POLYMORPHOUS BILLIARD AS A NEW TYPE OF BILLIARDS WITH CHAOTIC RAY DYNAMICS

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A new type of chaotic billiards is introduced. Unlike the known ones, it contains scattering as well as focusing regions of the boundary and has no neutral components. The dumbbell-like from polymorphous billiards family is proposed. Its characteristic phase dynamics (at control parameter changes) is studied and Lyapunov exponent as well as invariant reflections density on the boundary are calculated. The chaotic behavior of the beams and their uniform stationary distribution are proved.

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1. INTRODUCTION

Billiards, i.e. systems with elastic or mirror reflections, occupy the central position in the deterministic chaos theory and have numerous physical applications. Chaotic billiards, in which the dynamics of all beams is chaotic, have gained a special popularity in deterministic chaos theory. In them, similar trajectories exponentially diverge in phase space, which causes stirring and ergodicity. Among chaotic billiards scattering Sinai billiards [1-3] and defocusing Bunimovich billiards [4-6] are distinguished. A distinctive geometric feature of scattering billiards is that all their boundary consists only of scattering components with negative curvature $K < 0$. The reflection from scattering components is always accompanied by angle widening of the bundle of initially close trajectories. This, eventually, leads to their chaotization. The boundary of defocusing billiards includes components with non-negative curvature $K \geq 0$. In these billiards, chaotization is assured by defocusing of the beams, which are first focused at reflection from convex components of billiard boundary. Defocusing mechanism is an alternative to scattering mechanism. Absolute chaos in billiards with only concave components ($K > 0$) is found only in case their absolute defocusing [6]. In smooth convex billiards this condition is not true. For defocusing mechanism, boundary singularities, such like return points, are needed. In stadiums with convex boundary, defocusing works due to the existence of rectilinear regions with zero curvature $K = 0$.

Chaotic billiards with mixed scattering character, i.e. billiards, whose boundary includes components of positive as well as of negative curvature, have never been considered before. It is obvious that beams scattering and (de)focusing are contrary by their nature. In general case, they will compete. So full chaos should be expected from billiards of only special type, in which these mechanisms are not in each other's way, but supplement each other. Such chaotic billiards, whose boundary includes components with curvature of different sign, really exist. Among them are polymorphous billiards, introduced below.

2. POLYMORPHOUS AND DUMBBELL-LIKE BILLIARS

Let us smoothly join (so that the tangent has no discontinuities) an even number of arcs taken from one circle to get a closed curve. We shall call the billiard limited by such a curve *polymorphous billiards*. Its boundary is formed by the arcs of the same circle and has everywhere constant curvature to sign. Some examples of such billiards one can see in Fig. 1.

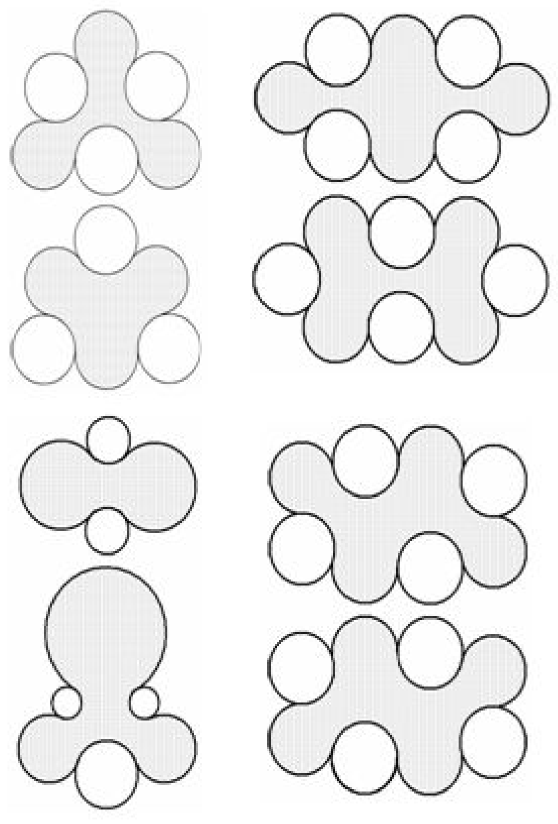

Fig. 1. *Geometry of polymorphous billiards: symmetric and asymmetric; 6(on the left) and 8 (on the right) orders*

