INTEGRABLE STRING MODELS AND SIGMA-MODELS OF HYDRODYNAMIC TYPE IN TERMS OF INVARIANT CHIRAL CURRENTS

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We considered two types of string models: on the Riemann space of string coordinates with null torsion and on the Riemann-Cartan space of string coordinates with constant torsion. We used the hydrodynamic approach of Dubrovin, Novikov to integrable systems and the Dubrovin solutions of the WDVV associativity equation to construct new integrable string models of hydrodynamic type on the torsion less Riemann space of chiral currents in the first case. We used the invariant local chiral currents of principal chiral models for SU(n), SO(n), SP(n) groups to construct new integrable string models of hydrodynamic type on the Riemann-Cartan space of invariant chiral currents and on the Casimir operators, considered as the Hamiltonians, in the second case.

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1. INTRODUCTION

String theory is a very promising candidate for a unified quantum theory of gravity and all the other forces of nature. For quantum description of string model we must have classical solutions of the string in the background fields. String theory in suitable space-time backgrounds can be considered as principal chiral model. The integrability of the classical principal chiral model is manifested through an infinite set of conserved charges, which can form non-abelian algebra. Any charge from the commuting subset of charges and any Casimir operator of charge algebra can be considered as Hamiltonian in bi-Hamiltonian approach to integrable models.

Magri [1] initiated the bi-Hamiltonian approach to integrable systems. Two Poisson brackets (PB's):

$$\{\varphi^a(x),\varphi^b(y)\}_0 = P_0^{ab}(x,y)(\varphi),\qquad \{\varphi^a(x),\varphi^b(y)\}_1 = P_1^{ab}(x,y)(\varphi) \tag{1}$$

are called compatible, if any linear combination of these PB's $\{*,*\}_0+\lambda\{*,*\}_1$ is PB also for arbitrary constant λ. The functions $\varphi^a(t,x),\ a=1,2,...,n$ are local coordinates on a certain given smooth n-dimensional manifold M^n. The Hamiltonian operators

$$P_0^{ab}(x,y)(\varphi),\ P_1^{ab}(x,y)(\varphi)$$

are the functions of local coordinates $\varphi^a(x)$. It is possible to find such Hamiltonians H_0 and H_1, which satisfy the bi-Hamiltonian condition [2]:

$$\frac{d\varphi^a(x)}{dt} = \{\varphi^a(x),H_0\}_0 = \{\varphi^a(x),H_1\}_1\ , \tag{2}$$

where $H_M = \int_0^{2\pi} h_M(\varphi(y))dy,\ M=0,1$.

Two branches of hierarchies arise under two equations of motion under two different parameters of evolution t_{0M} and t_{M0} [2]:

$$\frac{d\varphi^a(x)}{dt_{01}} = \{\varphi^a(x),H_0\}_1 = \int_0^{2\pi} P_1^{ab}(x,y)\frac{\partial h_0}{\partial\varphi^b(y)}dy = \int_0^{2\pi} R_c^a(x,z)P_0^{cb}(z,y)\frac{\partial h_0}{\partial\varphi^b(y)}dy\ ;$$

$$\frac{d\varphi^a(x)}{dt_{10}} = \{\varphi^a(x),H_1\}_0 = \int_0^{2\pi} P_0^{ab}(x,y)\frac{\partial h_1}{\partial\varphi^b(y)}dy = \int_0^{2\pi}(R^{-1})_c^a(x,z)P_0^{cb}(z,y)\frac{\partial h_0}{\partial\varphi^b(y)}dy\ .$$

$R_b^a(x,y)$ is recursion operator and $(R^{-1})_b^a(x,y)$ is its inverse:

$$R_c^a(x,y) = \int_0^{2\pi} P_1^{ab}(x,z)(P_0)^{-1}_{bc}(z,y)dz\ . \tag{3}$$

The first branch of the hierarchy of dynamical systems has the following form:

$$\frac{d\varphi^a(x)}{dt_{0M}} = \int_0^{2\pi}(R(x,y_1)...R(y_{M-2},y_{M-1}))_c^a \times P_0^{cb}(y_{M-1},y_M)\frac{\partial h_0}{\partial\varphi^b(y_M)}dy_1...dy_M\ .$$

