Ukrainian Mathematical Bulletin
Volume 6 **(2009)**, № 1, 93 – 107

Necessary and sufficient conditions for the existence of the unique solution of a homogeneous system of linear random equations over the field $GF(3)$

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Abstract. Two theorems on the conditions of existence of the unique solution depending on intervals of the distribution of coefficients of the system are proved for a homogeneous system of linear random equations over the field $GF(3)$.

2000 MSC. 34K06, 60H10.

Key words and phrases. Linear random equations, field $GF(3)$, unique solution.

1. Main results

Let the system

$$\sum_{j=1}^{n}{}_3 a_j^{(\mu)} x_j = 0, \quad \mu \in J, \tag{1.1}$$

be given over the field $GF(3)$, where $J = \{1, 2, \dots, T\}$, $T \geq 1$, and $\sum_3$ is the symbol of addition over the field $GF(3)$, which satisfies condition (A).

Condition (A): The coefficients $a_j^{(\mu)}$, $1 \leq j \leq n$, $\mu \in J$ are independent random quantities with the distribution $P\{a_j^{(\mu)} = a\} = p_\mu, a \in GF(3)$, $a \neq 0$ and $P\{a_j^{(\mu)} = 0\} = 1 - 2p_\mu$.

Let ν_n denote the number of solutions $\bar{x}$, $\bar{x} \in V_n$, of system (1.1) with the number $|\bar{x}|$ of nonzero components is greater than zero, and $|\bar{x}| > 0$ (here, V_n is the set of all n-dimensional vectors over the field $GF(3)$) .

Received 1.02.2009

ISSN 1812 – 3309.

Theorem 1.1. *Let conditions* (A) *and*

$$\frac{\ln n+z}{n}\le p_\mu\le\frac{1}{2}-\frac{\ln n+z}{n},\quad \mu\in J, \tag{1.2}$$

where $z=o(\ln n)$, $n\to\infty$, be satisfied.

Then the condition

$$\frac{T}{n}\ge 1+\gamma_n, \tag{1.3}$$

where $n^\varepsilon\gamma_n\to\infty$, $\gamma_n\to 0$, $n\to\infty$, $\varepsilon=\mathrm{const}$, $0<\varepsilon<1$, is sufficient, and the condition

$$\frac{T}{n}\ge\frac{\ln 1,8}{\ln 3}, \tag{1.4}$$

is necessary in order that

$$P\{\nu_n>0\}=o(1),\quad n\to\infty. \tag{1.5}$$

Theorem 1.2. *Let conditions* (A) *and*

$$\frac{E_n\ln n}{n}\le p_\mu\le\frac{1}{2}-\frac{E_n\ln n}{n},\quad \mu\in J, \tag{1.6}$$

where $E_n\to\infty$ as $n\to\infty$, be satisfied.

Then the condition

$$T=n+A_n, \tag{1.7}$$

where $A_n\to\infty$ as $n\to\infty$, is sufficient, and the condition (1.4) *is necessary in order that relation* (1.5) *be valid.*

2. Auxiliary statements

Lemma 2.1. *Let ξ be a random quantity that is given by $\xi=\xi_1+_3\cdots+_3\xi_k$, where $\xi_1,\dots,\xi_k$ are independent identically distributed random quantities; $P\{\xi_s=0\}=1-2p^*$, $P\{\xi_s=a\}=p^*$, $a\in GF(3), a\ne 0$, $s=1,\dots,k$, $1\le k<\infty$, $+_3$ is the operation of summation in the field $GF(3)$. Then*

$$P\{\xi=a\}=\frac{1}{3}-\frac{1}{3}(1-3p^*)^k,\quad a\in GF(3),\ a\ne 0.$$

Proof. The proof of Lemma 2.1 can be realized by the method of mathematical induction on the parameter $k\ge 1$. □

Lemma 2.2. *If condition* (A) *is satisfied, then the expectation $E\nu_n$ of the random quantity ν_n is equal to*

$$E\nu_n=3^{-T}\sum_{t=1}^{n}\binom{n}{t}2^tQ, \tag{2.1}$$

$$Q = \prod_{\mu=1}^{T} (1 + 2(1 - 3p_\mu)^t). \tag{2.2}$$

Proof. Let $\xi(\bar{x})$ is the indicator of an event which consists in that the vector $\bar{x}$, $\bar{x} \in V_n$, is a solution of system (1.1). With regard for condition (A), we have

$$E\nu_n = \sum_{\bar{x}:|\bar{x}|\geq 1} E\xi(\bar{x}) = \sum_{\bar{x}:|\bar{x}|\geq 1} \prod_{\mu=1}^{T} P\Big(\sum\nolimits_3{}_{j=1}^{n} a_j^{(\mu)} x_j = 0\Big). \tag{2.3}$$

