ON SOME PROPERTIES OF 2-D WEYL EQUATION FOR CHARGED MASSLESS SPIN ½ PARTICLE

I.E. Ovcharenko[1] and Yu.P. Stepanovsky[2]

[1]Institute of Problems of Mechanical Engineering of the Russian Academy of Sciences, St. Petersburg, Russia;
e-mail: ieo@rol.ru;
[2]National Science Center "Kharkov Institute of Physics and Technology", Kharkov, Ukraine;
e-mail: yustep@kipt.kharkov.ua

We discuss some properties of Dirac equation and 2-D Weyl equation for a charged massless spin ½ particle in a stationary inhomogeneous magnetic field such as Bloch theorem and Aharonov-Casher theorem and its relation to supersymmetric quantum mechanics.

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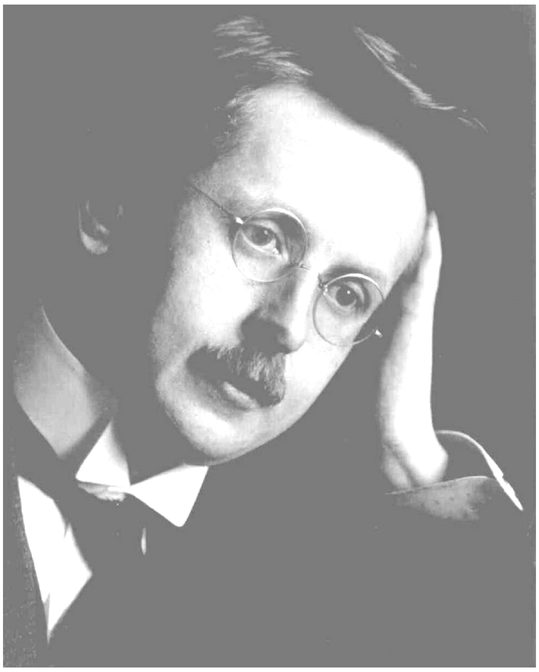
1. INTRODUCTION

Fifty years ago both of us (I. O. and Yu. S.) attended wonderful lectures of two outstanding scientists and two great teachers, two Akhiezer brothers Alexander Il'ich and Naum Il'ich and we thank one's lucky stars for that. In the persons of Akhiezer brothers the destiny happily united two beautiful sciences – physics and mathematics – and both brothers were our heroes all our life from student years.

But almost throughout our lives both of us have had another common heroes, great mathematician Hermann Weyl and great physicist Paul Dirac. They attracted (and do attract up to now) our attention not merely as great scientists but also as great hunters for beauty. **"My work has always tried to unite the true with the beautiful and when I had to choose one or the other, I usually chose the beautiful"**, – wrote Weyl [1]. **"Physical laws should have mathematical beauty"**, – wrote Dirac on the blackboard in the Moscow University in the fall of 1955. Reason for the mysteries that most of the time truth and beauty are the same, that there need not to be conflict between them, discusses David J. Gross in his assay [1] in detail: "…**the mathematical structures that mathematicians arrive at are not artificial creations of the human mind but rather have a naturalness to them as if they were as real as the structures created by physicists to describe the so-called real world. Mathematicians, in other words, are not inventing new mathematics, they are discovering it … we might expect that physical and mathematical structures would share the characteristics that we call beauty. Our minds have surely evolved to find natural patterns pleasing".**

Our teachers Akhizer brothers Alexander Il'ich and Naum Il'ich

Hunters for beauty and truth Herman Weyl and Paul Dirac

2. DIRAC AND WEYL EQUATIONS, BLOCH AND AHARONOV-CACHER THEOREMS, AND SUPERSYMMETRIC QUANTUM MECHANICS

2.1. DIRAC AND WEYL EQUATIONS

On January 2, 1928 editors of "Proceeding of Royal Society of London A" obtained the paper of 25-old Dirac "The quantum theory of the electron", that was published in February, 1928 [2]. This paper contained the greatest equation of theoretical physics after Maxwell equations and Einstein equations of general relativity – Dirac equation (we use the units $\hbar = c = 1$),

$$\left(\gamma_\mu(\frac{\partial}{\partial x_\mu} - ieA_\mu) + m\right)\Psi = 0, \quad (1)$$

where γ_μ – four 4×4 Dirac matrices, A_μ – 4-vector of the electromagnetic potential. Dirac matrices satisfy to anticommutation relations

