SOURCEWISE REPRESENTED GREEN'S FUNCTION OF THE CIRCULAR WAVEGUIDE

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Singular part of the Green's function of unbounded space is singled out in explicit form and contains all singularities, including a delta-shaped singularity. The problem of construction of Green's function for a field is solved, as a problem for diffraction of potential and rotational components electric field intensity of a point current source on the circular waveguide walls. The singling out of the electric field intensity singularity in an explicit form about a source enables to develop an effective algorithm of Green's function calculation at any distance between the source point and observation point in a circular waveguide.

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1. INTRODUCTION

Singular and hypersingular integral equations [1] with a kernel in the form of Green's function are a highly efficient apparatus of mathematical physics. These equations are applied in problems of the microwave electronics and accelerating engineering, for example, for calculation of electromagnetic fields in a coaxial girotron [2], an accelerating structure of H-type [3] and a bunch accumulator of the charged particles [4] (p.80).

Advantages of singular and hypersingular equations are connected with the use of well stipulated matrixes providing a high accuracy and stability of calculations. However, then the Green's function is to be calculated at short distances between the source points and the observation points.

Let the accelerating structure is a system of metal radio-frequency (rf) electrodes in a circular waveguide. Then the integral equations use the electric Green's function of a circular waveguide for the field $\widehat{G}_e(k,r,r')$ relative to the surface density of the force of the electric current which flows only on the electrode surface. Upon expansion using the system TE and TH waves, $\widehat{G}_e(k,r,r')$ is described by double series which diverge. This is because $\widehat{G}_e(k,r,r')$ in an implicit form includes the electric Green's function of unbounded space for a field $\widehat{G}_e^S(k,r,r')$ which has singularities $1/|\vec{r}-\vec{r'}|$, $1/|\vec{r}-\vec{r'}|^2$,$1/|\vec{r}-\vec{r'}|^3$, $\delta(\vec{r}-\vec{r'})$.

The problem of construction of Green's function for a field is solved, as a problem for diffraction of potential and rotational components of a tensor spherical wave of the electric field intensity of the point current source (a current point source is delta-shaped location current) on the circular waveguide walls. Thus the singularity of electric field intensity was singled out in an explicit form about a current source that allowed us to create an effective algorithm of electric Green's function calculation at any electric length of nonhomogeneities in the circular waveguide.

The use of the Green's function $\widehat{G}_e(k,r,r')$ with an explicit singularity enables the numerical solution of two-dimensional hypersingular integral equations instead of the three-dimensional equations. Hypersingularity and two-dimensionality of the equations can provide an increased accuracy and reduced time of calculations respectively.

2. THE BASIC PART

2.1. ANALYTICAL RELATIONS

The Green's electric function for a field is defined by the formula of [5],[6]

$$\widehat{G}_e^S(k,r,r') = \left(\hat{I} + \frac{1}{k^2} grad(r) div(r)\right) \widehat{G}_E(k,r,r'), \tag{1}$$

where $\widehat{G}_E(k,r,r')$ is the Green's function for a vector potential.

In case of a circular waveguide the function $\widehat{G}_e(k,r,r')$ is constructed by the system TE and TH waves in [7].

The Green's function of a circular waveguide for a field is obtained in the form of TE and TH waves and in the form of superpositions

$$\widehat{G}_e(k,r,r') = \widehat{G}_e^p(k,r,r') + \widehat{G}_e^r(k,r,r'), \tag{2}$$

$$\widehat{G}_e(k,r,r') = \widehat{G}_e^S(k,r,r') + \widehat{G}_e^R(k,r,r'), \tag{3}$$

where

$$\widehat{G}_e^p(k,r,r') = \widehat{G}_e^{sp}(k,r,r') + \widehat{G}_e^{Rp}(k,r,r'), \tag{4}$$

$$\widehat{G}_e^r(k,r,r') = \widehat{G}_e^{sr}(k,r,r') + \widehat{G}_e^{Rr}(k,r,r'), \tag{5}$$

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$$\widehat{G}_e^S(k,r,r') = \widehat{G}_e^{Sp}(k,r,r') + \widehat{G}_e^{Sr}(k,r,r'), \quad (6)$$

$$\widehat{G}_e^R(k,r,r') = \widehat{G}_e^{Rp}(k,r,r') + \widehat{G}_e^{Rr}(k,r,r'). \quad (7)$$