The number of nonzero terms in $\sum_3$ on the right-hand side of (2.3) is equal to t, where t is the total number of nonzero components of the vector $\bar{x}, |\bar{x}| \geq 1$. Then, using (2.3) and Lemma 2.1, we get (2.1). □

For arbitrary vectors $\bar{x}^{(q)} \in V_n$, $\bar{x}^{(q)} = (x_1^{(q)}, \dots, x_n^{(q)})$, $q = 1, 2$, we denote, by $i_{c_1 c_2}$, the number of components of the vector $\bar{x}^{(1)}$ which are equal to c_1. In the vector $\bar{x}^{(2)}$, they correspond to components which equal c_2, where $c_1, c_2 \in GF(3)$, $0 \leq i_{c_1 c_2} \leq n$.

Let $I = \{i_{\mathbf{01}}, i_{\mathbf{02}}, i_{\mathbf{10}}, i_{\mathbf{20}}, i_{\mathbf{11}}, i_{\mathbf{22}}, i_{\mathbf{12}}, i_{\mathbf{21}}\}$, $i = i_{\mathbf{01}} + i_{\mathbf{02}}$, $l = i_{\mathbf{10}} + i_{\mathbf{20}}$, $t = \sum\limits_{j \in I} j$, $E\nu_n^{[2]} = E\nu_n(\nu_n - 1)$.

Lemma 2.3. *If condition* (A) *is satisfied, then*

$$E\nu_n^{[2]} = 9^{-T} \sum_{t=1}^{n} \binom{n}{t} \sum \frac{t!}{\prod\limits_{j \in I} j!} Q^*, \tag{2.4}$$

where

$$Q^* = \prod_{\mu=1}^{T} \left(1 + 2\Big(\sum_{r=1}^{4} (1 - 3p_\mu)^{\Gamma^{(r)}}\Big)\right), \tag{2.5}$$

the summation $\sum$ *is realized over all* $j \in I$ *so that* $\sum_{j \in I} j = t$; *in equality* (2.4), *elements of the set* I *satisfy the relations*

$$t - i \geq 1, \tag{2.6}$$

$$t - l \geq 1, \tag{2.7}$$

$$i + l + i_{\mathbf{12}} + i_{\mathbf{21}} \geq 1; \tag{2.8}$$

and the parameters $\Gamma^{(k)}$, $k = 1, 2, 3, 4$, *are determined, respectively, by the equalities*

$$\Gamma^{(1)} = i + l, \tag{2.9}$$

$$\Gamma^{(2)} = t - l, \tag{2.10}$$

$$\Gamma^{(3)} = t - i, \tag{2.11}$$

$$\Gamma^{(4)} = t. \tag{2.12}$$

Proof. Using condition (A) and the relation

$$E\nu_n^{[2]} = \sum\nolimits^* E\xi(\bar{x}^{(1)})\xi(\bar{x}^{(2)}),$$

where the summation $\sum^*$ is executed over all pairs $(\bar{x}^{(1)}, \bar{x}^{(2)})$ of the vectors $\bar{x}^{(q)} \in V_n$ such that $|\bar{x}^{(q)}| \geq 1$, $q = 1, 2$, $\bar{x}^{(1)} \neq \bar{x}^{(2)}$, we get

$$\begin{aligned} &E\nu_n^{[2]} \\ &= \sum\nolimits^* \prod_{\mu=1}^{T} P\{\cup\{A^{(\mu)}(\bar{x}^{(k)}) = y_k, A^{(\mu)}(\bar{x}^{(1)}, \bar{x}^{(2)}) = y_{12}, k = 1, 2\}\} \\ &\qquad = \sum\nolimits^* \prod_{\mu=1}^{T} \sum\nolimits^{**} P\{A^{(\mu)}(\bar{x}^{(1)}, \bar{x}^{(2)}) = y_{12}\} \\ &\qquad\qquad \times \prod_{k=1,2} P\{A^{(\mu)}(\bar{x}^{(k)}) = y_k\}, \end{aligned} \tag{2.13}$$

where the symbol $\cup$ / $\sum^{**}$ / union /summation/ is applied to all solutions of the system of equations

$$\left\{ \begin{array}{l} y_1 +_3 y_{12} = 0, \\ y_2 +_3 y_{12} = 0 \end{array} \right. .$$