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = \delta_{\mu\nu}I, \quad \mu,\nu = 1,2,3,4. \quad (2)$$

Introducing the fifth matrix $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$, we evidently obtain

$$\gamma_a\gamma_b + \gamma_b\gamma_a = \delta_{ab}I, \quad a,b = 1,2,3,4,5. \quad (3)$$

Dirac took matrices γ_μ in the following explicit form (he did not yet consider matrix γ_5)

$$\gamma_k = \begin{pmatrix} 0 & -i\sigma_k \\ i\sigma_k & 0 \end{pmatrix}, \gamma_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \left[\gamma_5 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\right], \quad (4)$$

σ_k, $k = 1,2,3,$ are the Pauli matrices.

In his famous book "The theory of groups and quantum mechanics" published in German in 1928 [3] Weyl claimed: **"The importance of the standpoint afforded by the theory of groups for the discovery of the general laws of quantum theory has of late become more and more apparent".** But it was not the case for the discovery of the Dirac equation. Dirac himself indicated that he obtained his equation merely by **"playing around with three** 2×2 **matrices"** [4]. Dirac **"made breakthrough, a new method of doing physics"** [5]. Dirac said about his new method of doing discoveries: **"I like to play about with equations, just looking for beautiful mathematical relations which maybe don't have any physical meaning at all. Sometimes they do"** [6].

Group theory was the reason of misunderstanding between Weyl and Dirac. In 1934 Dirac gave a seminar in Princeton, **"at the end of which Weyl protested that Dirac had said he would make no use of group theory but that in fact most of his arguments were applications of group theory. Dirac replied, "I said that I would obtain the results without previous knowledge of group theory!"** [7]. Indeed, it is well known "**that everything one needs in the theory of groups can be discovered by the light of nature provided one knows how to multiply two matrices"** [8]. In 1928 Weyl made a very important contribution to theoretical physics also just multiplying two matrices: on the page 171 of his book [2] he introduced Dirac matrices in the following explicit form (Weyl as Dirac did not consider matrix γ_5)

$$\gamma_k = \begin{pmatrix} 0 & i\sigma_k \\ -i\sigma_k & 0 \end{pmatrix}, \gamma_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \left[\gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right]. \quad (5)$$

Let us now consider a 4-component Dirac wave function Ψ as composed of two spinors φ and χ.

Then we obtain from the Dirac Eq. (1) two following equations:

$$\left(\frac{\partial}{\partial t}+ieA_0+\vec{\sigma}(\frac{\partial}{\partial\vec{x}}-ie\vec{A})\right)\varphi+im\chi=0;$$
$$\left(\frac{\partial}{\partial t}+ieA_0-\vec{\sigma}(\frac{\partial}{\partial\vec{x}}-ie\vec{A})\right)\chi+im\varphi=0. \quad (6)$$

Putting in Eqs. (6) $m=0$, we obtain Weyl equations:

$$\left(\frac{\partial}{\partial t}+ieA_0+\vec{\sigma}(\frac{\partial}{\partial\vec{x}}-ie\vec{A})\right)\varphi=0;$$
$$\left(\frac{\partial}{\partial t}+ieA_0-\vec{\sigma}(\frac{\partial}{\partial\vec{x}}-ie\vec{A})\right)\chi=0. \quad (7)$$

The Weyl Eqs. (7) describe high energy electrons with right and left helicities. Weyl discovered a very important property of the Dirac equation: two spinors φ and χ transform independently (and by different ways) under the Lorentz transformations (without reflections of space and time) and change places $\varphi\leftrightarrow\chi$ under coordinate inversion $\vec{x}\to-\vec{x}$.

Let us point out that Hamiltonian for the electron with right (left) helicity is

$$H=\mp i\vec{\sigma}(\frac{\partial}{\partial\vec{x}}-ie\vec{A})+eA_0. \quad (8)$$

Weyl's derivation of Eqs. (6-7) was very simple and clear. But in the next editions of his book [2] Weyl radically altered the exposition of this matter. The new exposition was based on the group theory and on the variation principle. Simplicity was lost. The exposition became very beautiful but "gemeinunverständlich" (not understandable to all)[1].

