ELECTRON-POSITRON PLASMA: KINETIC SYMMETRIES AND EXACT SOLUTIONS

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Kinetic model for the high frequency electron-positron plasma waves is considered. Continuous symmetry transformations are found for the model by the indirect method, i.e. from the symmetries of the infinite set of equations for the moments of distribution functions. These symmetries, in combination with discrete ones, determine the form of the possible exact solutions, e.g. traveling waves and self similar local solutions determining the wave structure in the vicinity of its critical points. It is shown that the simplest form of extended symmetry is characteristic for this model. However, the extended symmetry is removed by the indirect moments method, since this symmetry is incompatible with the global conditions of the distribution functions non negativity and the existence of their moments. PACS: 52.65.Ff, 52.65.Kj

1. INTRODUCTION

Plasma theory is a relatively young branch of physics. It is based on Maxwell equations with sources determined by different kinetic or hydrodynamic models. Simplifying assumptions are made to build up the models describing correctly concrete experiments. Due to such simplifications, some symmetry properties are lost, but some new appear and must be investigated. In the linear theory, symmetry allows us to obtain an infinite set of eigenfunctions and to expand solutions on this basis. Even in the nonlinear case, symmetries are very useful as they allow us to clarify general properties of plasma theory models, to obtain some important exact solutions and conservation laws. In hydrodynamic plasma models this can be done by the usual Lie group theory, since these models are based on the partial differential equations. The problem is more complicated for the integro differential equations of the kinetic plasma theory. In this case, a method was proposed [1] to obtain kinetic symmetries from the symmetries of an infinite set of partial differential equations for the moments of distribution functions. This method was successfully applied to the different plasma models (see [1-5]). Other direct and indirect methods were proposed (see the review article [6]). On the other hand, in [4, 5] it was shown that there exist more general extended symmetry transformations which can also help us to solve nonlinear plasma theory problems.

In the present paper, the symmetry properties and their consequences are considered for the high frequency waves in the electron-positron plasma. In the section 2, this model is shortly described. Symmetries and their possible extensions are presented in the Section 3. Conclusions are made in the last Section.

2. MODEL

The model considered includes Vlasov equations for the positron f(t, x, v) and electron g(t, x, v) distribution functions in the electric field E(t, x) and the reduced Maxwell equations:

$$\frac{\partial f}{\partial t}+v\frac{\partial f}{\partial x}+E\frac{\partial f}{\partial v}=0, \quad \frac{\partial g}{\partial t}+v\frac{\partial g}{\partial x}-E\frac{\partial g}{\partial v}=0, \tag{1}$$

$$\frac{\partial E}{\partial x}=\int_{-\infty}^{\infty}(f-g)dv, \quad \frac{\partial E}{\partial t}+\int_{-\infty}^{\infty}v(f-g)dv=0.$$

Introducing the moments of distribution functions for the positrons

$$M_k(t,x)=\int_{-\infty}^{\infty}v^k f(t,x,v)dv$$

and for the electrons

$$N_k(t,x)=\int_{-\infty}^{\infty}v^k g(t,x,v)dv$$

we obtain an infinite set of equations (k=0, 1, …):

$$\frac{\partial M_k}{\partial t}+\frac{\partial M_{k+1}}{\partial x}-kEM_{k-1}=0,$$

$$\frac{\partial N_k}{\partial t}+\frac{\partial N_{k+1}}{\partial x}+kEN_{k-1}=0,$$

and

$$\frac{\partial E}{\partial x}=M_0-N_0, \quad \frac{\partial E}{\partial t}+M_1-N_1=0.$$

It must be noted that the equation for $\partial E/\partial t$ holds as a consequence of other equations of the model if we consider spatially localized perturbations.

According to [1], the system of equations for moments can be treated by the usual Lie formalism for finite number k of equation. Then, we can consider the limit $k\to\infty$ and restore kinetic symmetries as a final result. Namely, adding to the equations the differential identities:

$$dE-E_t dt-E_x dx=0,$$

$$dM_k-M_{k,t}dt-M_{k,x}dx=0,$$

$$dN_k-N_{k,t}dt-N_{k,x}dx=0,$$

and performing the transformations of independent variables and the functions:

t’ = t + $\mathbf{t}$(t, x, E, M_k, N_k), x’ = x + $\mathbf{x}$(t, x, E, M_k, N_k),

E’ = E + $\mathbf{E}$(t, x, E, M_k, N_k),

M_k’ = M_k + $\mathbf{M_k}$(t, x, E, M_k, N_k),

N_k’ = N_k + $\mathbf{N_k}$(t, x, E, M_k, N_k), we obtain the continuation formulas determining the corresponding transformations of derivatives:

$\mathbf{E_t} = d\mathbf{E}/dt - E_t\, d\mathbf{t}/dt - E_x\, d\mathbf{x}/dt,$
$\mathbf{E_x} = d\mathbf{E}/dx - E_t\, d\mathbf{t}/dx - E_x\, d\mathbf{x}/dx,$
$\mathbf{M_{k,t}} = d\mathbf{M_k}/dt - M_{k,t}\, d\mathbf{t}/dt - M_{k,x}\, d\mathbf{x}/dt,$
$\mathbf{M_{k,x}} = d\mathbf{M_k}/dx - M_{k,t}\, d\mathbf{t}/dx - M_{k,x}\, d\mathbf{x}/dx,$
$\mathbf{N_{k,t}} = d\mathbf{N_k}/dt - N_{k,t}\, d\mathbf{t}/dt - N_{k,x}\, d\mathbf{x}/dt,$
$\mathbf{N_{k,x}} = d\mathbf{N_k}/dx - N_{k,t}\, d\mathbf{t}/dx - N_{k,x}\, d\mathbf{x}/dx,$

where the full derivatives are defined as follows:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + E_t \frac{\partial}{\partial E} + \sum_{k=0}^{\infty} (M_{k,t} \frac{\partial}{\partial M_k} + N_{k,t} \frac{\partial}{\partial N_k}) ,$$

$$\frac{d}{dx} = \frac{\partial}{\partial x} + E_x \frac{\partial}{\partial E} + \sum_{k=0}^{\infty} (M_{k,x} \frac{\partial}{\partial M_k} + N_{k,x} \frac{\partial}{\partial N_k}) .$$

These formulas allow us to find continuous symmetry transformations of the moment equations by the use of the classical Lie method.