over the field $GF(3)$; for $\mu \in J$,

$$A^{(\mu)}(\bar{x}^{(1)}, \bar{x}^{(2)}) = \sum\nolimits_{\omega \in E^{(12)}}{}_3\, a_\omega^{(\mu)}, \quad A^{(\mu)}(\bar{x}^{(q)}) = \sum\nolimits_{\omega \in E^{(q)}}{}_3\, a_\omega^{(\mu)}, \quad q = 1, 2,$$

where

$$E^{(12)} = \{j,\ 1 \leq j \leq n : x_j^{(q)} \neq 0,\ q = 1, 2\},$$

$$E^{(q)} = \{j,\ 1 \leq j \leq n : x_j^{(q)} \neq 0,\ x_j^{(q^*)} = 0\},$$

$q \in \{1, 2\}$, $q^* \in \{1, 2\}$, $q^* \neq q$.

Let $\gamma^{(1)}$, $\gamma^{(2)}$, and $\gamma^{(3)}$ be the numbers of elements of the sets, respectively, $E^{(1)}$, $E^{(2)}$, and $E^{(12)}$.

We set

$$\Gamma^{(1)} = \gamma^{(1)} + \gamma^{(2)}, \quad \Gamma^{(2)} = \gamma^{(2)} + \gamma^{(3)},$$

$$\Gamma^{(3)} = \gamma^{(1)} + \gamma^{(3)}, \quad \Gamma^{(4)} = \gamma^{(1)} + \gamma^{(2)} + \gamma^{(3)}.$$

By condition (A) and Lemma 2.1, relation (2.13) can be rewritten as

$$E\nu_n^{[2]} = 9^{-T} \sum\nolimits^{*} \prod_{\mu=1}^{T} \left(1 + 2\left(\sum_{k=1}^{4} (1-3p_\mu)^{\Gamma^{(k)}}\right)\right). \tag{2.14}$$

The summation $\sum^{*}$ on the right-hand side of (2.14) over all pairs $(\bar{x}^{(1)}, \bar{x}^{(2)})$ such that $\bar{x}^{(1)} \neq \bar{x}^{(2)}$, $|\bar{x}^{(q)}| \geq 1$, $q = 1, 2$, is equivalent to the summation over all parameters $j \in I$ on the right-hand side of (2.4). Inequalities (2.6), (2.7), and (2.8) guarantee for the relations $|\bar{x}^{(1)}| \geq 1$, $|\bar{x}^{(2)}| \geq 1$, and $\bar{x}^{(1)} \neq \bar{x}^{(2)}$, respectively, to be satisfied.

Then we will verify equality (2.9). Indeed, because the sum $i_{\mathbf{10}}+i_{\mathbf{20}}$ is the number of nonzero components of the vector $\bar{x}^{(1)}$ which correspond to zero components of the vector $\bar{x}^{(2)}$, we have found $|E^{(1)}| = \gamma^{(1)} = i_{\mathbf{10}} + i_{\mathbf{20}}$. By analogy, we get $|E^{(2)}| = \gamma^{(2)} = i_{\mathbf{01}} + i_{\mathbf{02}}$. That is why, $\Gamma^{(1)} = \gamma^{(1)} + \gamma^{(2)} = i + l$, which proves (2.9). In the same way, we verify equalities (2.10)–(2.12). □

Lemma 2.4. *If conditions* (A) *and*

$$p_\mu \leq \frac{1}{2} - \upsilon, \tag{2.15}$$

where $0 < \upsilon \leq \frac{1}{2}$, $\mu \in J$, *are satisfied, then*

$$E\nu_n > 0. \tag{2.16}$$

Proof. To prove relation (2.16), it is sufficient to show with regard for (2.1) and (2.2) that, for $n \geq 1$,

$$Q > 0. \tag{2.17}$$

To this end, we represent the product Q which is determined by equality (2.2) in the form

$$Q = \prod_{r=1}^{3} Q_r, \tag{2.18}$$

where Q_r denote the product of all multipliers on the right-hand side of (2.2), for which the parameter μ belongs to the set W_r, $r = 1, 2, 3$. Here,

$$W_1 = \{\mu, \quad 1 \leq \mu \leq T : p_\mu \leq \frac{1}{3}\},$$
$$W_2 = \{\mu, \quad 1 \leq \mu \leq T : \frac{1}{3} < p_\mu \leq \frac{1}{2} - \upsilon, \quad t \text{ even}\},$$
$$W_3 = \{\mu, \quad 1 \leq \mu \leq T : \frac{1}{3} < p_\mu \leq \frac{1}{2} - \upsilon, \quad t \text{ odd}\},$$

where $t \geq 1$, and t is the parameter from the right-hand side of equality (2.2).