Here $\widehat{G}_e^p(k,r,r')$ and $\widehat{G}_e^r(k,r,r')$ are, respectively, the potential and rotational components of the electric field intensity of the point current source in the circular waveguide, $\widehat{G}_e^S(k,r,r')$ and $\widehat{G}_e^R(k,r,r')$ are, respectively, the intensity of the electric field in the unbounded space and of the field reflected from the walls of the circular waveguide of the point current source; $\widehat{G}_e^{Sr}(k,r,r')$ and $\widehat{G}_e^{Sr}(k,r,r')$ are, respectively, the potential and rotational components of the electric field intensity of the point current source in the unbounded space; $\widehat{G}_e^{Rp}(k,r,r')$ and $\widehat{G}_e^{Rr}(k,r,r')$ are, respectively, the potential and rotational components of the intensities of the electric field of the point current source reflected from the waveguide walls. The potential and rotational components of the electric field intensity are stipulated by the scalar and vector potentials, respectively or by the distributions of charges and currents in the source point.

The tensor function $\widehat{G}_e^S(k,r,r')$ in an explicit form describes singularities of intensity of an electric field of a current point source. It is found 9 components of $\widehat{G}_e(k,r,r')$ in the form of TE and TH waves of a waveguide and in the form of (2)- (7).

In particular,, by the system of TE and TH waves of a circular waveguide, $\widehat{G}_{e_{11'}}(k,r,r')$ is

$$\widehat{G}_{e_{11'}}(k,r,r') = \sum_{m=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty} e^{im(\phi-\phi')}\times$$
$$(A_{mn}\left(1-\frac{(k_{mn}^h)^2}{k^2}\right) J'_m(k_{mn}^h\rho)J'_m(k_{mn}^h\rho')\times$$
$$f_{mn}(z,z') + B_{mn}\frac{m^2}{(k_{mn}^e)^2\rho\rho'}J_m(k_{mn}^e\rho)\times$$
$$J_m(k_{mn}^e\rho')l_{mn}(z,z')), \quad (8)$$

where

$$\begin{aligned}
A_{mn} &= \frac{1}{\pi J_{m+1}^2(k_{mn}^e R)R^2}, \\
B_{mn} &= \frac{1}{\pi J_m^2(k_{mn}^e R)R^2\left(1-\frac{m^2}{(k_{mn}^e R)^2}\right)}, \\
l_{mn}(z,z') &= e^{-\gamma_{mn}|z-z'|}/2\gamma_{mn}, \\
\gamma_{mn} &= \sqrt{(k_{mn}^e)^2 - k^2}, \\
k_{mn}^e &= \frac{\mu_{mn}}{R}, \\
f_{mn}(z,z') &= e^{-\beta_{mn}|z-z'|}/2\beta_{mn}, \\
\beta_{mn} &= \sqrt{(k_{mn}^h)^2 - k^2}, \\
k_{mn}^h &= \frac{\nu_{mn}}{R},
\end{aligned}$$

γ_{mn}, μ_{mn} are the roots of the equations $J_m(z) = 0$ and $J'_m(z) = 0$, respectively, and R is the waveguide radius.

It is shown, that

$$div(r_\perp)\hat{G}_E(k_{mn}^e, r_\perp, r'_\perp; z, z') = 0, \quad (9)$$

i. e. TE or TH waves in the circular waveguide are rotational waves relative to the transverse coordinates.

The singular components of $G_{e_{11'}}^S(k,r,r')$, $G_{e_{11'}}^{sp}(k,r,r')$, $G_{e_{11'}}^{sr}(k,r,r')$ are described in the form of (2-7) by the formulas

$$G_{e_{11'}}^S(k,r,r') = G_{e_{11'}}^{sp}(k,r,r') + G_{e_{11'}}^{sr}(k,r,r'), \quad (10)$$

$$G_{e_{11'}}^{sp}(k,r,r') = G_{e_{xx'}}^{sp}(k,r,r')\cos\varphi\cos\varphi' +$$
$$G_{e_{yx'}}^{sp}(k,r,r')\sin\varphi\cos\varphi' + G_{e_{xy'}}^{sp}(k,r,r')\times$$
$$\cos\varphi\sin\varphi' + G_{e_{yy'}}^{sp}(k,r,r')\sin\varphi\sin\varphi', \quad (11)$$