The second branch of the hierarchy can be obtained by replacement $R\to R^{-1}$ and $t_{0M}\to t_{M0}$. Dubrovin, Novikov [3, 4] and Tsarev [5] introduced the local PB of hydrodynamical type for Hamiltonian description of equations of hydrodynamics. Ferapontov [6] and Mokhov, Ferapontov [7] generalized it on the non-local PB's of hydrodynamic type. Integrable systems of hydrodynamic type are described by Hamiltonians of hydrodynamic type, which are not depending of derivatives of local coordinates. Integrable bi-Hamiltonian systems of hydrodynamic type were considered by Maltsev [8], Ferapontov [9], Mokhov [11], Pavlov [12], Maltsev, Novikov [13]. Polynomials of local chiral currents were considered by Goldshmidt and Witten [14] (see also

[15]). Local conserved chiral charges in principal chiral models were considered by Evans, Hassan, MacKay, Mountain [23]. Integrable string models of hydrodynamic type were considered by author [16,17].

2. STRING MODEL OF SIGMA-MODEL TYPE

String model is described by the Lagrangian

$$L = \frac{1}{2}\int_0^{2\pi} \eta^{\alpha\beta} g_{ab}(\varphi(t,x)) \frac{\partial\varphi^a(t,x)}{\partial x^\alpha}\frac{\partial\varphi^b(t,x)}{\partial x^\beta} dx$$

and by two first kind constraints:

$$g_{ab}(\varphi(x))[\frac{\partial\varphi^a(x)}{\partial t}\frac{\partial\varphi^b(x)}{\partial t} + \frac{\partial\varphi^a(x)}{\partial x}\frac{\partial\varphi^b(x)}{\partial x}] \approx 0,$$

$$g_{ab}(\varphi(x))\frac{\partial\varphi^a(x)}{\partial t}\frac{\partial\varphi^b(x)}{\partial x} \approx 0 .$$

The target space of local coordinates $\varphi^a(x)$, $a = 1,...,n$ belong to certain given smooth n-dimensional manifold M^n with nongenerated metric tensor $g_{ab}(\varphi(x)) = \eta_{\mu\nu} e^\mu_a(\varphi(x)) e^\nu_b(\varphi(x))$, where $\mu,\nu = 1,...,n$ are indexes of tangent space for the manifold M^n in some point $\varphi^a(x)$. The veilbein $e^\mu_a(\varphi)$ and its inverse $e^a_\mu(\varphi)$ satisfy to the conditions: $e^\mu_a e^b_\mu = \delta^b_a$, $e^\mu_a e^{a\nu} = \eta^{\mu\nu}$.

The coordinates x^α ($x^0 = t$, $x^1 = x$) belong to the world sheet with metric tensor $g_{\alpha\beta}(x)$ in the conformal gauge. String equations of motion have the form:

$$\eta^{\alpha\beta}[\partial_{\alpha\beta}\varphi^a + \Gamma^a_{bc}(\varphi)\partial_\alpha\varphi^b\partial_\beta\varphi^c] = 0 ,$$

where $\Gamma^a_{bc}(\varphi) = \frac{1}{2} e^a_\mu [\frac{\partial e^\mu_b}{\partial\varphi^c} + \frac{\partial e^\mu_c}{\partial\varphi^b}]$ is connection.

In terms of the canonical chiral currents $J^\mu_\alpha(\varphi) = e^\mu_a(\varphi)\partial_\alpha\varphi^a$, $\partial_\alpha = \frac{\partial}{\partial x^\alpha}$ equations of motion have the form:

$$\eta^{\alpha\beta}\partial_\alpha J^\mu_\beta(\varphi(t,x)) = 0 ,$$

$$\partial_\alpha J^\mu_\beta(\varphi) - \partial_\beta J^\mu_\alpha(\varphi) - C^\mu_{\nu\lambda}(\varphi) J^\nu_\alpha(\varphi) J^\lambda_\beta(\varphi) = 0 ,$$

where $C^\mu_{\nu\lambda}(\varphi) = \frac{1}{2} e^a_\nu e^b_\lambda [\frac{\partial e^\mu_a}{\partial\varphi^b} - \frac{\partial e^\mu_b}{\partial\varphi^a}]$ is torsion. The Hamiltonian has form:

$$H = \frac{1}{2}\int_0^{2\pi} [\eta^{\mu\nu} J_{0\mu} J_{0\nu} + \eta_{\mu\nu} J^\mu_1 J^\nu_1] dx , \qquad (4)$$

where $J_{0\mu}(\varphi) = e^a_\mu(\varphi) p_a$, $J^\mu_1(\varphi) = e^\mu_a(\varphi)\frac{\partial\varphi^a}{\partial x}$ and $p_a(t,x)$ is canonical momentum. The canonical commutation relations of currents are the following:

$$\{J_{0\mu}(x), J_{0\nu}(y)\} = C^\lambda_{\mu\nu}(\varphi(x)) J_{0\lambda}(\varphi(x))\delta(x-y) ,$$

$$\{J_{0\mu}(x), J^\nu_1(y)\} = C^\nu_{\mu\lambda}(\varphi(x)) J^\lambda_1(\varphi(x))\delta(x-y) - \frac{1}{2}\delta^\nu_\mu \frac{\partial}{\partial x}\delta(x-y) ,$$

$$\{J^\mu_1(x), J^\nu_1(y)\} = 0 .$$

Let us introduce chiral currents:

$$U^\mu = \eta^{\mu\nu} J_{0\nu} + J^\mu_1 \ , V^\mu = \eta^{\mu\nu} J_{0\nu} - J^\mu_1 .$$

The commutation relations of the chiral currents are the following:

$$\{U^\mu(x), U^\nu(y)\} = C^{\mu\nu}_\lambda [\frac{3}{2} U^\lambda(x) - \frac{1}{2} V^\lambda(x)]\delta(x-y) - \eta^{\mu\nu}\frac{\partial}{\partial x}\delta(x-y) ,$$

$$\{U^\mu(x), V^\nu(y)\} = C^{\mu\nu}_\lambda [U^\lambda(x) + V^\lambda(x)]\delta(x-y) , \qquad (5)$$

$$\{V^\mu(x), V^\nu(y)\} = C^{\mu\nu}_\lambda [\frac{3}{2} V^\lambda(x) - \frac{1}{2} U^\lambda(x)]\delta(x-y) + \eta^{\mu\nu}\frac{\partial}{\partial x}\delta(x-y) .$$

The equations of motion in the light-cone coordinates $x^\pm = \frac{1}{2}(t \pm x)$, $\frac{\partial}{\partial x^\pm} = \frac{\partial}{\partial t} \pm \frac{\partial}{\partial x}$ have the form:

$$\partial_- U^\mu = C^\mu_{\nu\lambda} U^\nu V^\lambda , \partial_+ V^\mu = C^\mu_{\nu\lambda} V^\nu U^\lambda .$$

In the case of null torsion:

$$C^\mu_{\nu\lambda}(\varphi) = 0 , e^\mu_a(\varphi) = \frac{\partial e^\mu}{\partial\varphi^a} ,$$

$$\Gamma^a_{bc}(\varphi) = e^a_\mu \frac{\partial^2 e^\mu}{\partial\varphi^b\partial\varphi^c} = 0 , R^\mu_{\nu\lambda\rho}(\varphi) = 0$$

string model is integrable one. The Hamiltonian (4) describes two independent left and right movers: $U^\mu(t+x)$ and $V^\mu(t-x)$.

3. INTEGRABLE STRING MODELS OF HYDRODYNAMIC TYPE WITH NULL TORSION

We want to construct new integrable string models with Hamiltonians, which are polynomials of the initial chiral currents $U^\mu(\varphi(x))$. The PB of chiral currents $U^\mu(x)$ coincides with the flat PB of Dubrovin, Novikov

$$\{U^\mu(x), U^\nu(y)\}_0 = -\eta^{\mu\nu}\frac{\partial}{\partial x}\delta(x-y) .$$

Let us introduce local Dubrovin, Novikov PB [3,4]. It has the form:

$$\{U^\mu(x), U^\nu(y)\} = g^{\mu\nu}(U(x))\frac{\partial}{\partial x}\delta(x-y) - \Gamma^{\mu\nu}_\lambda(U(x))\frac{\partial U^\lambda(x)}{\partial x}\delta(x-y) . \qquad (6)$$