2.2. BLOCH THEOREM

Let us consider solutions of Eqs. (7) for the case of massless charged particle with definite energy E in the stationary inhomogeneous magnetic field $\vec{B}(\vec{x})$, $\vec{B}(\vec{x})=\text{rot }\vec{A}(\vec{x})$,

$$\left(E+i\vec{\sigma}(\frac{\partial}{\partial\vec{x}}-ie\vec{A})\right)\varphi=0;$$
$$\left(E-i\vec{\sigma}(\frac{\partial}{\partial\vec{x}}-ie\vec{A})\right)\chi=0. \quad (9)$$

We see that there are two fundamentally different solutions of the Weyl Eqs. (9) (that is solutions of the Dirac equations in massless limit): $\varphi\neq0,\chi=0$ (an electron with the right helicity), $\varphi=0,\chi\neq0$ (an electron with the left helicity). In other words **all the nonzero energy eigenstates of the "massless" electron in an arbitrary stationary magnetic field are degenerate with respect to the spin flip.** In 1982 F. Bloch proved [10] that "masslessness" is of no importance and in the general case **all the nonzero energy eigenstates of an electron in an arbitrary stationary magnetic field are degenerate under the spin flip.** To demonstrate the validity of the Bloch theorem let us restore the rejected mass in Eqs. (9)

$$\left(E+i\vec{\sigma}(\frac{\partial}{\partial\vec{x}}-ie\vec{A})\right)\varphi+m\chi=0;$$
$$\left(E-i\vec{\sigma}(\frac{\partial}{\partial\vec{x}}-ie\vec{A})\right)\chi+m\varphi=0. \quad (10)$$

Excluding χ from Eqs. (9) we obtain

$$\left(E^2-m^2+\left[\vec{\sigma}(\frac{\partial}{\partial\vec{x}}-ie\vec{A})\right]^2\right)\varphi=0. \quad (11)$$

There are two different solutions of Eq. (11), corresponding to two different signs in Eq. (8), $\varepsilon=|\sqrt{E^2-m^2}|$,

$$\left(\varepsilon\pm i\vec{\sigma}(\frac{\partial}{\partial\vec{x}}-ie\vec{A})\right)\varphi=0. \quad (12)$$

We see that the Bloch degeneration is reduced to the degeneration of two Weyl Eqs. (9).

Note that in the 4-component form the Hamiltonian for a "massless" electron in an arbitrary electromagnetic field

$$H=\gamma_4\vec{\gamma}(\frac{\partial}{\partial\vec{x}}-ie\vec{A})+eA_0 \quad (13)$$

commutes with the matrix γ_5. So the eigenvalues of the matrix γ_5,

$$\gamma_5\Psi=\mp\Psi, \quad (14)$$

are conserved quantum numbers for "massless" (high energy) electron (– and + correspond to the right and left helicities).

2.3. AHARONOV-CASHER THEOREMS

In their paper [11] Y. Aharonov and A. Casher proved two theorems for the case of a 2-D magnetic field. F. Bloch was inspired by the second of them and the Bloch theorem is the word by the word the same as the second 2-D Aharonov-Casher theorem.

The first theorem by Y. Aharonov and A. Casher states that **an electron moving in a plane under the influence of a perpendicular inhomogeneous magnetic field has N ground-energy states, where N is the integral part of the total flux Φ in units of the flux quantum Φ_0** $(\Phi_0=2\pi/e\equiv hc/e)$, $N=\{\Phi/\Phi_0\}$.

