DISPERSION RELATIONS FOR FIELD-ALIGNED CYCLOTRON WAVES IN THE LABORATORY DIPOLE MAGNETOSPHERIC PLASMAS WITH ANISOTROPIC TEMPERATURE

N.I. Grishanov[1,2], N.A. Azarenkov[1], A.G. Kovalenko[2]

[1]V.N. Karazin Kharkov National University, Department of Physics and Technology, Ukraine;
[2]Ukrainian State Academy of Railway Transport, Department of Physics, Ukraine

Dispersion relations are derived for field aligned circularly-polarized waves in a laboratory magnetic dipole plasma. The steady-state bi-maxwellian distribution function is used to model the energetic particles with anisotropic temperature. The growth rate of cyclotron waves in the laboratory magnetosphere is defined by contribution of the resonant trapped and untrapped particles to the imaginary part of the transverse permittivity components.

1. INTRODUCTION

Plasma confined in the levitated magnetic dipole [1] is an alternative configuration for realization of the controlled thermonuclear fusion and suitable to model some phenomena in the Earth's magnetosphere. As well known, the energetic particles with a non-equilibrium distribution function can excite a wide class of the wave instabilities in any magnetized plasma. These instabilities in the two-dimensional (2D) axisymmetric traps could be described in the scope of the 2D kinetic wave theory by solving the Maxwell's equations with a proper 'kinetic' dielectric tensor. Moreover, describing the wave-particle interaction in a laboratory dipole plasma, we should account that there are two entirely different groups of the so-called trapped and untrapped particles [2]. In this paper, we derive the dispersion relations of the field-aligned waves in the Levitated Dipole eXperiment (LDX) plasma having the energetic particles, e.g. electrons under the electron cyclotron resonance heating [3], with anisotropic temperature. To simplify a problem the Vlasov equation is solved neglecting the drift effects, the finite Larmor radius corrections and the finite orbit widths of the trapped and untrapped particles in the LDX plasma.

2. REDUCED VLASOV EQUATION

To describe a 2D axisymmetric LDX-like plasma we use the quasi-toroidal coordinates (r,θ,ϕ) connected with cylindrical ones (ρ,ϕ,z) as $\rho = a + r\cos\theta$, $z = -r\sin\theta$, $\phi = \phi$. In this case, the cylindrical components of the stationary magnetic field, $\mathbf{H}_0 = (H_{0\rho}, 0, H_{0z})$, are

$$H_{0\rho} = \frac{2Ir\sin\theta}{c(a+r\cos\theta)\sqrt{r^2+4a^2+4ar\cos\theta}} \times \left[K(\kappa) - \frac{r^2+2a^2+2ar\cos\theta}{r^2}E(\kappa)\right], \quad (1)$$

$$H_{0z} = \frac{2I/c}{\sqrt{r^2+4a^2+4ar\cos\theta}}\left[K(\kappa) - \frac{r+2a\cos\theta}{r}E(\kappa)\right]$$

where a is the current ring radius, I is the ring current, c is the speed of light; $K(\kappa)$, $E(\kappa)$ and $\kappa = 4a(a+r\cos\theta)/(r^2+4a^2+4ar\cos\theta)$ are the complete elliptic integrals of the first and second kind and their argument, respectively.

Introducing the variables (v,μ,L) instead of $(v_\parallel, v_\perp, r)$

$$v = \sqrt{v_\parallel^2 + v_\perp^2}, \qquad \mu = \frac{v_\perp^2 H_0(r,0)}{v^2 H_0(r,\theta)},$$

$$L = \frac{\pi a}{\sqrt{r^2+4a^2+4ar\cos\theta}\left[(2-\kappa)K(\kappa) - 2E(\kappa)\right]}, \quad (2)$$

the linearized Vlasov equation for the first harmonics of the perturbed distribution function,

$$f(t,\mathbf{r},\mathbf{v}) = \sum_s^{\pm 1}\sum_l^{\pm\infty} f_l^s(L,\theta,v,\mu)\exp(-i\omega t + im\phi - il\sigma), \quad (3)$$

in the zero-order of a magnetization parameter can be reduced to the set of the first order differential equations with respect to a single θ-variable:

$$\frac{\sqrt{1-\mu b(L,\theta)}}{\delta(L,\theta)}\frac{\partial f_l^s}{\partial\theta} - is\frac{La}{v}\left[\omega - l\Omega_{c0}b(L,\theta)\right]f_l^s = Q_l^s. \quad (4)$$