The PB is skew-symmetric if $g^{\mu\nu}(U) = g^{\nu\mu}(U)$ and it satisfy to Jacobi identity [18] if

$$\Gamma^a_{bc}(U)=\Gamma^a_{cb}(U)\,,\; C^a_{bc}(U)=0\,,\; R^a_{bcd}(U)=0\,.$$

In the case of non-zero curvature tensor we must include [6,7] Weingarten operators to right side of PB with the step-function

$$\mathrm{sgn}(x-y)=(\frac{d}{dx})^{-1}\delta(x-y)=\nu(x-y)\,.$$

The PB's $\{*,*\}_0$ and $\{*,*\}_1$ are compatible by Magri [1] if pencil $\{*,*\}_0+\lambda\{*,*\}_1$ is PB also. As a result, Mokhov [11,10] have obtained compatible pair of PB's:

$$P_{0\mu\nu}(U)(x,y)=-\eta_{\mu\nu}\frac{\partial}{\partial x}\delta(x-y)\,; \tag{7}$$

$$P_{1\mu\nu}(U)(x,y)=2\frac{\partial^2F(U(x))}{\partial U^\mu\partial U^\nu}\frac{\partial}{\partial x}\delta(x-y) + \frac{\partial^3F(U(x))}{\partial U^\mu\partial U^\nu\partial U^\lambda}\frac{\partial U^\lambda}{\partial x}\delta(x-y)\,. \tag{8}$$

The function F(U) satisfies equation:

$$\frac{\partial^3F(U)}{\partial U^\mu\partial U^\rho\partial U^\lambda}\frac{\partial^3F(U)}{\partial U^\nu\partial U^\sigma\partial U^\lambda} = \frac{\partial^3F(U)}{\partial U^\nu\partial U^\rho\partial U^\lambda}\frac{\partial^3F(U)}{\partial U^\mu\partial U^\sigma\partial U^\lambda}\,. \tag{9}$$

This equation is WDVV [19,20] associativity equation and it was obtained in 2D topological field theory. Dubrovin [21,22] obtained a lot of solutions of WDVV equation. He showed, that local fields $U^\mu(x)$ must belong to Frobenius manifolds to solve the WDVV equation and gave examples of Frobenius structures. Associative Frobenius algebra may be written in the following form:

$$\frac{\partial}{\partial U^\mu}*\frac{\partial}{\partial U^\nu}:=d^\lambda_{\mu\nu}(U)\frac{\partial}{\partial U^\lambda}\,.$$

Totally symmetric structure function has form:

$$d_{\mu\nu\lambda}(U)=\frac{\partial F(U)}{\partial U^\mu\partial U^\nu\partial U^\lambda}\,,\; \mu=1,...,n.$$

The associativity condition

$$(\frac{\partial}{\partial U^\mu}*\frac{\partial}{\partial U^\nu})*\frac{\partial}{\partial U^\lambda}=\frac{\partial}{\partial U^\mu}*(\frac{\partial}{\partial U^\nu}*\frac{\partial}{\partial U^\lambda})$$

leads to the WDVV equation. Function *F(U)* is quasi-homogeneous function of its variables:

$$F(\lambda^{d_1}U_1,...,\lambda^{d_n}U_n)=\lambda^{d_F}F(U_1,...,U_n)\,.$$

Frobenius manifolds can be realized as Coxeter groups (group of reflections). Coxeter groups of correspondent simple Lie algebras (SU(n),SO(n),SP(n)) are Weyl groups. Dubrovin examples of certain solutions of WDVV equation are:

$$n=1,\, F(U)=U_1^3\,;\; n=2,\, F(U)=\frac{1}{2}U_1^2U_2+e^{U_2}\,, \quad n=3,\, F(U)=d_{\mu\nu\lambda}U^\mu U^\nu U^\lambda\,,\; \mu=1,2,3. \tag{10}$$