Let η_r be the number of elements of the set W_r, $\eta_r = |W_r|$, $r = 1, 2, 3$. Then

$$\sum_{r=1}^{3} \eta_r = T. \tag{2.19}$$

From the definition of the products Q_1 and Q_2, we get

$$Q_1 \geq 1, \quad Q_2 \geq 1. \tag{2.20}$$

Using condition (2.15), we find

$$Q_3 \geq (6\upsilon)^{\eta_3}. \tag{2.21}$$

From (2.18)–(2.21), we have

$$Q \geq (6\upsilon)^{\eta_3},$$

which gives (2.17) and, hence, (2.16). □

We denote

$$p_{\max} = \max_{1 \leq \mu \leq T} p_\mu, \quad p_{\min} = \min_{1 \leq \mu \leq T} p_\mu.$$

Lemma 2.5. *Let conditions* (A), (1.2), *and*

$$\frac{T}{n} < \frac{\ln 1,8}{\ln 3} - \gamma, \tag{2.22}$$

where γ is a fixed positive number, be satisfied.

Then, for arbitrary $t \in F$, where $F = [[\frac{2}{3}n] - [\frac{n}{\ln n}]; n]$, the relation

$$Q \geq a_1 \tag{2.23}$$

holds as $n \to \infty$, where $[d]$ is the integer part of a number d. Here and below, a_z is a fixed positive number, $a_z < \infty$, $z = 1, 2, \ldots$

Proof. By virtue of (2.18), for proving (2.23), it is sufficient to show that, for $t \in F$ and $n \geq 1$, there exists a_2 such that

$$Q_r \geq a_2, \quad r = 1, 2, 3. \tag{2.24}$$

With the help of (1.2) for $\mu \in W_1$ and $t \in F$ for $n \to \infty$, we get

$$(1 - 3p_\mu)^t \geq (1 - 3p_{max})^t \geq -a_3 2^{-\frac{2}{3}n - \frac{n}{\ln n}} n^{-4(1+o(1))}. \tag{2.25}$$

Using (2.22) and (2.25) as $n \to \infty$, we get

$$Q_1 \geq (1 - a_4 2^{-\frac{2}{3}n - \frac{n}{\ln n}} n^{-4(1+o(1))})^{(\frac{\ln 1,8}{\ln 3} - \gamma)n},$$

which yields (2.24) for $r = 1$.

We now check relation (2.24) for $r = 2$. Indeed, taking (1.2) into account for $\mu \in W_2$ and $t \in F$, we find, as $n \to \infty$,

$$(1 - 3p_\mu)^t \geq (1 - 3p_{min})^t \geq -a_5 n^{-3(1+o(1))}. \tag{2.26}$$

Using (2.22) and (2.26) for $n \to \infty$, we get

$$Q_2 \geq (1 - a_6 n^{-3(1+o(1))})^{(\frac{\ln 1,8}{\ln 3} - \gamma)n}. \tag{2.27}$$

With the help of (2.27), we get (2.24) for $r = 2$.

Inequality (2.24) for $r = 3$ follows from (2.2) and from the relations

$$(1 - 3p_\mu)^t \geq -(3p_{max} - 1)^t \geq -a_3 2^{-\frac{2}{3}n - \frac{n}{\ln n}} n^{-4(1+o(1))}$$

for $\mu \in W_3$, $t \in F$, and $n \to \infty$. Whence, taking (2.18) and (2.24) into account, we get relation (2.23). □

Lemma 2.6. *If conditions* (A), (1.2), *and* (1.3) *are satisfied, then, as* $n \to \infty$,

$$D_1 = o(1), \tag{2.28}$$

where

$$D_1 = 3^{-T} \sum_{t=1}^{[\varepsilon_1 \frac{n}{\ln n}]} \binom{n}{t} 2^t Q.$$

Here, ε_q *is a sufficiently small fixed positive number, and* $q \geq 1$.

