(\alpha) = k_y/k_x$ and $\Omega = ck$.

Now let us consider the movement equations for a point vortex, which is situated near the wall under the influence of the given potential wave. To derive these equations, we use the approach, developed in [13,14]. It is generally known that the vortex is trapped in the liquid, and so, the vortex velocity coincides with the liquid velocity (V_x, V_y) in the point of its location, which means that

$$\frac{dx_1}{dt} = V_x\Big|_{x=x_1 y=y_1},$$

$$\frac{dy_1}{dt} = V_y\Big|_{x=x_1 y=y_1}.$$

Here (x_1, y_1) is the vortex location in the positive half-space $y > 0$. According to Helmholtz theorem, the velocity field can be split to a sum of vortex and potential components $\vec{V} = \vec{v}_v + \vec{v}_p$. The potential component consists of a given external flow and waves induced by the vortex movements. But the contributions connected with the induced potential waves are proportional to square Mach number (see. [11]) and can be neglected in the general approximation [13,14].

The vortex velocity field for a single vortex near a solid wall is well known [10] $\vec{v}_v = (\Gamma/4\pi y, 0)$, and the potential component of the velocity $\vec{v}_p = \nabla\varphi$ (the velocity potential is shown above). Then the equations of the vortex movement in the field of the given falling and reflected wave (1) can be written in the following form

$$\frac{dX}{d\tau} = \frac{\delta}{Y} - 1 - \varepsilon\cos\left(Y\right)\sin(X) \tag{2}$$

$$\frac{dY}{d\tau} = \Delta\sin\left(Y\right)\cos(X) \tag{3}$$

where dimensionless variables $\tau = \Omega t$, $Y = yk_y$ are introduced and the transition to moving base is made $X = k_x x - \tau$. In this case, dimensionless parameters $\varepsilon = \dfrac{a_0 k_x^2}{\Omega}$, $\Delta = \dfrac{a_0 k_y^2}{\Omega}$ and $\delta = \dfrac{\Gamma k_x k_y}{4\pi ck}$, which characterize the wave velocity components amplitudes and the vortex intensity respectively, appear.

In the absence of potential oscillations, this equations system is reduced to the well-known vortex movement equations near a solid wall (see, for example, [10]). The character of vortex movement in this case is very simple. The vortex moves along the wall preserving the distance to it and with constant speed $V_v = \Gamma/4\pi y_0$, depending on the vorticity value Γ and the initial distance to the wall. Using the equations obtained, let us pass to the analysis of the possible vortex movement modes under the influence of a potential wave near a solid wall.

3. VORTEX MOVEMENT MODES IN THE FIELD OF FALLING AND REFLECTED SOUND WAVES

Let us qualitatively analyze the equations system (2), (3). The coordinates of fixed points are defined by the zeros of the right parts in these equations

$$\frac{\delta}{Y} - 1 - \varepsilon\cos\left(Y\right)\sin(X) = 0, \tag{4}$$

$$\Delta\sin\left(Y\right)\cos(X) = 0. \tag{5}$$

The equation (5) has two types of solutions and, so, there appear two sets of fixed points. We shall mark their coordinates by indices A and B. The fixed points form a point system, which is periodical along x axis with the period 2π. This means the periodicity of the phase portraits for the system (2), (3) along x axis with the period 2π.

In case of A, x coordinates of the fixes points are

$$X_A^* = \frac{\pi}{2} \pm k\pi \tag{6}$$

here $k = 0, 1, 2, \ldots$.

From the equation (5), we can also find coordinates of the fixed points B

$$Y_B^* = \pm n\pi, \tag{7}$$

where $n = 1, 2, \ldots$. It is natural that only fixed points with the coordinates $y > 0$, situated in the part filled with liquid, have physical meaning. So, in the case of B one should limit oneself with only $Y_B^* = n\pi$.