We used local fields U_μ with the low indexes there for convenience. One of Dubrovin polynomial solution is:

$$F(U)=\frac{1}{2}(U_1^2U_3+U_1U_2^2)+\frac{1}{4}U_2^2U_3^2+\frac{1}{60}U_3^5\,. \tag{11}$$

In the bi-Hamiltonian approach to integrable string model we must construct the recursion operator to generate hierarchy of PB's and hierarchy of Hamiltonians:

$$R^\mu_\nu(x,y)=\int_0^{2\pi}P_1^{\mu\lambda}(x,z)(P_0^{-1})_{\lambda\nu}(z,y)dz = 2\frac{\partial^2F(U)}{\partial U^\mu\partial U^\nu}\delta(x-y)+\frac{d}{dx}\frac{\partial^2F(U)}{\partial U^\mu\partial U^\nu}\nu(x-y)\,.$$

The Hamiltonian equation of motion with Hamiltonian H_0 is the following:

$$H_0=\int_0^{2\pi}\eta_{\mu\nu}U^\mu(x)U^\nu(x)dx\,,\; \frac{\partial U^\mu}{\partial t}=\frac{\partial U^\mu}{\partial x}\,. \tag{12}$$

First of new equations of motion under new time t_{01} has the form [11]:

$$\frac{\partial U^\mu}{\partial t_{01}}=\int_0^{2\pi}R^\mu_\nu(x,y)\frac{\partial U^\nu(y)}{\partial y}dy=\eta^{\mu\nu}\frac{d}{dx}(\frac{\partial F(x)}{\partial U^\nu})\,. \tag{13}$$

This equation of motion can be obtained as Hamiltonian equation with the new Hamiltonian H_1:

$$H_1=\int_0^{2\pi}\frac{\partial F(U(x))}{\partial U^\mu}U^\mu(x)dx\,, \tag{14}$$

where F(U) is each of the Dubrovin solutions of WDVV associativity equation (10), (11).

Any system of the following hierarchy [11]:

$$\frac{\partial U^\mu}{\partial t_{0M}}=\int_0^{2\pi}(R(x,y_1)...R(y_{M-1},y_M))^\mu_\nu\frac{\partial U^\nu}{\partial y_M}\prod_{k=1}^M dy_k$$

is integrable system. As a result, we will obtain chiral currents $U^\mu(\varphi(t_{0M},x))=f^\mu(\varphi(t_{0M},x))$, where $f^\mu(\varphi)$ is solution of equation of motion. In the case of Hamiltonian H_1 and the equation of motion (27) one can introduce new currents:

$$J_0^\mu(t_{01},x)=U^\mu(t_{01},x),$$
$$J_1^\mu(t_{01},x)=\eta^{\mu\nu}\frac{\partial F(t_{01},x)}{\partial U^\nu}\,.$$

Consequently, we can introduce new metric tensor and new velbein depending on new time coordinate. New string equation of motion has form:

$$e^\mu_a(\varphi(t_{01},x))\frac{\partial\varphi^a(t_{01},x)}{\partial x}=\frac{de^\mu(\varphi(t_{01},x))}{dx} =\eta^{\mu\nu}\frac{\partial F(f(\varphi(t_{01},x)))}{\partial f^\nu(\varphi(t_{01},x))}\,.$$

4. INTEGRABLE STRING MODELS WITH CONSTANT TORSION

Let us come back to commutation relations of chiral currents. Let torsion $C^\mu_{\nu\lambda}(\varphi(x))\neq 0$ and $C^\mu_{\nu\lambda}$ are structure constant of simple Lie algebra. We will consider string model with constant torsion in light-cone gauge in target space. This model coincides to principal chiral model on compact simple Lie group. We cannot divide motion on right and left mover because the chiral currents $\partial_- U^{\mu} = C^{\mu}_{\nu\lambda} U^{\nu} V^{\lambda}$, $\partial_+ V^{\mu} = C^{\mu}_{\nu\lambda} V^{\nu} U^{\lambda}$ do not conserve. The correspondent charges are not Casimirs. Evans, Hassan, MacKay, Mountain ([27] and other references therein) constructed local invariant chiral currents as polynomial of initial chiral currents of SU(n), SO(n), SP(n) principal chiral models and they found commutative combination of them. Correspondent charges are Casimir operators of these dynamical systems. This paper was based on the paper of de Azcarraga, Macfarlane, MacKay, Perez Buena ([24] and other references therein) about invariant tensors for simple Lie group.