Here

$$\delta = \frac{0.5cr^2 H_0(r,\theta)(a+r\cos\theta)\sqrt{r^2+4a^2+4ar\cos\theta}}{ILa\left[(r^2+2a^2+3ar\cos\theta)E(\kappa) - r(r+a\cos\theta)K(\kappa)\right]},$$

$$b(L,\theta) = \frac{H_0(r(L,\theta),\theta)}{H_0(r(L,0),0)}, \quad \Omega_{c0} = \frac{eH_0(r(L,0),0)}{Mc},$$

$$Q_0^s = \frac{2ev}{Mv_{T\parallel}^2}LaF\sqrt{1-\mu b(L,\theta)}E_\parallel, \quad v_{T\parallel} = \sqrt{\frac{2T_\parallel}{M}}, \quad (5)$$

$$Q_{\pm 1}^s = \frac{seLa}{Mv_{T\parallel}^2}\sqrt{\mu}F\left[\frac{E_{\pm 1}}{\sqrt{b(L,\theta)}}\left(b(L,\theta) + 1 - \frac{T_\parallel}{T_\perp}\right) - i\frac{sv\sqrt{1-\mu b(L,\theta)}}{\omega La\delta(L,\theta)}\left(1-\frac{T_\parallel}{T_\perp}\right)\frac{\partial}{\partial\theta}\frac{E_{\pm 1}}{\sqrt{b(L,\theta)}}\right],$$

$$F = \frac{N\pi^{-1.5}}{v_{T\parallel}v_{T\perp}^2}\exp\left(-\frac{v^2}{v_{T\parallel}^2}\left(1-\mu\left(1-\frac{T_\parallel}{T_\perp}\right)\right)\right), \quad v_{T\perp} = \sqrt{\frac{2T_\perp}{M}},$$

where F is the bi-maxwellian distribution function of particles with density N, mass M, charge e, parallel and transverse temperatures $T_\parallel$ and $T_\perp$; Ω_{c0} is the minimal cyclotron frequency of plasma particles at the considered (by L) magnetic field line; $H_0 = \sqrt{H_{0\rho}^2 + H_{0z}^2}$. Here, $E_{\pm 1} = E_n \pm iE_b$ describe the transverse electric field components with the left- and right-hand polarization, where E_n, E_b and $E_\parallel$ are the normal, binormal and parallel perturbed **E**-field components with respect to $\mathbf{H}_0$. By $s=\pm 1$ we distinguish the particles with positive and negative parallel velocity relatively $\mathbf{H}_0$:

$$v_{\parallel} = sv\sqrt{1-\mu b(L,\theta)}\ .$$

Since LDX plasma is a configuration with one minimum of $\mathbf{H}_0$, the plasma particles should be separated in the two populations of the trapped and untrapped particles: 1) $0 \le \mu \le \mu_0$, $-\pi \le \theta \le \pi$ for untrapped particles, where $\mu_0 = 1/b(L,\pi)$ is the inverse mirror ratio of the given (by L) magnetic field line, and 2) $\mu_0 \le \mu \le 1$, $-\theta_t \le \theta \le \theta_t$ for trapped particles, where $\pm\theta_t(\mu,L)$ are the reflection points (or stop points, or mirror points) of the trapped particles, which are defined by the zeros of parallel velocity: $v_{\parallel}(v,\mu,L,\pm\theta_t)=0$.

3. DISPERSION RELATIONS

After solving Eq. (4), the 2D transverse (with respect to $\mathbf{H}_0$) current density components, $j_{\pm 1}$, can be found as

$$j_l(L,\theta) = \frac{\pi e}{2} b^{1.5}(L,\theta)\sum_s^{\pm 1}\int_0^\infty v^3 dv\left[\int_0^{\mu_0}\frac{f_{l,u}^s\sqrt{\mu}d\mu}{\sqrt{1-\mu b(L,\theta)}} + \right.$$
$$\left. + \int_{\mu_0}^{1/b(L,\theta)}\frac{f_{l,t}^s\sqrt{\mu}d\mu}{\sqrt{1-\mu b(L,\theta)}}\right],\quad l=\pm 1,\quad (6)$$