$$G_{e_{11}}^{sr}(k,r,r') = G_{e_{xx'}}^{sr}(k,r,r')\cos\varphi\cos\varphi' +$$
$$G_{e_{yy'}}^{sr}(k,r,r')\sin\varphi\sin\varphi' = G_{E_{11}}^{sr}(k,r,r') =$$
$$\frac{1}{4\pi}\frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}\cos(\varphi-\varphi'), \quad (12)$$

$$G_{e_{xx'}}^{sp}(k,r,r') = \frac{1}{4\pi k^2}\, e^{ik|\vec{r}-\vec{r}'|}\times$$
$$\left\{\frac{3(\rho\cos\varphi-\rho'\cos\varphi')^2}{|\vec{r}-\vec{r}'|^5} - \frac{1}{|\vec{r}-\vec{r}'|^3} - \right.$$
$$\frac{4\pi}{3}\,\delta(r,r') - \frac{3ik(\rho\cos\varphi-\rho'\cos\varphi')}{|\vec{r}-\vec{r}'|^4} -$$
$$\left.\frac{k^2(\rho\cos\varphi-\rho'\cos\varphi')^2}{|\vec{r}-\vec{r}'|^3} + \frac{ik}{|\vec{r}-\vec{r}'|^2}\right\}, \quad (13)$$

$$G_{e_{xx'}}^{sr}(k,r,r') = \frac{1}{4\pi}\frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}, \quad (14)$$

$$G_{e_{xy'}}^{sp}(k,r,r') = G_{e_{yx'}}^{sp}(k,r,r') = \frac{1}{4\pi k^2}\, e^{ik|\vec{r}-\vec{r}'|}\times$$
$$(\rho\cos\varphi-\rho'\cos\varphi')(\rho\sin\varphi-\rho'\sin\varphi')\times$$
$$\left\{\frac{3}{|\vec{r}-\vec{r}'|^5} - \frac{3ik}{|\vec{r}-\vec{r}'|^4} - \frac{k^2}{|\vec{r}-\vec{r}'|^3}\right\}, \quad (15)$$

$$G_{e_{yy'}}^{sp}(k,r,r') = \frac{1}{4\pi k^2}\, e^{ik|\vec{r}-\vec{r}'|}\times$$
$$\left\{\frac{3(\rho\sin\varphi-\rho'\sin\varphi')^2}{|\vec{r}-\vec{r}'|^5} - \frac{1}{|\vec{r}-\vec{r}'|^3} - \right.$$
$$\frac{4\pi}{3}\,\delta(r,r') - \frac{3ik(\rho\sin\varphi-\rho'\sin\varphi')^2}{|\vec{r}-\vec{r}'|^4} -$$
$$\left.\frac{k^2(\rho\sin\varphi-\rho'\sin\varphi')^2}{|\vec{r}-\vec{r}'|^3} + \frac{ik}{|\vec{r}-\vec{r}'|^2}\right\}, \quad (16)$$

$$G_{e_{xx'}}^{sr}(k,r,r') = G_{e_{yy'}}^{sr}(k,r,r') = \frac{1}{4\pi}\frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}, \quad (17)$$

The potential $G_{e_{11'}}^{Rp}(k,r,r')$ and rotational $G_{e_{11'}}^{Rr}(k,r,r')$ components of the regular Green's function of a circular waveguide are in the form of (2)-(7)

$$G_{e_{11'}}^{Rp}(k,r,r') = \frac{i}{8\pi}\sum_{m=-\infty}^{+\infty} e^{im(\varphi-\varphi')}\times$$
$$\int_{-\infty}^{+\infty}\frac{e^{i\eta(z-z')}\nu^2(\eta)}{k^2}J'_m(\nu(\eta)\rho)J'_m(\nu(\eta)\rho')\frac{H_m^{(1)}(\nu(\eta)R)}{J_m(\nu(\eta)R)}d\eta, \quad (18)$$

$$G^{Rr}_{e_{11'}}(k,r,r') = -\frac{i}{8\pi}\sum_{m=-\infty}^{+\infty} e^{im(\varphi-\varphi')}\times$$