Let t_{μ} be a matrix representation of generators Lie algebra: $[t_{\mu}, t_{\nu}] = C^{\lambda}_{\mu\nu} t_{\lambda}$, $Tr(t_{\mu} t_{\nu}) = -\frac{1}{2}\delta_{\mu\nu}$.

There is additional relation for SU(n) algebra:

$$\{t_{\mu}, t_{\nu}\} = -\frac{1}{n}\delta_{\mu\nu} - i d^{\lambda}_{\mu\nu} t_{\lambda} \ , \ \mu = 1,...,n^2-1.$$

Invariant tensors have the following form:

$$d_{\mu_1 \cdots \mu_M} = STr(t_{\mu_1} ... t_{\mu_M}) = d^{k_1}_{(\mu_1\mu_2} d^{k_2}_{\mu_3 k_1} ... d_{k_{M-3}\mu_{M-1}\mu_M)} \ ,$$

where STr means Tr of completely symmetrized product of matrix and $d_{\mu_1 \cdots \mu_M}$ is totally symmetric tensor. There are n-1 primitive tensors for SU(n). The invariant tensors for $M \geq n$ are functions of primitive tensors.

De Azcarraga et al. gave some examples of these functions and they gave general method to calculate them. Evans et al. introduced local chiral currents based on the symmetric tensors of simple Lie algebra:

$$J_M(U) = STr\, U^M = d_{\mu_1 \cdots \mu_M} U^{\mu_1} \cdots U^{\mu_M} \ ,$$

where $U = t_{\mu} U^{\mu}$ $\mu = 1,...,n^2-1$. The commutation relations of invariant currents SU(n) [23] are PB's of hydrodynamics type:

$$\{J_M(x), J_N(x)\} = -[MN\, J_{M+N-2}(x) - \frac{MN}{n} J_{M-1}(x) J_{N-1}(x)] \frac{\partial}{\partial x}\delta(x-y) - [\frac{MN(N-1)}{M+N-2} \frac{\partial J_{M+N-2}(x)}{\partial x} - \frac{MN}{n} J_{M-1}(x) \frac{\partial J_{N-1}(x)}{\partial x}] \delta(x-y) \, .$$

Let us note, that ultra local term in commutation relation of chiral currents U^{μ} (5) does not contribute to the commutation relations of invariant chiral currents because of totally symmetric invariant tensors $d_{\mu_1 \cdots \mu_M}$.

Therefore, chiral currents $J_M(U(x))$ form closed algebra under canonical PB. Evans et.al. found conserved combination of invariant chiral currents:

$$K_2(U) = J_2(U) \, , \, K_3(U) = J_3(U) \, ,$$

$$K_4(U) = J_4(U) - \frac{3}{2n} J_2^2(U) \, ,$$

$$K_5(U) = J_5(U) - \frac{10}{3n} J_2(U) J_3(U) \, ,$$

$$K_6(U) = J_6(U) - \frac{5}{3n} J_3^2(U) - \frac{15}{4n} J_4(U) J_2(U) + \frac{25}{8n^2} J_2^3(U).$$

Corresponding charges of chiral currents $K_M(U)$ are Casimir operators.

Let us apply the hydrodynamics approach to integrable string models with constant torsion. In this case we must consider the conserved chiral currents $K_M(U(x))$, $M = 2,3,...,n-1$ as the local fields of Riemann-Cartan manifold. The corresponding local charges form the hierarchy of new Hamiltonians in bi-Hamiltonian approach to integrable systems. The commutation relations of invariant chiral currents are local PB's of hydrodynamics type. The metric tensor $g_{MN}(K)$ for the SU(3) group has the following form:

$$(g)_{MN}(K) = \begin{pmatrix} -4K_2 & -6K_3 \\ -6K_3 & -c_0 K_2^2 \end{pmatrix}.$$