where the indexes u and t correspond to the untrapped and trapped particles, respectively. To describe the bounce-periodic motion of the trapped and untrapped particles along the $\mathbf{H}_0$-field line, it is convenient to introduce the new time-like variable τ instead of the poloidal angle θ,

$$\tau(\theta) = \int_0^\theta \frac{\delta(L,\eta)}{\sqrt{1-\mu b(L,\eta)}}d\eta\ , \quad (7)$$

taking into account that the transit-time and bounce-period of the u- and t-particles are proportional to $\tau_{b,u}=2\tau(\pi)$ and $\tau_{b,t}=4\tau(\theta_t)$, respectively. After this, the distribution functions of u- and t-particles can be defined by the corresponding Fourier series ($\beta=u,t$):

$$f_{l,\beta}^s = \sum_p^{\pm\infty} f_{l,\beta}^{s,p}\exp\left[ip\frac{2\pi}{\tau_{b,\beta}}\tau(\theta) + isl\frac{La}{v}\int_0^\theta\frac{\widetilde{\Omega}_{c,\beta}\delta(L,\eta)}{\sqrt{1-\mu b(L,\eta)}}d\eta\right],$$

where $\widetilde{\Omega}_{c,\beta} = \Omega_{c0}b(L,\theta) - \overline{\Omega}_{c,\beta}$ is the oscillating parts of cyclotron frequencies of untrapped ($\beta=u$) and trapped ($\beta=t$) particles; $\overline{\Omega}_{c,u}$ and $\overline{\Omega}_{c,t}$ are the bounce-averaged cyclotron frequencies of u- and t-particles:

$$\overline{\Omega}_{c,u} = \frac{2\Omega_{c0}}{\tau_{b,u}}\int_0^\pi\frac{b(L,\theta)\delta(L,\theta)}{\sqrt{1-\mu b(L,\theta)}}d\theta\ , \quad (8)$$

$$\overline{\Omega}_{c,t} = \frac{4\Omega_{c0}}{\tau_{b,t}}\int_0^{\theta_t}\frac{b(L,\theta)\delta(L,\theta)}{\sqrt{1-\mu b(L,\theta)}}d\theta\ . \quad (9)$$

To evaluate the dielectric tensor elements we use the Fourier expansions of the 2D perturbed electric field and current density components over the variable λ varying along the $\mathbf{H}_0$-field line: $\lambda(\theta) = \int_0^\theta \delta(L,\eta)d\eta$. In this case

$$\frac{j_l(L,\theta)}{\sqrt{b(L,\theta)}} = \sum_n^{\pm\infty} j_l^{(n)}(L)\exp\left(i\pi n\frac{\lambda(\theta)}{\lambda_0}\right), \quad (10)$$

$$\frac{E_l(L,\theta)}{\sqrt{b(L,\theta)}} = \sum_{n'}^{\pm\infty} E_l^{(n')}(L)\exp\left(i\pi n'\frac{\lambda(\theta)}{\lambda_0}\right), \quad (11)$$

where $\lambda_0 = \lambda(\pi)$, so that $La\lambda_0$ is the half-length of the given (by L) magnetic field line. As a result,

$$\frac{4\pi i}{\omega}j_l^{(n)} = \frac{2i}{\omega}\int_{-\pi}^{\pi}\frac{j_l(L,\theta)}{\sqrt{b(L,\theta)}}\exp\left(-i\pi n\frac{\lambda(\theta)}{\lambda_0}\right)d\theta =$$
$$= \sum_{n'}^{\pm\infty}\left(\varepsilon_{l,u}^{n,n'} + \varepsilon_{l,t}^{n,n'}\right)E_l^{(n')},\quad l=\pm 1. \quad (12)$$