Proof. Using (1.2) and the relation $p_{min}t \in (0;1)$ for $n \to \infty$ and $t \in [1; [\varepsilon_1 \frac{n}{\ln n}]]$, we get

$$Q \leq 3^T (1 - 2p_{min}t + 3(p_{min}t)^2)^T. \tag{2.29}$$

Using (2.29), we find, as $n \to \infty$,

$$D_1 \leq \sum_{t=1}^{[\varepsilon_1 \frac{n}{\ln n}]} \frac{(2n)^t}{t!} \exp\Big\{ -2Ttp_{\min}\Big(1 - \frac{3}{2}\varepsilon_1 \frac{n}{\ln n} p_{\min}\Big)\Big\}. \tag{2.30}$$

With the help of (1.2) for all $t \in [1; [\varepsilon_1 \frac{n}{\ln n}]]$, we have

$$\Big(\frac{2}{e^{2Tp_{\min}(1 - \frac{3}{2}\varepsilon_1 \frac{n}{\ln n} p_{\min}) - \ln n}}\Big)^t \leq \Big(\frac{2}{e^{2\frac{T}{n}(\ln n + z)(1 - \frac{3}{2}\varepsilon_1(1 + \frac{z}{\ln n})) - \ln n}}\Big)^t.$$

Then, taking (1.3) and (2.30) into account, we obtain the estimate

$$D_1 \le \sum_{t=1}^{[\varepsilon_1 \frac{n}{\ln n}]} \frac{1}{t!} \exp\{-t(\ln n + 2z)(1 - 3\varepsilon_1 + o(1))\}, \quad n \to \infty. \tag{2.31}$$

This relation yields obviously (2.28). □

Lemma 2.7. *If conditions* (A), (1.2), *and* (1.3) *are satisfied, then, as* $n \to \infty$,

$$D_2 = o(1), \tag{2.32}$$

where

$$D_2 = 3^{-T} \sum_{t=[\varepsilon_1 \frac{n}{\ln n}]+1}^{[\varepsilon_2 n]} \binom{n}{t} 2^t Q.$$

Proof. For $t \in [[\varepsilon_1 \frac{n}{\ln n}] + 1, [\varepsilon_2 n]]$, we find

$$Q \le \Big(1 + 2\exp\Big\{ - 3p_{\min}\Big(\Big[\varepsilon_1 \frac{n}{\ln n}\Big] + 1\Big)\Big\}\Big)^T. \tag{2.33}$$

As $n \to \infty$, relations (1.2) and (2.33) yield

$$D_2 \le \left(\frac{e^{\sigma_1(\varepsilon_2)\frac{n}{T}}(1 + \frac{2}{e^{3\varepsilon_1(1+\frac{z}{\ln n})}})}{3} \right)^T, \tag{2.34}$$

where $\sigma_r(\varepsilon) \to 0(\varepsilon \to 0)$, $r \ge 1$. In view of assumption (1.3), condition (1.2) for $z = o(\ln n)$, and inequality (2.34), we get (2.32). □

Lemma 2.8. *If conditions* (A), (1.2), *and* (1.3) *are satisfied, then, as* $n \to \infty$,

$$D_3 = o(1), \tag{2.35}$$

where

$$D_3 = 3^{-T} \sum_{t=[\varepsilon_2 n]+1}^{n} \binom{n}{t} 2^t Q.$$

Proof. For all $t \in [[\varepsilon_2 n] + 1, n]$,

$$Q \le (1 + 2\exp\{-3p_{\min}([\varepsilon_2 n] + 1)\})^T. \tag{2.36}$$

Using (1.2) and (2.36), we have

$$D_3 \le \frac{3^n}{3^T} \exp\Big\{ \frac{2T}{\exp\{3\varepsilon_2(\ln n)(1 + \frac{z}{\ln n})\}} \Big\}. \tag{2.37}$$

With the help of relations $z = o(\ln n)$, $n^{\varepsilon}\gamma_n \to \infty$ as $n \to \infty$, and (2.37), we get

$$D_3 \le \exp\Big\{ -T\gamma_n(\ln 3)\Big\{ -\frac{2}{\gamma_n(\ln 3)n^{3\varepsilon_2(1+o(1))}} + 1 + O(\gamma_n)\Big\}\Big\}.$$

This directly yields (2.35). □

Lemma 2.9. *If conditions* (A), (1.6), *and* (1.7) *are satisfied, then relation* (2.35) *is valid.*

Proof. To prove the validity of (2.35), we note that, first, the product Q satisfies estimate (2.36). Second, using condition (1.6) for $n \to \infty$, we get

$$D_3 \le \frac{3^n}{3^T}\Big(1 + \frac{2}{n^{3\varepsilon_2 E_n}}\Big)^T.$$

Then, taking (1.7) into account, we have, as $n \to \infty$,

$$D_3 \le \exp\Big\{\frac{2n}{n^{3\varepsilon_2 E_n}}\Big\}\Big(\frac{\exp\{\frac{2}{n^{3\varepsilon_2 E_n}}\}}{3}\Big)^{A_n}. \tag{2.38}$$