The metric tensor for SU(4) group is the following:

$$(g)_{MN}(K) = \begin{pmatrix} -4K_2 & | -6K_3 & | -8K_4 + 3K_2^2 \\ -6K_3 & | -9K_4 - \frac{45}{8}K_2^2 & | -\frac{1}{2}K_3 K_2 \\ -8K_4 + 3K_2^2 & | -\frac{1}{2}K_3 K_2 & | c_1 K_3^2 + c_2 K_4 K_2 \\ +c_3 K_2^3 & & \end{pmatrix}.$$

The equations of motion for local fields $K_M(U(x))$ are the following:

$$\frac{\partial K_M(U(x))}{\partial t_P} = \{K_M(U(x)), \int_0^{2\pi} K_P(U(y)) dy \, ,$$

$$M = 2,3,...,n-1 \, ; P \geq 2 \, .$$

The similar method of construction of chiral currents for SO(2n+1), SP(n) groups was used by Evans et al. on the basis of symmetric invariant tensors of Azcarraga et al. Symmetric product of three generators of these algebras let to introduce symmetric structure tensor:

$$t_{(\mu} t_{\nu} t_{\lambda)} = V^{\rho}_{\mu\nu\lambda} t_{\rho} \ .$$

Invariant symmetric tensors have the form:

$$V_{\mu_1 \ldots \mu_{2M}} = V^{\nu_1}_{(\mu_1\mu_2\mu_3} V^{\nu_2}_{\mu_4\mu_5\nu_1} ... V_{\nu_{2M-3}\mu_{2M-2}\mu_{2M-1}\mu_{2M})} \ .$$

The invariant chiral currents J_{2M} can be constructed from invariant symmetric tensor and initial chiral currents U^{μ} :

$$J_{2M} = V_{\mu_1 \ldots \mu_{2M}} U^{\mu_1} ... U^{\mu_{2M}} \ .$$

The commutation relations of invariant chiral currents are PB's of hydrodynamics type:

$$\{J_M(x), J_N(x)\} = -MN\, J_{M+N-2}(x)\frac{\partial}{\partial x}\delta(x-y)$$
$$-\frac{MN(N-1)}{M+N-2}\frac{\partial J_{M+N-2}(x)}{\partial x}\delta(x-y).$$

Conserved densities of Casimir operators of SU(2n+1), SP(n) group have the form:

$$K_2(U) = J_2,\ K_4(U) = J_4 - \frac{1}{2}(3\alpha)J_2^2,$$
$$K_6(U) = J_6 - \frac{3}{4}(5\alpha)J_4 J_2 + \frac{1}{8}(5\alpha)^2 J_2^3,$$
$$K_8(U) = J_8 - \frac{2}{3}(7\alpha)J_6 J_2 - \frac{1}{4}(7\alpha)J_4^2 +$$
$$(7\alpha)^2 J_4 J_2^2 - \frac{1}{48}(7\alpha)^3 J_2^4.$$

Constant parameter α is arbitrary one.

5. INTEGRABLE STRING MODELS IN TERMS OF POLMEYER TENSOR NONLOCAL CURRENTS

In the case of flat space $C^{\mu}_{\nu\lambda} = 0$, there are nonlocal tensor totally symmetric chiral currents, such called as "Polmeyer" currents [25-27]:

$$R^{(M)}(U(x)) \equiv R^{\mu_1 \cdots \mu_M}(U(x))$$
$$= U^{(\mu_1}(x)\int_0^x U^{\mu_2}(x_1)dx_1 \ldots \int_0^{x_{M-2}} U^{\mu_M)}(x_{M-1})dx_{M-1},$$

where round brackets mean totally symmetric product of the chiral currents $U^{\mu}(x)$. New Hamiltonians have the following form:

$$H^{(M)} = \frac{1}{2}\int_0^{2\pi} R^{(M)}(U(x))\, M\, R^{(M)}(U(x))dx,$$

where M is totally symmetric invariant constant tensor, which can be constructed from Kronecker deltas. For example:

$$R^{(2)} \equiv R^{\mu\nu} = \frac{1}{2}[U^{\mu}(x)\int_0^{2\pi} U^{\nu}(x_1)dx_1$$
$$+U^{\nu}(x)\int_0^{2\pi} U^{\mu}(x_1)dx_1],$$
$$H^{(2)} = \frac{1}{2}\int_0^{2\pi}[U^{\mu}(x)U^{\mu}(x)\int_0^{2\pi} U^{\nu}(x_1)dx_1\int_0^{2\pi} U^{\nu}(x_2)dx_2$$
$$+U^{\mu}(x)U^{\nu}(x)\int_0^{2\pi} U^{\mu}(x_1)dx_1\int_0^{2\pi} U^{\nu}(x_2)dx_2]dx.$$