Here $\varepsilon_{l,u}^{n,n'}$ and $\varepsilon_{l,t}^{n,n'}$ are the contribution of u- and t-particles to the transverse permittivity elements:

$$\varepsilon_{l,u}^{n,n'} = \frac{\omega_p^2 La\pi^{-1.5}T_\parallel}{8\omega\lambda_0 v_{T\parallel}T_\perp}\sum_{p=-\infty}^{\infty}\int_0^{\mu_0}\mu d\mu\int_{-\infty}^{\infty}\exp\left[-u^2\left(1-\mu\left(1-\frac{T_\parallel}{T_\perp}\right)\right)\right]$$
$$\times A_{l,p}^n(u,\mu)B_{l,p}^{n'}(u,\mu)\frac{u^4 du}{pu - Z_{l,u}}\ , \quad (13)$$

$$\varepsilon_{l,t}^{n,n'} = \frac{\omega_p^2 La\pi^{-1.5}T_\parallel}{8\omega\lambda_0 v_{T\parallel}T_\perp}\sum_{p=-\infty}^{\infty}\int_{\mu_0}^{1}\mu d\mu\int_{-\infty}^{\infty}\exp\left[-u^2\left(1-\mu\left(1-\frac{T_\parallel}{T_\perp}\right)\right)\right]$$
$$\times C_{l,p}^n(u,\mu)D_{l,p}^{n'}(u,\mu)\frac{u^4 du}{pu - Z_{l,t}}\ , \quad (14)$$

where

$$A_{l,p}^n(u,\mu) = \int_{-\pi}^{\pi}\cos\left[\Psi_{l,p}^n(u,\mu,\theta)\right]\frac{b(L,\theta)\delta(L,\theta)}{\sqrt{1-\mu b(L,\theta)}}d\theta\ ,$$

$$B_{l,p}^n(u,\mu) = \int_{-\pi}^{\pi}\left[b(L,\theta) - 1 + \frac{T_\parallel}{T_\perp} + \frac{\pi n v_{T\parallel}u}{\omega La\lambda_0}\left(1-\frac{T_\parallel}{T_\perp}\right)\times\right.$$
$$\left.\times\sqrt{1-\mu b(L,\theta)}\right]\cos\left[\Psi_{l,p}^n(u,\mu,\theta)\right]\frac{\delta(L,\theta)\ d\theta}{\sqrt{1-\mu b(L,\theta)}}\ ,$$

$$C_{l,p}^n(u,\mu) = \int_{-\theta_t}^{\theta_t}\cos\left[\Phi_{l,p}^n(u,\mu,\theta)\right]\frac{b(L,\theta)\delta(L,\theta)}{\sqrt{1-\mu b(L,\theta)}}d\theta\ ,$$

$$D_{l,p}^n(u,\mu) = \int_{-\theta_t}^{\theta_t}\left[b(L,\theta) - 1 + \frac{T_\parallel}{T_\perp} + \frac{\pi n v_{T\parallel}u}{\omega La\lambda_0}\left(1-\frac{T_\parallel}{T_\perp}\right)\times\right.$$
$$\left.\times\sqrt{1-\mu b(L,\theta)}\right]\cos\left[\Phi_{l,p}^n(u,\mu,\theta)\right]\frac{\delta(L,\theta)\ d\theta}{\sqrt{1-\mu b(L,\theta)}} +$$
$$+(-1)^p\int_{-\theta_t}^{\theta_t}\left[b(L,\theta) - 1 + \frac{T_\parallel}{T_\perp} + \frac{\pi n v_{T\parallel}u}{\omega La\lambda_0}\left(1-\frac{T_\parallel}{T_\perp}\right)\times\right.$$
$$\left.\times\sqrt{1-\mu b(L,\theta)}\right]\cos\left[\Phi_{l,-p}^n(-u,\mu,\theta)\right]\frac{\delta(L,\theta)\ d\theta}{\sqrt{1-\mu b(L,\theta)}}\ ,$$

$$\Psi_{l,p}^n(u,\mu,\theta) = n\pi\frac{\lambda(\theta)}{\lambda_0} - \left(p\frac{2\pi}{\tau_{b,u}} - l\frac{La}{uv_{T\parallel}}\overline{\Omega}_{c,u}\right)\tau(\theta) -$$
$$- l\frac{La}{uv_{T\parallel}}\Omega_{c0}\int_0^\theta\frac{b(L,\eta)\lambda(L,\eta)}{\sqrt{1-\mu b(L,\eta)}}d\eta\ ,$$