For closed boundary of polymorphous billiard it is necessary that initial number of circles arches should be even and not less than four. The distinctive feature of their boundary is that its curvature alternately changes its sign, i.e. $\operatorname{sgn} K_j = (-1)^{j+1}$, where (?). If radiuses of all the discs and circles which limit them are equal to one, then $K_j = (-1)^{j+1}$. The curvature here is constant by its absolute value. The leap in the contact points of neighboring components affects only the vector of normal, unlike the everywhere continuous tangent field.

The simplest polymorphous billiard is a dumbbell-like billiard, the boundary of which is formed by arcs of four circles (Fig. 2). A smaller number of arcs is impossible, otherwise the smoothness of the boundary obtained would have been broken.

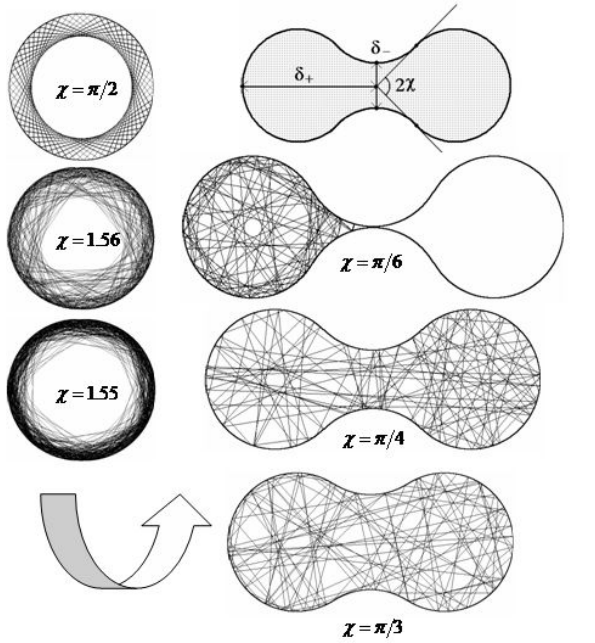

Fig. 2. *Geometrical portrait of family of dumbbell-like billiards. It is shown the typical billiard trajectories*

To study the "dumbbell" dynamics, we use geometric-dynamical approach [7-10], in which beams dynamics is described in a special symmetric phase space. Let us chose angle χ between the axis, connecting the centers of the convex components of the border and the beam, drawn to the point of contact between the convex and concave components as the control parameter of the dynamic system. This angle χ corresponds to the width of the middle of the dumbbell. It is changed from $\pi/2$ to $\pi/6$. At $\chi = \pi/2$ there is no narrow middle and we have a billiard in a circle instead of the dumbbell one. At $\chi = \pi/3$ the circles corresponding to the convex parts of dumbbell boundary, contact (inside of the billiard). At $\chi = \pi/2$ we have the most symmetric configuration. At $\chi = \pi/6$ the middle reaches its maximum and billiard falls into two ones.

3. CHAOTIC DYNAMICS OF "DUMBBELL"

The phase portrait gives us important information that billiard in a dumbbell has developed chaos. Fig. 3 shows changes in "dumbbell" phase portrait at changes of control parameter $\chi \in [\pi/6, \pi/2]$. At $\chi \in [0, \pi/6]$ the dumbbell falls into a pair of symmetric billiards in the form of "drops" with the identical phase portrait (Fig. 4).