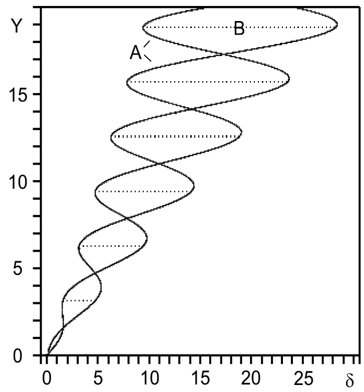

Fig. 1. *The position of fixed points A and B along y axis at different values of δ and fixed value of $\varepsilon = 0.5$. The solid lines on the diagram correspond to A type fixed points, and dotted line – to B type points. The dotted line corresponds to two B fixed points, because they have the same y coordinates, but different x coordinates. It is easy to see that the change of the fixed points number with δ increase is connected with the birth and annihilation of the fixed points*

The location of the fixed points A along y axis is determined by the equation

$$\frac{\delta}{Y_A^*} = 1 - \alpha_k \varepsilon\cos(Y_A^*), \tag{8}$$

where $\alpha_k = \sin(\frac{\pi}{2} \pm k\pi) = (-1)^k$. The number of the solutions of this equation and, so, the number of fixed points A per period, essentially depends on the parameters ε and δ. In Fig. 1 it is shown how the number of fixed A type points changes with the change of parameter δ and fixed ε. In fact, the figure can be regarded as a bifurcational diagram at δ changing. The bifurcations leading to the change of fixed points number are the birth or annihilation of fixed point A and B pairs. Analyzing the equation (8), one can find areas with different numbers of A type fixed points in the parameter plane.

Analogously, the equation defining the fixed B points location and number, has the form

$$\sin(X_B^*) = \frac{\delta - \pi n}{\pi n \varepsilon} \alpha_n , \tag{9}$$

where $\alpha_n = \cos(n\pi)$. This equation has solutions at fixed n if the following inequality is true: $|1 - \delta / \pi n| \le |\varepsilon|$. These conditions at different n values, define the areas on the parameter plane with different number of fixed B points on a period of the phase portrait. The locations of the fixed points B on Y axis at different values of δ and fixed ε, are also shown in Fig. 1.

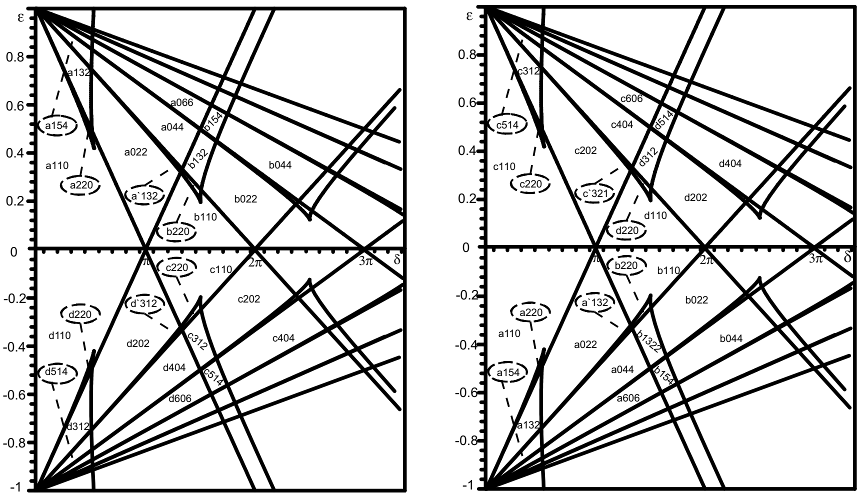

Fig. 2. The spaces of parameters (ε, δ)*, defining the number and type of fixed points in the phase portraits, are shown. On the left there are the parameter spaces at* $\Delta > 0$*, on the right — at* $\Delta < 0$*. The first digit in the space number denotes the number of fixed points* A *with hyperbolic type, the second one shows the number of elliptic fixed points* A*, and the third digit means the number of hyperbolic fixed points* B *for numbers starting with* a *and* b *or the number of nodes for numbers starting from* c *and* d*. Phase portraits with the same numbers, but different letters differ one from another with their shift along* x *axis on a half-period equal to* π*, for example,* $a \leftrightarrow b$*,* $c \leftrightarrow d$ *at* π *shift*