$$\int_{-\infty}^{+\infty} e^{i\eta(z-z')} J'_m(\nu(\eta)\rho) J'_m(\nu(\eta)\rho')\frac{H^{(1)}_m(\nu(\eta)R)}{J'_m(\nu(\eta)R)} +$$

$$\frac{m^2}{\nu^2\rho\rho'} J_m(\nu(\eta)\rho) J_m(\nu(\eta)\rho')\frac{H^{(1)'}_m(\nu(\eta)R)}{J'_m(\nu(\eta)R)} d\eta\,. \quad (19)$$

Notice, that the problem of construction of the Green's function for a field is solved, as a problem of diffraction of the potential and rotational components of the intensity of electric field divergent spherical wave of a point current source on the circular waveguide walls. As this takes place, the potential and rotational components correspond to the scalar and vector potentials, respectively. We used the representation of a spherical wave in the form of a spectrum of non-uniform cylindrical waves diverging in two opposite directions along the radius, i.e. the sourcewise representation of a spherical wave in the radial direction of [8](p.42)

2.2. NUMERICAL RESULTS

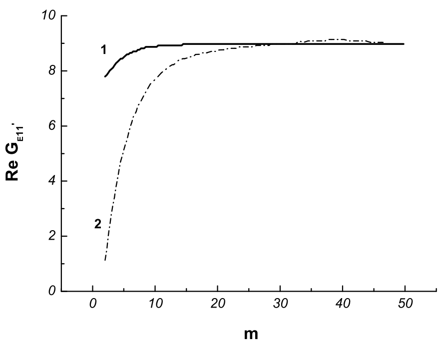

Fig.1. *Rotational component of Green's function* $Re\, G^r_{e_{11'}}(k,r,r')$ *of a circular waveguide* ($k = 12.56\, m^{-1}$; $R = 0.0755\, m$; $|\vec{r}-\vec{r}'|/\lambda = 0,02$)

The algorithm of calculation of $\hat{G}_e(k,r,r')$ in the form of TE and TH waves of a circular waveguide and in the form of (2)- (7) is developed. Singularities of the tensor Green's function are singled out in an explicit form for representation of (2)- (7). The efficiency of calculations of $\hat{G}_e(k,r,r')$ in the form of (2)- (7) is illustrated by the plots in Fig.1-4.

The real part of the rotational component $Re\, G^r_{e_{11'}}(k,r,r')$ of the Green's function and its derivative $\partial ReG^r_{e_{11'}}(k,r,r')/\partial x^1$ (x^1 is the radial coordinate) for the cutoff circular waveguide versus the number of an azimuthal harmonic m is shown in Fig.1 and Fig.3 for $k = 12.56\, m^{-1}$; $R = 0.0755\, m$; $\rho = 0.07\, m$; $\rho' = 0,08\, m$; $z = z'$; $\varphi = \varphi'$; $|\vec{r}-\vec{r}'|/\lambda = 0,02$, and in Fig.2 and Fig.4 for $k = 12,56\, m^{-1}$; $R = 0,0755\, m$; $\rho = 0.07\, m$; $\rho' = 0.08\, m$; $z = z'$; $\varphi = \varphi' + \pi$; $|\vec{r}-\vec{r}'|/\lambda = 0.26$.

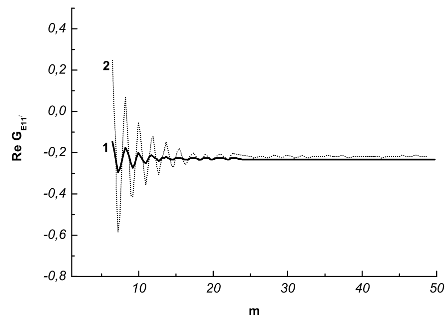

Fig.2. *Rotational component of Green's function* $Re\, G^r_{e_{11'}}(k,r,r')$ *of a circular waveguide* ($k = 12.56\, m^{-1}$; $R = 0.0755\, m$; $|\vec{r}-\vec{r}'|/\lambda = 0,26$)