$$\Phi_{l,p}^n(u,\mu,\theta) = n\pi\frac{\lambda(\theta)}{\lambda_0} - \left(p\frac{2\pi}{\tau_{b,t}} - l\frac{La}{uv_{T\parallel}}\overline{\Omega}_{c,t}\right)\tau(\theta) -$$
$$- l\frac{La}{uv_{T\parallel}}\Omega_{c0}\int_0^\theta\frac{b(L,\eta)\lambda(L,\eta)d\eta}{\sqrt{1-\mu b(L,\eta)}}\ ,$$

$$Z_{l,u} = \frac{La\tau_{b,u}}{2\pi v_{T\parallel}}\left(\omega - l\overline{\Omega}_{c,u}\right),\quad Z_{l,t} = \frac{La\tau_{b,t}}{2\pi v_{T\parallel}}\left(\omega - l\overline{\Omega}_{c,t}\right).$$

Note that Eqs. (13,14) describe the contribution of any kind of the u- and t-particles to $\varepsilon_l^{n,n'} = \varepsilon_{l,u}^{n,n'} + \varepsilon_{l,t}^{n,n'}$. The corresponding expressions for plasma electrons and ions can be obtained from Eqs. (13,14) by replacing the temperatures $T_{\|}$, $T_{\perp}$, density N, mass M, charge e by the electron $T_{\|e}$, $T_{\perp e}$, N_e, m_e, e_e and ion $T_{\|i}$, $T_{\perp i}$, N_i, M_i, e_i parameters, respectively.

To have analogy with the linear theory of cyclotron waves in the straight magnetic field let us assume that the $E^n_{\pm 1}$-harmonic of **E**-field gives the main contribution to $j^n_{\pm 1}$. In this case, for the field-aligned cyclotron waves (m=0, $\partial / \partial L = 0$, $E_{\|} = 0$, $H_{\|} = 0$), we get the following dispersion equation from the Maxwell's equations:

$$\left(\frac{\pi n c}{L a \lambda_0 \omega} \right)^2 = 1 + 2 \sum_{\alpha}^{e, i_1, i_2, \ldots} \varepsilon^{n,n}_{l,(\alpha)}(L), \qquad (15)$$

where α denotes the particle species (electron, proton, heavy ions), $n\pi c/(La\lambda_0\omega)$ is the non-dimensional parallel refractive index. Further, to define the conditions of the wave instabilities in the LDX-like plasmas with anisotropic temperature, Eq. (15) should be resolved numerically for the real and imaginary parts of the wave frequency, $\omega = \mathrm{Re}\,\omega + i\,\mathrm{Im}\,\omega$. As usual, the growth (damping) rate of the cyclotron waves, $\mathrm{Im}\,\omega$, is defined by the contribution of the resonant particles to the imaginary part of the transverse permittivity elements:

$$\mathrm{Im}\,\varepsilon^{n,n}_{l,(\alpha)} = \mathrm{Im}\,\varepsilon^{n,n}_{l,u,(\alpha)} + \mathrm{Im}\,\varepsilon^{n,n}_{l,t,(\alpha)} = \sum_{p=1}^{\infty} \left(\mathrm{Im}\,\varepsilon^{n,n}_{l,p,u,\alpha} + \mathrm{Im}\,\varepsilon^{n,n}_{l,p,t,\alpha} \right)$$

where

$$\mathrm{Im}\,\varepsilon^{n,n}_{l,p,u,\alpha} = \frac{0.125\omega_p^2 L\, a\, T_{\|\alpha}}{\omega\sqrt{\pi}\lambda_0 v_{T\|} T_{\perp\alpha} p^5} \int_0^{\mu_0} \exp\left[-\frac{Z^2_{l,u,\alpha}}{p^2} \left(1 - \mu \left(1 - \frac{T_{\|\alpha}}{T_{\perp\alpha}} \right) \right) \right] \times$$

$$\times A^n_{l,p}\left(\frac{Z_{l,u,\alpha}}{p}, \mu \right) B^n_{l,p}\left(\frac{Z_{l,u,\alpha}}{p}, \mu \right) Z^4_{l,u,\alpha} \mu d\mu, \qquad (16)$$

$$\mathrm{Im}\,\varepsilon^{n,n}_{l,p,t,\alpha} = \frac{0.125\omega_p^2 L\, a\, T_{\|\alpha}}{\omega\sqrt{\pi}\lambda_0 v_{T\|} T_{\perp\alpha} p^5} \int_{\mu_0}^{1} \exp\left[-\frac{Z^2_{l,t,\alpha}}{p^2} \left(1 - \mu \left(1 - \frac{T_{\|\alpha}}{T_{\perp\alpha}} \right) \right) \right]$$

$$\times C^n_{l,p}\left(\frac{Z_{l,t,\alpha}}{p}, \mu \right) D^n_{l,p}\left(\frac{Z_{l,t,\alpha}}{p}, \mu \right) Z^4_{l,t,\alpha} \mu d\mu \qquad (17)$$

are the separate contributions of the untrapped and trapped particles to the bounce resonance terms of $\mathrm{Im}\,\varepsilon^{n,n}_{l,(\sigma)}$.