Now let us discuss the types of fixed points. The characteristic equation for fixed points (X_A^*, Y_A^*) has the form $\lambda_A^2 = \alpha_k \Delta \sin(Y_A^*) \left(\frac{\delta}{Y_A^{*2}} - \alpha_k \varepsilon \sin(Y_A^*) \right)$. It is easy to see that fixed points A can be only elliptic (in case $\alpha_k \Delta \sin(Y_A^*) \left(\frac{\delta}{Y_A^{*2}} - \alpha_k \varepsilon \sin(Y_A^*) \right) < 0$ is true) or hyperbolic (at $\alpha_k \Delta \sin(Y_A^*) \left(\frac{\delta}{Y_A^{*2}} - \alpha_k \varepsilon \sin(Y_A^*) \right) > 0$). Parameter Δ is included in these condition trivially, and only its sign matters. When the sign of Δ changes, hyperbolic fixed points turn into elliptic ones, and elliptic — into hyperbolic without changing their position in the phase space.

The type of fixed points B is defined analogously by the following characteristic equation

$$\lambda^2 - \alpha_n \lambda \cos(X_B^*)(\Delta - \varepsilon) - \Delta\varepsilon \cos^2(X_B^*) = 0$$

the solutions of which have the form:

$$\lambda_B^{\pm} = \frac{\cos(X_B^*)}{2} [\alpha_n (\Delta - \varepsilon) \pm (\Delta + \varepsilon)],$$

whence $\Delta > 0, \varepsilon > 0$ and $\Delta < 0, \varepsilon < 0$ points have hyperbolic type, and in case of $\Delta < 0, \varepsilon > 0$ or $\Delta > 0, \varepsilon < 0$ the fixed points are stable and unstable nodes. In this case the influence of Δ is more essential. Summing up the information about the number of fixed points on a period and about their types, one can plot the parameter spaces for any phase portraits types realized in this mode (see Fig. 2). All the parameter space is divided into an infinite number of areas with different phase portraits types. A part of these areas, for comparatively simple phase portraits, is shown in Fig. 2. The choice of spaces is defined by physically sensible restrictions $|\varepsilon| \le 1$ and not too large δ (for example, $|1 - \delta / 4\pi| \le \varepsilon$). Plotting all the rest of the spaces and their geometric position can be easily continued on all the parameter plane. The areas numeration is chosen according to the type and number of the fixed points existing in the phase portrait at these values of parameters.

Let us now describe the modes of vortex movement in the moving base. Of course, the complexity of the phase portraits and, so, that of the vortex movement modes, increases with the increase of fixed points number on a period. The phase space division with separatrices into separate cells is common. The number of such cells increases with the increase of fixed points number on a period. In this case, only three types of behavior can be realized in the cells. These types are non-linear oscillations of trapped vortexes with zero velocity of movement along x axis, non-linear oscillations of passing vortexes with nonzero average velocity of motion along x axis and nonlinear relaxation into a stable node inside the cell. The last behavior type is unusual. During the relaxation process, the memory about the initial vortex state is entirely lost. Such a behavior is typical for dissipative systems. In (Fig.3) simple examples of phase portraits, characteristic for different parameter spaces, are shown.

Phase portraits in Fig. 3 on the left, are plotted at $\Delta > 0, \varepsilon > 0$ and $\Delta < 0, \varepsilon < 0$, and on the right – at $\Delta < 0, \varepsilon > 0$ and $\Delta > 0, \varepsilon < 0$; they have a certain symmetry.