For the electron ground state $E=m$ we obtain from Eqs. (10-11)

$$\chi=\varphi,\quad \vec{\sigma}(\frac{\partial}{\partial\vec{x}}-ie\vec{A})\varphi=0. \quad (15)$$

In the 2-D case we have from Eqs. (15)

$$\left[\sigma_1(\frac{\partial}{\partial x_1}-ieA_1)+\sigma_2(\frac{\partial}{\partial x_2}-ieA_2)\right]\varphi\equiv$$
$$\equiv\left[\sigma_x(\frac{\partial}{\partial x}-ieA_x)+\sigma_y(\frac{\partial}{\partial y}-ieA_y)\right]\varphi=0. \quad (16)$$

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[1] Einstein joked about his first popular book on relativity that it is necessary the book subtitle **"**Gemeinverständlich" (understandable to all) to change on subtitle "Gemeinunverständlich" (not understandable to all**).**

It is remarkable that we can "remove" magnetic field from Eq. (16) by a simple transformation

$$\psi = \exp(e\phi\sigma_z)\varphi, \quad (17)$$

where ϕ satisfies the relations

$$\frac{\partial\phi}{\partial x} = A_y, \quad \frac{\partial\phi}{\partial y} = -A_x \quad (18)$$

and φ is the eigenfunction of σ_z, $\sigma_z\varphi_\sigma = \sigma\varphi_\sigma$. As

$$\frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_x = B(x,y) \quad (19)$$

then

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = B(x,y) \quad (20)$$

and we have that asymptotically for $r \to \infty$

$$\phi(x,y) = \frac{\Phi}{2\pi}\ln\frac{r}{r_0}, \quad (21)$$

where

$$\Phi = \int B(x,y)dxdy \quad (22)$$

is the total magnetic flux through the (x,y)-plane, r_0 is some real constant. So we obtain that asymptotically

$$\varphi_\sigma = \left(\frac{r_0}{r}\right)^{\frac{\sigma\Phi}{\Phi_0}}\psi_\sigma(z), \quad (23)$$

where $z = x + i\sigma y$ and $\psi_\sigma(z)$ is an entire function of z because after "removing" of the magnetic field from the Dirac-Weyl Eq. (16) this equation changed into very simple equation

$$\left(\frac{\partial}{\partial x} + i\sigma\frac{\partial}{\partial y}\right)\psi_\sigma(x,y) = 0. \quad (24)$$

In order that φ_σ to be square-integrable we should consider $\sigma\Phi > 0$ and ψ_σ has to be a polynomial whose degree is not greater than *N-1*, where $N = \{\Phi/\Phi_0\}$. So we have N independent solutions for ψ_σ, $1, z, z^2, ..., z^{N-1}$.

2.4. 2-D WEYL EQUATION AND SUPERSYMMETRIC QUANTUM MECHANICS

In this section we would like to point out a close connection of the 2-D Weyl equation to supersymmetric quantum mechanics [12], and to indicate that the SUSY energy degeneration is a special case of the Bloch-Aharonov-Casher degeneration.

Let us consider 2-D Weyl Eq. (12) for the case of $eA_x = V(y), A_y = A_z = 0, \varphi = \varphi(y)$. Then Eq. (12) gives for $\varphi(y)$

$$\left(\pm\varepsilon + i\sigma_y\frac{d}{dy} + V\sigma_x\right)\varphi(y) = 0. \quad (25)$$

From Eq. (25) we easily obtain

$$\left(-\frac{d^2}{dy^2} + V^2 + V'\sigma_z\right)\varphi = E\varphi, \ E = \varepsilon^2. \quad (26)$$

In the matrix form Eqs. (25-26) give the well known pair of Darboux conjugate equations of supersymmetric quantum mechanics,

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\frac{d\varphi}{dy} + \begin{pmatrix} 0 & V \\ V & 0 \end{pmatrix}\varphi = \mp\varepsilon\varphi; \quad (27)$$

$$\left(-\frac{d^2}{dy^2} + \begin{pmatrix} V^2 + V' & 0 \\ 0 & V^2 - V' \end{pmatrix}\right)\varphi = E\varphi, \quad (28)$$

and are investigated by mathematicians as a "canonical system" [13]. As we saw in section 2.2 **all the nonzero energy eigenstates of the "massless" electron in an arbitrary stationary magnetic field are degenerate with respect to the spin flip** precisely to the flip *right helicity* ↔ *left helicity* $(R \leftrightarrow L)$. It is clear that in the 2-D case an arbitrary superposition $\varphi = \alpha\varphi_R + \beta\varphi_L$ is a solution to Eq. (26) and Eq. (28). It is evident that

$$\varphi_R = \sigma_z\varphi_L, \quad \varphi_L = \sigma_z\varphi_R. \quad (29)$$

So an arbitrary superposition $\varphi = \gamma\varphi + \delta\sigma_z\varphi$ is also a solution to Eq. (26) and Eq. (28). The solutions

$$\varphi_\pm = (1 \pm \sigma_z)\varphi \quad (30)$$

are evidently the Darboux conjugate solutions of the SUSY Schrödinger-Pauli Eq. (26) and Eq. (28).