With the help of conditions $E_n \to \infty$, $A_n \to \infty$ for $n \to \infty$, and (2.38), we find (2.35) for $n \to \infty$. □

3. Proof of theorems

Proof of Theorem 1.1. *Sufficiency.* We will show that if relation (1.3) is valid, then

$$E\nu_n = o(1), \quad n \to \infty. \tag{3.1}$$

In view of (2.1) and (2.2), the expectation $E\nu_n$ can be written as

$$E\nu_n = \sum_{h=1}^{3} D_h, \tag{3.2}$$

where

$$D_h = 3^{-T}\sum_{t\in R_h}\binom{n}{t}2^t Q, \quad h = 1, 2, 3.$$

The closed segments R_h, $h = 1, 2, 3$, whose ends are integers, are as follows: $R_1 = [1, [\varepsilon_1\frac{n}{\ln n}]]$, $R_2 = [[\varepsilon_1\frac{n}{\ln n}] + 1, [\varepsilon_2 n]]$, $R_3 = [[\varepsilon_2 n] + 1, n]$.

To prove (3.1) with the help of (3.2), it is sufficient to be convinced that, for $n \to \infty$,

$$D_h = o(1) \tag{3.3}$$

for $h = 1, 2, 3$. Using (2.28), (2.32), and (2.35), we get (3.3) for $h = 1, 2, 3$.

Relations (3.2) and (3.3) yield (3.1). Taking (3.1) and the Chebyshev inequality into account, we get (1.5).

Necessity. As $n \to \infty$, let the probability $P(\nu_n > 0)$ tend to zero, i.e.,

$$P(\nu_n > 0) \to 0, \quad n \to \infty. \tag{3.4}$$

We will show that (1.4) is satisfied. Let us assume that equality (1.4) does not hold, i.e. relation (2.22) has place. We will show that, in this case,

$$P(\nu_n > 0) > 0 \tag{3.5}$$

as $n \to \infty$. That is, the nonzero solutions exist with a positive probability. To that end, we will check the estimates for $n \to \infty$,

$$(E\nu_n)^{-1} \leq a_7, \tag{3.6}$$

$$E\nu_n^{[2]}(E\nu_n)^{-2} \leq a_8, \tag{3.7}$$

with their subsequent use in the inequality [1]

$$P(\nu_n > 0) \geq ((E\nu_n)^{-1} + E\nu_n^{[2]}(E\nu_n)^{-2})^{-1}. \tag{3.8}$$

Indeed, with the help of relations (2.1), (2.2), and Lemma 2.4 for $t \in F$ and $n \to \infty$, we have

$$(E\nu_n)^{-1} \leq 3^{T-n}\delta_n, \tag{3.9}$$

where

$$\delta_n \leq a_1^{-1}(3^{-n}\sum_{t\in F}\binom{n}{t}2^t)^{-1}. \tag{3.10}$$

Using the equality $3^{-n}\sum_{t=0}^{n}\binom{n}{t}2^t = 1$, we get

$$3^{-n}\sum_{t=0}^{[\frac{2}{3}n]-[\frac{n}{\ln n}]}\binom{n}{t}2^t \leq \exp\Big\{-\frac{n}{\ln^2 n}\Big(\frac{9}{4} + O((\ln n)^{-1})\Big)\Big\} \to 0,$$

$n \to \infty$, whence

$$3^{-n}\sum_{t\in F}\binom{n}{t}2^t \to 1, \quad n \to \infty. \tag{3.11}$$

Relations (3.9)–(3.11) yield (3.6).

We now show that, as $n \to \infty$, there exists a number a_9, for which

$$(3^{T-n}E\nu_n)^{-1} \leq a_9. \tag{3.12}$$

Indeed, taking (3.9) into account, we obtain

$$(3^{T-n}E\nu_n)^{-1} \le \delta_n \tag{3.13}$$

as $n \to \infty$. But, using (3.10) and (3.11), we get the inequality $\overline{\lim}_{n\to\infty}\delta_n \le a_1^{-1}$, which together with (3.13) proves (3.12).

Relation (3.12) implies that, to order to prove (3.7), it suffices to show that the relation

$$9^{T-n}E\nu_n^{[2]} \le a_{10}, \quad n \to \infty, \tag{3.14}$$

holds.