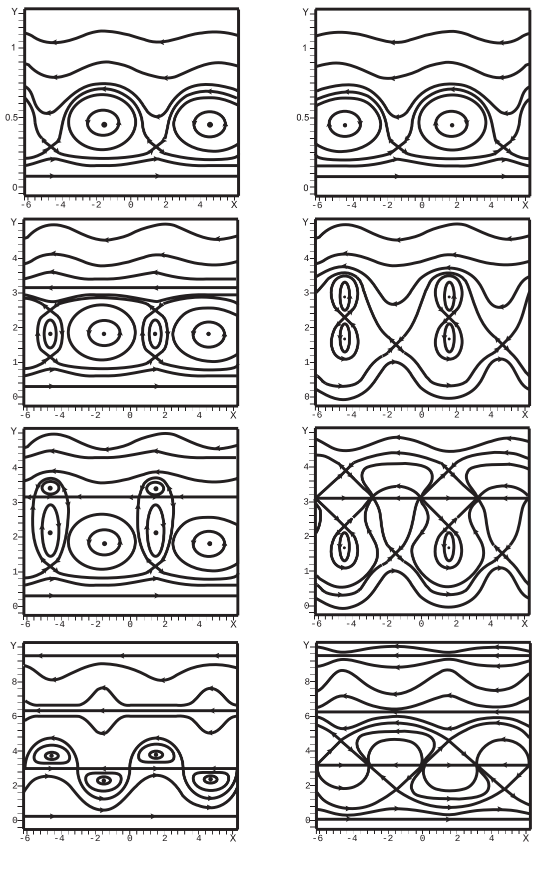

Fig. 3. *In the left figure (top-down) one can see the typical phase portraits for values of parameters belonging to the spaces* $a110$ *(*$\delta = 1, \varepsilon = 0.4, \Delta = 0.5$*),* $a220$ *(*$\delta = 1.5$*,* $\varepsilon = 0.5$*,* $\Delta = 0.5$*),* $a132$ *(*$\delta = 1.2$*,* $\varepsilon = 0.7$*,* $\Delta = 0.5$*),* $a022$ *(*$\delta = 3$*,* $\varepsilon = 0.4$*,* $\Delta = 0.5$*). On the right* $c110$ *(*$\delta = 3$*,* $\varepsilon = 0.4$*,* $\Delta = -0.5$*),* $c220$ *(*$\delta = 4$*,* $\varepsilon = 0.5$*,* $\Delta = -0.5$*),* $c312$ *(*$\delta = 1.2$*,* $\varepsilon = 0.7$*,* $\Delta = -0.5$*),* $c202$ *(*$\delta = 3$*,* $\varepsilon = 0.4$*,* $\Delta = -0.5$*)*

It is easy to notice, that they turn into each other in case of replacement of hyperbolic points with elliptic ones and vice versa. Let us start from the description of the phase portraits shown in the left figure. We can see that the phase portraits are periodical along x axis with period 2π and consist of cells, limited with separatrices of hyperbolic fixed points. Inside these cells there are elliptic fixed points. The vortex, whose initial coordinates are in such a cell, is trapped and starts nonlinear oscillations lengthwise and crosswise near the elliptic point. If the initial location of the vortex is higher of lower than the cell, then the vortex moves along the boundary, making nonlinear oscillations crosswise.

The phase portraits on the right (see Fig. 3) also consist of cells. On the two top phase portraits on the right such cells are analogous to those on the left, so the vortex movement modes are analogous. On the two bottom phase portraits on the right, there appear cells of a different type. In the vertexes of these cells there are hyperbolic fixed points, and on two separatrices, joining the vertexes, there are a stable node and an unstable one. Such areas can be distinctly seen on the bottom phase portrait. The appearance of the nodes means that a vortex with the initial conditions, which belong to the corresponding cell, will inevitable be drawn into the stable node. This is the manifestation of one more vortex movement mode — its trapping by a stable node, followed by changing the vortex velocity to zero (in a moving base).