Let us make some historical comment to Eq. (8). H. Weyl was the first who established that Hamiltonian for a free massless spin ½ particle is

$$H = \mp i\left(\vec{\sigma}\frac{\partial}{\partial\vec{x}}\right) = -\frac{i}{\lambda}\left(\frac{\partial}{\partial\vec{x}}\vec{S}\right), \quad (31)$$

where $\vec{S} = \frac{\vec{\sigma}}{2}$ is the spin operator and $\lambda = \pm\frac{1}{2}$ is the helicity of the particle. It is known that the operator (31) may be used as a Hamiltonian in the case of arbitrary spin [14] (see also [15]).

This year is the year of centenary of two brilliant physicists Ettore Majorana and Matvei Petrovich Bronstein both passed away in 1938 [16]. E. Majorana and M.P. Bronstein were the first who considered a generalization of the Weyl equation in the form

$$\left[\frac{\partial}{\partial t} + \frac{1}{\lambda}\left(\frac{\partial}{\partial\vec{x}}\vec{S}\right)\right]\psi = 0. \quad (32)$$

They studied the case of spin one (Majorana) and spin two (Bronstein) and used Eq. (32) for the quantum treatment of photons and gravitons.

CONCLUSION

And in summary we would like to give some generalization of one non mathematical equation.

In February, 2002, one of Dirac admirers, physicist Graham Farmelo, asked the readers of "The Guardian": **"Who was the 20th century's greatest English-speaking poet? T.S. Eliot, W.B. Yeats, Sylvia Plath? Not for me; my nomination is the theoretician Paul**

Dirac, honorary poet laureate of modern physics". Graham Farmelo proposed the following equation:

Physics + Dirac = poetry.

But we believe that the Farmelo equation is incomplete and there is a good reason to extend it:

Physics + Dirac = poetry = Mathematics + Weyl.

It does not contradict to the well known Dirac claim about science and poetry: **"They are in opposition. In science you want to say something nobody knew before, in words which everyone can understand. In poetry you are bound to say something that everybody knows already in words that nobody can understand".** Poetry is many-sided and according to American poet John Crowe Ransom, **"Poetry is the kind of knowledge by which we must know that we have arranged that we shall not know otherwise"** [17]**.** Study of Dirac's and Weyl's papers is the best way to recognize truth and beauty of physics and mathematics, as well as poetry, is the best way to recognize truth and beauty of the real world.

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О НЕКОТОРЫХ СВОЙСТВАХ ДВУХМЕРНОГО УРАВНЕНИЯ ВЕЙЛЯ ДЛЯ ЗАРЯЖЕННОЙ ЧАСТИЦЫ СО СПИНОМ 1/2

И.Е. Овчаренко, Ю.П. Степановский

Обсуждаются некоторые свойства уравнения Дирака и двухмерного уравнения Вейля для заряженной частицы со спином ½ в стационарном неоднородном магнитном поле, такие, как теорема Блоха и теорема Ааронова-Кашера, и их связь с суперсимметричной квантовой механикой.

ПРО ДЕЯКІ ВЛАСТИВОСТІ ДВОВИМІРНОГО РІВНЯННЯ ВЕЙЛЯ ДЛЯ ЗАРЯДЖЕНОЇ ЧАСТИНКИ ІЗ СПІНОМ 1/2

І.Е. Овчаренко, Ю.П. Степановський

Обговорюються деякі властивості рівняння Дірака та двовимірного рівняння Вейля для зарядженої частинки із спіном ½ у стаціонарному неоднорідному магнітному полі, такі, як теорема Блоха та теорема Ааронова-Кашера, та їх зв’язок з суперсиметричною квантовою механікою.