4. CONCLUSIONS

In conclusion, let us summarized the main results of the paper. The contribution of the trapped and untrapped particles to the transverse permittivity elements in the LDX plasma with anisotropic temperature are expressed by summation of the bounce-resonant terms including the double integration in velocity space, the resonant denominators, and the corresponding phase coefficients.

Due to two-dimensional $\mathbf{H_0}$-field nonuniformity, the bounce resonance conditions for trapped and untrapped particles in the LDX-like plasmas are different from ones in the straight magnetic field; the whole spectrum of the electric field is present in the given current density harmonic; the left-hand and right-hand polarized waves are coupled in the general case.

The dispersion equations for field-aligned cyclotron waves in the LDX-like plasma are derived which are suitable to analyze the instabilities of both the electron-cyclotron (l=-1) and ion-cyclotron (l=1) waves. As in the uniform plasma confined in the straight magnetic field, the growth/damping rates of the cyclotron waves in the LDX-like plasmas are defined by the contribution of the resonant particles to the imaginary part of the transverse permittivity elements.

REFERENCES

1. J. Kesner, M. Mauel. Plasma confinement in a levitated magnetic dipole// *Plasma Phys. Reports*. 1997, v. 23, N9, p. 742-750 .
2. N.I. Grishanov, A.F.D. Loula, C.A. de Azevedo, J. Pereira Neto, Dielectric characteristics for radio frequency waves in a laboratory dipole plasma// *Contrib. Plasma Phys.* 2006, v. 46, N4, p. 265-279.
3. D.T. Garnier, A. Hansen et al. Production and study of high-beta plasma confined by a superconducting dipole magnet// *Phys. Plasmas.* 2006, v.13, N5, p. 056111.

Article received 22.09.08.

ДИСПЕРСИОННЫЕ УРАВНЕНИЯ ЦИКЛОТРОННЫХ ВОЛН ВДОЛЬ МАГНИТНОГО ПОЛЯ В ЛАБОРАТОРНОЙ МАГНИТОСФЕРНОЙ ПЛАЗМЕ С АНИЗОТРОПНОЙ ТЕМПЕРАТУРОЙ

Н.И. Гришанов, Н.А. Азаренков, А.Г. Коваленко

Получены дисперсионные уравнения циркулярно-поляризованный волн, распространяющихся вдоль магнитного поля в лабораторной диполярной магнитосфере. В качестве модельного распределения энергичных частиц по скоростям использована бимаксвелловская функция с анизотропной температурой. Показано, что инкремент циклотронных волн в лабораторной магнитосфере определяется вкладом резонансных пролетных и запертых частиц в мнимую часть поперечных компонент тензора диэлектрической проницаемости.

ДИСПЕРСІЙНІ СПІВВІДНОШЕННЯ ЦИКЛОТРОННИХ ХВИЛЬ ВЗДОВЖ МАГНІТНОГО ПОЛЯ В ЛАБОРАТОРНІЙ МАГНІТОСФЕРНІЙ ПЛАЗМІ З АНИЗОТРОПНОЮ ТЕМПЕРАТУРОЮ

М.І. Гришанов, М.О. Азарєнков, О.Г. Коваленко

Отримані дисперсійні співвідношення циркулярно-поляризованих хвиль, що поширюються вздовж магнітного поля у лабораторній диполярній магнітосфері. У якості модельного розподілу енергійних частинок у просторі швидкостей використана бімаксвелівська функція з анізотропною температурою. Доведено що інкремент циклотронних хвиль в лабораторній магнітосфері визначається внеском резонансних пролітних та захоплених частинок в уявну частину поперечних компонент тензора діелектричної проникності.