3. SYMMETRIES

The symmetries found for the infinite system of the moments equations combined with Maxwell ones are as follows.

Time and space shifts

$$X_1 = \frac{\partial}{\partial t} \quad X_2 = \frac{\partial}{\partial x} .$$

Galilean transform

$$X_3 = t\frac{\partial}{\partial x} + \sum_{k=1}^{\infty} k(M_{k-1} \frac{\partial}{\partial M_k} + N_{k-1} \frac{\partial}{\partial N_k}) .$$

Self similarities

$$X_4 = t\frac{\partial}{\partial t} - 2E\frac{\partial}{\partial E} - \sum_{k=0}^{\infty} (k+2)(M_k \frac{\partial}{\partial M_k} + N_k \frac{\partial}{\partial N_k})$$

and

$$X_5 = x\frac{\partial}{\partial x} + E\frac{\partial}{\partial E} + \sum_{k=1}^{\infty} k(M_k \frac{\partial}{\partial M_k} + N_k \frac{\partial}{\partial N_k}) .$$

It can be readily seen that these transformations are produced by the corresponding kinetic symmetries

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial v},$$

and

$$X_4 = t\frac{\partial}{\partial t} - v\frac{\partial}{\partial v} - 2E\frac{\partial}{\partial E} - f\frac{\partial}{\partial f} - g\frac{\partial}{\partial g},$$

as well as

$$X_5 = x\frac{\partial}{\partial x} + v\frac{\partial}{\partial v} + E\frac{\partial}{\partial E} - f\frac{\partial}{\partial f} - g\frac{\partial}{\partial g} .$$

Discrete symmetries like the following one:

$$t' = -t, \quad x' = -x, \quad f' = g, \quad g' = f \qquad (2)$$

must be added to the continuous transformations found above. They are also important to complete symmetry analysis of the kinetic theory.

It is clear that the shift of distribution functions by the same constant

$$f' = f + \text{const}, \quad g' = g + \text{const}, \qquad (3)$$

leaves the kinetic model invariant if we calculate the expression (f – g) first and perform the integration later. So this transform is in some sense additional or extended symmetry.

CONCLUSIONS

It is shown that integro differential system of equations of the kinetic theory of electron-positron high frequency plasma waves (1) can be treated by the moments method [1].

Continuous symmetry group (X_1 – X_5) is found. It includes time and space translations, Galilei transform and two self similar transforms. Traveling waves and self similar solutions can exist due to these symmetries.

Discrete symmetries like (2) are also important for the determination of the possible solutions to the model.

All these symmetries are compatible with the global conditions that

a) distribution functions must be non negative;
b) moments of these functions, at least those present explicitly in Maxwell equations, must exist.

These global conditions are violated by the additional extended symmetry transform (3). It must be noted that the transform (3) is removed by the method [1] and is not present among the transformations of the continuous symmetry group (X_1 – X_5).

Nevertheless, this extended symmetry can be important in the construction of the possible exact solutions, like the extended transforms for the plasma containing particles with equal charge to mass ratio (see [4, 5]). In fact, (3) is the simplest form of the extended symmetries found in [4, 5].

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СИММЕТРИИ И ТОЧНЫЕ РЕШЕНИЯ КИНЕТИЧЕСКОЙ ТЕОРИИ ЭЛЕКТРОННО-ПОЗИТРОННОЙ ПЛАЗМЫ

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Рассмотрена кинетическая модель высокочастотных волн в электронно-позитронной плазме. Непрерывные преобразования симметрии этой модели найдены непрямым методом, т.е. получены из симметрий бесконечной системы уравнений для моментов функций распределения. Эти симметрии, в сочетании с симметриями дискретными, определяют форму возможных точных решений, например, бегущих волн и автомодельных локальных решений, описывающих структуру волн вблизи их критических точек. Показано, что простейшая форма расширенной симметрии характерна для этой модели. Эта расширенная симметрия удаляется непрямым методом моментов, так как она не совместима с глобальными условиями неотрицательности функций распределения и существования их моментов.

СИМЕТРІЇ І ТОЧНІ РОЗВ'ЯЗКИ КІНЕТИЧНОЇ ТЕОРІЇ ЕЛЕКТРОННО-ПОЗИТРОННОЇ ПЛАЗМИ

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Розглянута кінетична модель високочастотних хвиль у електронно-позитронній плазмі. Неперервні перетворення симетрії цієї моделі одержані непрямим методом, а саме через симетрії нескінченної системи рівнянь для моментів функцій розподілу. Такі симетрії, в сукупності із дискретними, визначають форму можливих точних розв'язків, наприклад, біжучих хвиль і автомодельних локальних розв'язків, що визначають структуру хвиль поблизу їхніх критичних точок. Показано, що найпростіша форма розширеної симетрії є характерною для цієї моделі. Така розширена симетрія видаляється непрямим методом моментів, тому що вона є несумісною із глобальними умовами невід'ємності функцій розподілу і вимозі існування їхніх моментів.