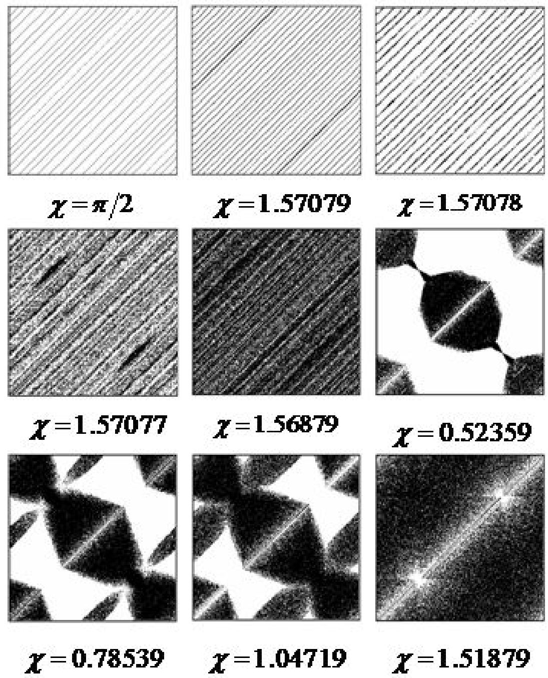

Fig. 3. *Phase portrait of family of dumbbell-like billiards in symmetric phase space at* $\pi/6 < \chi < \pi/2$

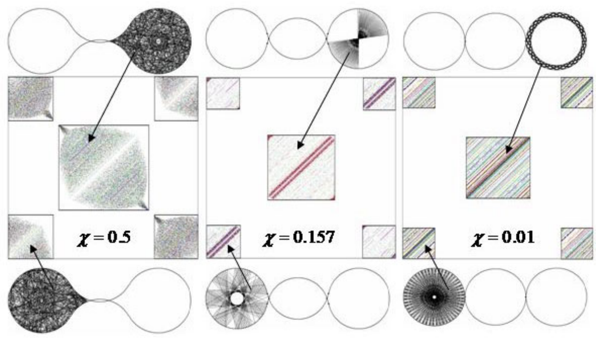

Fig. 4. *Geometrical and phase portrait of family of dumbbell-like billiards at* $0 < \chi < \pi/6$. *By arrows it is noted the typical trajectories in symmetric phase space*

At $\pi/6 < \chi < \pi/2$ there are two lacunas in the dumbbell phase space. So the billiard has N_6 topological type (sphere, stuck up with 6 Moebius loops). At $0 < \chi < \pi/6$ there is only one return lacuna. So topological type of "drop" is N_4. χ decreasing, the phase volume of "dumbbell" lacunas increases, but one of the "drop" lacuna decreases. Phase trajectories never get inside the lacunas. At intermediate $\chi = \pi/4$, lacunas overlap. Instead of a pair of isolated lacunas, in the symmetric phase space, there is one common region of classically illegal movement.

Trajectories dynamics at $\chi \neq \pi/2$ is always chaotic. In the symmetric phase space of the dumbbell there are no traces of any elliptic (intregrable) movement component. Regular trajectories like "whispering galleries" are ruined by lacunas, that appear at any "bottleneck", however small. As a result, there is one common ergodic movement component with stirring. The phase transition from integrable billiard in a circle to chaotic billiard in a dumbbell takes place by the scenario of coarse bifurcation. Arbitrarily small perturbance, defined by the parameter $\varepsilon = \pi/2 - \chi$, leads to global reconstruction of the phase portrait – all the neutral cycles of the billiard become unstable, and quasi-periodic orbits become chaotic.

The phase portrait corresponds to the billiard dynamic in the asymptotic limit at arbitrarily great number of iterations. The dumbbell dynamics studies on finite time intervals shows that at very small $\varepsilon \propto 10^{-3} \div 10^{-4}$ (see Fig. 3 at the control parameter change from $\chi = 1.57079$ to $\chi = 1.56879$) phase trajectories stay for a long time in intermediate rationally commensurable layers between the ruined invariant curves. For homogeneous filling of all the phase space, they need essentially larger number of iteration of the order $10^9 \ldots 10^{12}$ and more than at $\varepsilon \propto 10^{-1}$. This lets us conclude that the chaotization of the dumbbell starts by the scenario of forming cantori at ruining invariant curves near non-perturbed periodic orbits. Phase trajectories slowly ooze through these cantori. Invariant curves ruin the quickest near the periodic orbits with 2 period, joining the concave components of the dumbbell, i.e. near the phase points $(1/2, 3/2)$ and $(3/2, 1/2)$. This corresponds to more filled (dark) zones on the phase portrait. Let us note that the appearance of irregular invariant sets in the phase space of the dumbbell is connected with the breaks of involution and of its derivatives. The phase cascade rapidly multiplies these breaks, so non-perturbed KAM-tori (invariant curves) of the circle billiard turn into the fractal cantori of the dumbbell. Deformation parameter ε increasing, apparent Cantor-structure collapses. It is changed by a structure of lacunas, connected with the geometric shadow for the billiard beams. At $\varepsilon \propto 10^{-1}$ any phase trajectory fills the available phase space rather quickly during the time of the order about $10^4 \ldots 10^5$ iterations.