At the end of this part, we shall consider the parameter space, in which it is easy to prove the integrability of the vortex movement equations. Equations system (2), (3) can be reduced to a quasi-Hamiltonian form

$$\Delta \frac{dX}{dt} = \frac{\partial H}{\partial Y}, \tag{10}$$

$$\varepsilon \frac{dY}{dt} = -\frac{\partial H}{\partial X}. \tag{11}$$

The part of Hamiltonian H is played by the function $H = \Delta\delta \ln(Y) - \Delta Y - \varepsilon\Delta \sin(Y)\sin(X)$. In case of parameters $\Delta = \varepsilon$, the equations system (10),(11) becomes Hamiltonian, and its Hamiltonian does not depend on time. So, according to Liouville theorem about the integrability of Hamiltonian systems, this system is integrated in quadratures. Let us note that under given condition $\varepsilon = \Delta$ the integrable systems belong to the parameter spaces $(\varepsilon > 0, \delta > 0, \Delta > 0)$ and $(\varepsilon < 0, \delta > 0, \Delta < 0)$. In the general case, the initial equations system cannot be reduced to Hamiltonian form. It is easy to understand, taking into account the presence of modes with fixed points such as nodes, which cannot appear in Hamiltonian systems.

4. CONCLUSION

In summary, we shall discuss the main qualitative changes in vortex evolution influenced by a potential wave. First of all, we shall note that the state of vortex uniform motion with preserved distance to the wall is easily broken under the influence of the wave. In this case the distance changes as well as lengthwise and crosswise (to the wall) velocities. In the moving base, even the direction of the vortex movement can change to the opposite. Taking into account that a point vortex in a real liquid corresponds to a linear extended vortex, one can expect horseshoe-shaped and more complicated structures to appear due to the interaction of the linear vortex with heterogeneous wave packets of sound waves.

Now let us consider the influence of small corrections, appearing due to the potential waves influence induced by the vortex movement, on the vortex movement. Basing upon the general properties of dynamic systems, one can expect that taking the corrections into account should make separatrices to disappear and thin stochastic layers to appear about them. This, in its turn, means the possibility of vortex wandering about the overlapping stochastic layers. In the rest of phase space the qualitative picture of vortex behavior should not have any qualitative changes.

It is interesting to note that the modes causing the memory loss about the initial state of the vortex, lead to vortex component energy changes. This implies that in wave–vortex system there can an effect appear, which is analogous to collisionless wave attenuation in plasma. In other words, the interaction of a potential wave with point vortexes in the ideal liquid can lead to the change of its amplitude. But the analysis of such effects requires self-consistent description of waves and vortexes interaction in the quasilinear theory manner, which is beyond the subject of this work.

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ВЛИЯНИЕ ПАДАЮЩЕЙ И ОТРАЖЕННОЙ ПОТЕНЦИАЛЬНОЙ ВОЛНЫ НА ЭВОЛЮЦИЮ ТОЧЕЧНОГО ВИХРЯ У СТЕНКИ

К.Н. Кулик, А.В. Тур, В.В. Яновский

Изучено влияние потенциальных мод колебаний на движение точечного вихря вблизи твердой стенки. Получены уравнения, описывающие движение точечного вихря в заданном поле потенциальной волны при наличии твердой стенки. Показано, что характер движения вихря под влиянием потенциальной волны качественно меняется. Проанализированы все возможные режимы движения вихря под воздействием потенциальной волны.

ВПЛИВ ПАДАЮЧОЇ ТА ВІДБИТОЇ ПОТЕНЦІЙНОЇ ХВИЛІ НА ЕВОЛЮЦІЮ ТОЧКОВОГО ВИХОРА БІЛЯ СТІНКИ

К.М.Кулик, А.В.Тур, В.В.Яновський

Вивчена дія потенційних мод коливань на рух точкового вихору поблизу твердої стінки. Отримані рівняння, які описують рух точкового вихора у заданому полі потенційної хвилі у наявності твердої стінки. Показано, що характер руху вихору під дією потенційної хвилі змінюється. Проаналізовані всі можливі режими руху вихору під дією потенційної хвилі.