To that end, we rewrite the left-hand side of (3.14) with the help of (2.4) and (2.5) as

$$9^{T-n}E\nu_n^{[2]} = 9^{-n}S(n;Q^*), \tag{3.15}$$

where

$$S(n;Q^*) = \sum_{t=1}^{n}\binom{n}{t}\sum_{\sum\limits_{j\in I} j=t}\frac{t!}{\prod\limits_{j\in I} j!}Q^*. \tag{3.16}$$

We represent the sum $S(n;Q^*)$ as

$$S(n;Q^*) = S_1(n;Q^*) + S_2(n;Q^*), \tag{3.17}$$

where $S_1(n;Q^*)$ differs from $S(n;Q^*)$ by that the summation on the right-hand side of (3.16) is performed over all j, $j \in I$ such that

$$\Gamma^{(k)} \ge \varepsilon n, \tag{3.18}$$

where $\varepsilon = \mathrm{const}$, $0 < \varepsilon < 1$, $\Gamma^{(k)}$ are determined by equalities (2.9)–(2.12) for $k = 1,2,3,4$; and $S_2(n;Q^*)$ is the sum of terms from $S(n;Q^*)$ which do not enter $S_1(n;Q^*)$. Then, in view of (1.2), (2.5), (2.22), and (3.18), we get the estimate

$$S_1(n;Q^*) \le S_1(n;1)Q_1^*, \tag{3.19}$$

where $Q_1^* = (1+8n^{-3\varepsilon(1+\frac{z}{\ln n})})^{(\frac{\ln 1,8}{\ln 3}-\gamma)n}$. The inequality

$$S_1(n;1) \le 9^n \tag{3.20}$$

together with (3.19) give us

$$S_1(n;Q^*) \le a_{11}9^n(1+8n^{-3\varepsilon(1+\frac{z}{\ln n})})^{(\frac{\ln 1,8}{\ln 3}-\gamma)n}. \tag{3.21}$$

We represent the sum $S_2(n;Q^*)$ as

$$S_2(n;Q^*) = \sum_{r=1}^{4} S_{2;r}(n;Q^*), \tag{3.22}$$

where $S_{2;r}(n;Q^*)$ differs from $S_2(n;Q^*)$ by that the summation on the right-hand side of (3.16) is realized over all parameters j, $j \in I$ such that there exist $l_1,\dots,l_r \in \{1,2,3,4\}$, for which $\Gamma^{(l_h)} < \varepsilon n$, $\Gamma^{(k)} \geq \varepsilon n$, where $k \in \{1,2,3,4\}\backslash\{l_1,\dots,l_r\}$, $h = 1,\dots,r$, $r = 1,2,3,4$. Then, for $r = 1,2,3,4$, we can write $S_{2;r}(n;Q^*)$ in the form

$$S_{2;r}(n;Q^*) = \sum_{1\leq t_1<\dots<t_r\leq 4} S_{2;r;t_1,\dots,t_r}(n;Q^*), \tag{3.23}$$

where $S_{2;r;t_1,\dots,t_r}(n;Q^*)$ denotes the sum of all terms that belong to $S_{2;r}(n;Q^*)$ and for which $\Gamma^{(t_l)} < \varepsilon n$, $l = 1,\dots,r$, $\Gamma^{(t')} \geq \varepsilon n$, $t' \in \{1,2,3,4\}\backslash\{t_1,\dots,t_r\}$.

We now show that, for $r = 1$,

$$S_{2;r}(n;Q^*) \leq a_{12} 3^{(\frac{\ln 1,8}{\ln 3}-\gamma)n} e^{\sigma_2(\varepsilon)n} \times (1+2n^{-3\varepsilon(1+\frac{z}{\ln n})})^{(\frac{\ln 1,8}{\ln 3}-\gamma)n}(5^n+3^n+1). \tag{3.24}$$

Indeed, with the help of (1.2), (2.5), (2.22), and (3.23) for $r = 1$, we get

$$S_{2;r}(n;Q^*) \leq Q^*_{2;r} \sum_{l=1}^{4} S_{2;r;l}(n;1), \tag{3.25}$$

where $Q^*_{2;r} = a_{13} 3^{(\frac{\ln 1,8}{\ln 3}-\gamma)n}(1+2n^{-3\varepsilon(1+\frac{z}{\ln n})})^{(\frac{\ln 1,8}{\ln 3}-\gamma)n}$.

Then we estimate every of four terms $S_{2;r;l}(n;1)$, $l = 1,2,3,4$ on the right-hand side of (3.25).