4. LYAPUNOV EXPONENT

The phase portrait of the "dumbbell" apparently demonstrates that this is a chaotic billiard with static properties of phase trajectories. One of the main numeric characteristics of the deterministic chaos is Lyapunov index, i.e. the index of exponential divergence of the phase trajectories for the dynamic system. The positivity of Lyapunov index, $\lambda > 0$, is the most frequently used as a chaoticity criterion of a dynamic systems, including billiards, see. [11] etc.

Numeric dependence of the "dumbbell's" Lyapunov index is shown in Fig. 5. Lyapunov index is always strictly positive at any choice of the initial beam (the point of the symmetric phase space). In Fig. 5, the monotonous growth of the lacunas full volume, i.e. of phase image of the geometric shadow with parameter χ decrease is also shown. Besides, Fig. 5 shows, in double logarithmic scale, the scaling of $\lambda(\varepsilon) = A\varepsilon^{\gamma}$ at $\varepsilon = \pi/2 - \chi \ll 1$, i.e. in the point of transition form billiard in circle to billiard in the dumbbell with arbitrarily small "bottleneck". This bifurcation can be regarded as 2nd-tupe phase transition, for which Lyapunov index is the order parameter. The numeric values of the constants are $A \approx e^{0.4028}$ (renormalization constant) and $\gamma = 0.3565$ (critical index).

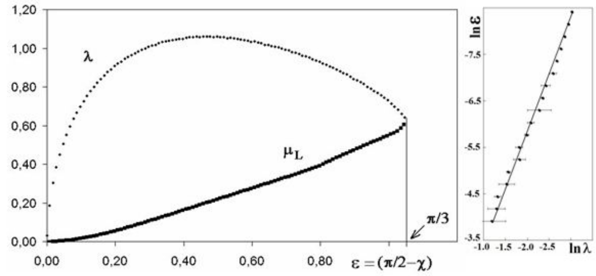

Fig. 5. *The dependence of Lyapunov exponent of the dumbbell billiards on the control parameter from "drop" form to circle form. On the right the Lyapunov index scaling for a dumbbell close to a circle billiard is shown*

The growth of lacunas phase volume affects the dumbbell dynamic in different ways. At $\pi/3 < \chi < \pi/2$ it strengthens the chaos, $d\lambda/d\mu_L > 0$, and at $\pi/6 < \chi < \pi/3$, inhibits it, $d\lambda/d\mu_L < 0$. The phase volume of the lacunas is rather crude (average) characteristic. So one should expect correlation here only for some averages of the billiard, e.g., average path length.

5. INVARIANT DISTRIBUTION

It is convenient to use the stationary reflection density, introduced in [12], for the description of statistic properties. In the symmetric phase space it is easier to calculate than full invariant measure. At small "bottleneck" of the dumbbell we shall decompose the billiard mapping by $\varepsilon = \pi/2 - \chi \ll 1$

$$\begin{pmatrix} \bar{\varphi}_1 \\ \bar{\varphi}_2 \end{pmatrix} = \mathbb{B}_\varepsilon \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & \alpha \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} + o(\varphi_1, \varphi_2) \qquad (1)$$

where $\alpha = \alpha(\varepsilon) > 2$ defines the average inclination of involution level; lines in the phase space. We shall limit ourselves to the main contribution and cast out the contributions to dynamics from the special set $\mathbb{S} = \bigcup_{n=0}^{+\infty} \mathbb{B}^{-n}(\partial L)$ (the full preimage of the trajectories visiting the lacunas boundary) of the vanishing measure, $\mu(\mathbb{S}) = O(\varepsilon)$. The equation for one-point invariant density will have the following form

$$\rho(\varphi) = \frac{1}{\alpha} \int_0^1 \rho(\psi)\, \rho\left(\frac{\varphi + \psi}{\alpha}\right) d\psi + O(\varepsilon), \qquad (2)$$