The Hamiltonian $H^{(2)}$ commutes with Hamiltonian $H^{(1)} = \frac{1}{2}\int_0^{2\pi} U^{\mu}(x)U^{\mu}(x)dx$ and it commutes with the Casimir $\int_0^{2\pi} U^{\mu}(x)dx$. The equation of motion under the Hamiltonian $H^{(2)}$ is the following:

$$\frac{\partial U^{\mu}(x)}{\partial t} = \frac{\partial}{\partial x}[U^{\mu}(x)\int_0^{2\pi} U^{\nu}(x_1)dx_1\int_0^{2\pi} U^{\nu}(x_2)dx_2$$
$$+U^{\nu}(x)\int_0^{2\pi} U^{\mu}(x_1)dx_1\int_0^{2\pi} U^{\nu}(x_2)dx_2]$$
$$-U^{\nu}(x)U^{\nu}(x)\int_0^{2\pi} U^{\mu}(x_1)dx_1 - U^{\mu}(x)U^{\nu}(x)\int_0^{2\pi} U^{\nu}(x_1)dx_1.$$

In the variables $S^{\mu}(x) = \int_0^x U^{\mu}(x_1)dx_1$ the last equation can be rewritten as the following:

$$\frac{\partial S^{\mu}}{\partial t} = \frac{\partial}{\partial x}[S^{\mu}(S^{\nu}S^{\nu})] + \int_0^x S^{\mu}(S^{\nu}\frac{\partial^2 S^{\nu}}{\partial^2 x_1})dx_1$$
$$\mu = 1, \ldots, n.$$

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ИНТЕГРИРОВАННЫЕ СТРУННЫЕ МОДЕЛИ И СИГМА-МОДЕЛИ ГИДРОДИНАМИЧЕСКОГО ТИПА В ТЕРМИНАХ ИНВАРИАНТНЫХ КИРАЛЬНЫХ ТОКОВ

В.Д. Гершун

Рассмотрены два типа струнных моделей: на пространстве Римана струнных координат с нулевым кручением и на пространстве Римана-Картана с постоянным кручением. В первом случае использовали гидродинамический подход Дубровина, Новикова к интегрированным системам и Дубровина решения ВДВВ уравнения ассоциативности, чтобы построить новые итегрированные струнные модели гидродинамического типа на пространстве Римана киральных токов с нулевым кручением. Во втором случае использовали локальные инвариантные киральные токи в модели главного кирального поля для SU(n), SO(n), SP(n)-групп, чтобы построить новые интегрированные струнные модели гидродинамического типа на Римана-Картана-пространстве инвариантных киральных токов и на операторах Казимира, рассматриваемых как гамильтонианы.

ІНТЕГРОВАНІ СТРУННІ МОДЕЛІ ТА СИГМА-МОДЕЛІ ГІДРОДИНАМІЧНОГО ТИПУ У ТЕРМІНАХ ІНВАРІАНТНИХ КІРАЛЬНИХ ТОКІВ

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Розглянуто два типу струнних моделей: на просторі Рімана струнних координат з нульовим скрутом та на просторі Рімана-Картана з постійним скрутом. В першому випадку, ми використали гідродинамічний підхід Дубровіна-Новікова до інтегрованих систем та розв’язок Дубровіна рівняння асоціативності ВДВВ, щоб побудувати нові інтегровані струнні моделі гідродинамічного типу на безскрутному просторі Рімана кіральних токів. У другому випадку використали інваріантні локальні кіральні токи SU(n), SO(n), SP(n)-моделі головного кірального поля, щоб побудувати нові інтегровані струнні моделі гідродинамічного типу на просторі Рімана-Картана інваріантних кіральних токів та на операторах Казіміра, розглянутих як гамільтоніани.