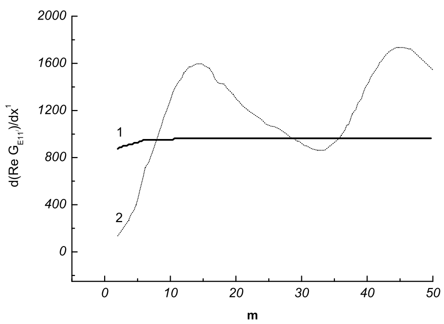

Fig.3. *Rotational component of Green's function* $Re\, \partial G^r_{e_{11'}}(k,r,r')/\partial x^1$ *of a circular waveguide* ($k = 12.56 m^{-1}$; $R = 0.0755 m$; $|\vec{r}-\vec{r}'|/\lambda = 0,2$)

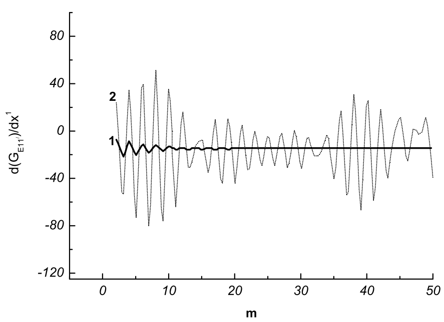

Fig.4. *Rotational component of Green's function* $Re\, \partial G^r_{e_{11'}}(k,r,r')/\partial x^1$ *of a circular waveguide* ($k = 12.56 m^{-1}$; $R = 0.0755 m$; $|\vec{r}-\vec{r}'|/\lambda = 0,26$)

The cases when the singularity is singled out and not singled out are described by curves 1 and 2 respectively. As follows from Fig.1-4 curves of 1 flattens out at $m > 20$ as for $Re\, G^r_{e_{11'}}(k,r,r')$ and for a derivative of $\partial ReG^r_{e_{11'}}(k,r,r')/\partial x^1$., i.e. series converges good at $m > 20$. Week oscillations take place for curves of 2 at $m > 20$ for $Re\, G^r_{e_{11'}}(k,r,r')$, i.e. series converges worse than for curves of 1. A sizable oscillations take place for curves of 2 at $m > 20$ for a derivative of $\partial ReG^r_{e_{11'}}(k,r,r')/\partial x^1$, i.e. series diverge.

3. CONCLUSIONS

For the first time the problem of construction of Green's function for a field is solved as a problem of diffraction of potential and rotational parts of the electric field intensity of a point current source on circular waveguide walls.

The potential and rotational components of the electric field intensity are conditioned by the scalar and vector potentials or distributions of a charge and a current, respectively, in the source point.

By singling out the singularity of the electric field intensity in an explicit form about of a source it is possible to develop the effective algorithm of calculation of the electric Green's function at any electric length of nonhomogeneities in a circular waveguide.

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ИСТОКООБРАЗНАЯ ФУНКЦИЯ ГРИНА КРУГЛОГО ВОЛНОВОДА

С.Д. Прийменко, Л.А. Бондаренко

Сингулярная часть функции Грина круглого волновода в форме функции Грина неограниченного пространства выделена в явном виде и содержит все особенности, включая дельта-образную особенность. Задача построения функции Грина для поля решена как задача дифракции потенциальной и вихревой частей напряженности электрического поля точечного источника тока на стенках круглого волновода. Выделение особенности напряженности электрического поля в явном виде в окрестности источника позволило разработать эффективный алгоритм расчета электрической функции Грина при произвольном расстоянии между точками источника и наблюдения в круглом волноводе.

ДЖЕРЕЛОПОДІБНА ФУНКЦІЯ ГРІНА КРУГЛОГО ХВИЛЕВОДУ

С.Д. Прийменко, Л.О.Бондаренко

Сінгулярна частина функції Гріна круглого хвилеводу у формі функції Гріна необмеженого простору виділена в явному вигляді й містить всі особливості, включаючи дельта-подібну особливість. Задача побудови функції Гріна для поля розв'яза як задача дифракції потенційної й вихрової частин напруженості електричного поля крапкового джерела струму на стінках круглого хвилеводу. Виділення особливості напруженості електричного поля в явному вигляді в околиці джерела дозволило розробити ефективний алгоритм розрахунку електричної функції Гріна при довільній відстані між крапками джерела й спостереження в круглому хвилеводі.