Solving the equation (2) by the successive iterations method, taking into account that $\alpha > 2$ (this assures the convergence of series), we shall have

$$\rho_\varepsilon(\varphi) = 1 + O(\varepsilon). \qquad (3)$$

The solution (3) is coordinated with the normalization only under the condition that $\rho_\varepsilon(\varphi) = 1$. For a chaotic billiard, due to its stirring, this is the only solution. The numeric calculation shows that stationary density $\rho(\varphi)$ also stays constant at arbitrary ε, see

Fig. 6. This is obviously connected with the constancy of the absolute curvature in the given billiard.

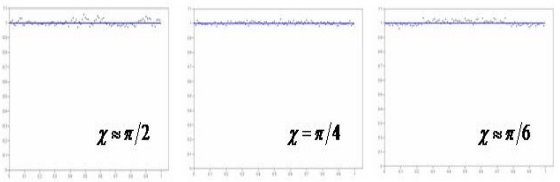

Fig. 6. *Stationary density of reflections on the billiard boundary. The full length of the billiard boundary is normalized to one*

So, in asymptotic limit, the beams visit the dumbbell boundary with approximately equal frequency. In the symmetric phase space, on the contrary, the phase points are distributed non-uniformly because of the lacunas presence.

6. CONCLUSIONS

A new type of polymorphous billiards with chaotic dynamics of the beams is proposed. Unlike the known chaotic billiards, their boundary includes scattering and focusing boundary components at the same time. So the chaotization mechanism in such billiards is of mixed character. The reconstruction dynamics and phase reconstructions in one-parametric family of typical polymorphous dumbbell-like billiards are studied. The chaotic behaviour of the beams and uniform distribution of the reflections are proved

One should expect analogous behavior of the beams in polymorphous billiard of arbitrary form. The chaotic properties found let one use polymorphous billiard in applications. In particular, “dumbbell” form can be used for example, in atomic [13], microwave [14] or semiconductor billiards [15]. Chaos peculiarities in polymorphous billiards can also influence the character of light pass in optic nanoceramics microclusters, formed due to coagulating of ball-like nanoparticels etc.

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ПОЛИМОРФНЫЙ БИЛЬЯРД КАК НОВЫЙ ТИП БИЛЬЯРДОВ С ХАОТИЧЕСКОЙ ДИНАМИКОЙ ЛУЧЕЙ

С.В. Найденов, Ю.Н. Масловский, В.В. Яновский

Введен новый тип хаотических полиморфных бильярдов. В отличие от других хаотических бильярдов они одновременно содержат рассеивающие и фокусирующие участки границы, но не содержат нейтральных компонент. Предложено семейство полиморфных бильярдов в форме гантели. Изучена их фазовая динамика и определены зависимость показателя Ляпунова (от управляющего параметра семейства бильярдов) и инвариантная плотность распределения отражений по границе бильярда. Компьютерные вычисления подтверждают наличие полного динамического хаоса и универсальность стационарной функции распределения в таких бильярдах.

ПОЛІМОРФНИЙ БІЛЬЯРД ЯК НОВИЙ ТИП БІЛЬЯРДІВ З ХАОТИЧНОЮ ДИНАМІКОЮ ПРОМЕНІВ

С.В. Найдьонов, Ю.М. Масловський, В.В. Яновський

Уведений новий тип хаотичних поліморфних більярдів. На відміну від інших хаотичних більярдів вони одночасно мають компоненти границі, що розсіюють та фокусують, але не є нейтральними. Запропоновано сімейство поліморфних більярдів у формі гантелі. Вивчена їх фазова динаміка та визначені залежність показника Ляпунова (від керуючого параметра сімейства більярдів) та інваріантна густина розподілу відбитків від границі більярду. Комп’ютерні обчислення доводять наявність повного динамічного хаосу та універсальність стаціонарної функції розподілу у таких більярдах.