The inequality $\Gamma^{(1)} < \varepsilon n$ and relation (2.9) imply that all parameters $j, j \in I^* \setminus \{i_{\mathbf{11}}, i_{\mathbf{22}}, i_{\mathbf{12}}, i_{\mathbf{21}}, i_{\mathbf{00}}\}$, $I^* = I \cup \{i_{\mathbf{00}}\}$ which are present in the definition of the sum $S(n;Q^*)$ do not exceed εn. Then, using the polynomial formula, we have

$$S_{2;1;1}(n;1) \leq 5^n \exp\{\sigma_3(\varepsilon)n\}. \tag{3.26}$$

In order to verify the validity of the estimate

$$S_{2;1;2}(n;1) \leq 3^n \exp\{\sigma_4(\varepsilon)n\}, \tag{3.27}$$

it is sufficient to observe, by taking (2.10) and the inequality $\Gamma^{(2)} < \varepsilon n$ into account, that all parameters j, $j \in I^* \setminus \{i_{\mathbf{10}}, i_{\mathbf{00}}, i_{\mathbf{20}}\}$ on the right-hand side of (3.16) do not exceed εn.

With the help of (2.11) by analogy with (3.27), we get

$$S_{2;1;3}(n;1) \leq 3^n \exp\{\sigma_5(\varepsilon)n\}. \tag{3.28}$$

The inequality $\Gamma^{(4)} < \varepsilon n$ and relation (2.12) imply that all parameters $j, j \in I^* \setminus \{i_{\mathbf{00}}\}$ which enter the sum $S(n; Q^*)$ do not exceed εn. This yields

$$S_{2;1;4}(n;1) \leq \exp\{\sigma_6(\varepsilon)n\}. \tag{3.29}$$

Using (3.25)–(3.29), we get

$$S_{2;1}(n;1) \leq a_{14} \exp\{\sigma_7(\varepsilon)n\}(5^n + 3^n + 1). \tag{3.30}$$

Relations (3.25) and (3.30) prove (3.24).

We will show that, for $r = 2$, the estimate

$$S_{2;r}(n;Q^*) \leq a_{15} e^{\sigma_8(\varepsilon)n} 5^{(\frac{\ln 1.8}{\ln 3} - \gamma)n} \times \left(1 + \frac{4}{5} n^{-3\varepsilon(1+\frac{z}{\ln n})}\right)^{(\frac{\ln 1.8}{\ln 3} - \gamma)n} \tag{3.31}$$

holds.

Indeed, with the help of (1.2), (2.5), (2.22), and (3.23) for $r = 2$, we get

$$S_{2;r}(n;Q^*) \leq Q^*_{2;r} \sum_{1 \leq t_1 < t_2 \leq 4} S_{2;r;t_1,t_2}(n;1), \tag{3.32}$$

where $Q^*_{2;r} = a_{16} 5^{(\frac{\ln 1,8}{\ln 3} - \gamma)n} (1 + \frac{4}{5} n^{-3\varepsilon(1+\frac{z}{\ln n})})^{(\frac{\ln 1,8}{\ln 3} - \gamma)n}$.

Using the inequalities $\Gamma^{(t_1)} < \varepsilon n$, $\Gamma^{(t_2)} < \varepsilon n$, $1 \leq t_1 < t_2 \leq 4$, and relations (2.9)–(2.12), we established that all parameters j, $j \in I^* \setminus \{i_{\mathbf{00}}\}$ on the right-hand side of (3.16) do not exceed εn. This allows us to write the estimate

$$\max_{1 \leq t_1 < t_2 \leq 4} S_{2;2;t_1,t_2}(n;1) \leq \exp\{\sigma_9(\varepsilon)n\}. \tag{3.33}$$

In view of (3.32) and (3.33), we have, for $r = 2$,

$$S_{2;r}(n;1) \leq a_{17} \exp\{\sigma_{10}(\varepsilon)n\}. \tag{3.34}$$

Relations (3.32) and (3.34) yield (3.31).

Let us verify that, for $r = 3$,

$$S_{2;r}(n;Q^*) \leq a_{18} e^{\sigma_{11}(\varepsilon)n} 7^{(\frac{\ln 1,8}{\ln 3} - \gamma)n} \times (1 + \frac{2}{7} n^{-3\varepsilon(1+\frac{z}{\ln n})})^{(\frac{\ln 1,8}{\ln 3} - \gamma)n}. \tag{3.35}$$

On the basis of (1.2), (2.5), (2.22), and (3.23) for $r = 3$, we find

$$S_{2;r}(n; Q^*) \leq Q^*_{2;r} \sum_{1\leq t_1<t_2<t_3\leq 4} S_{2;r;t_1,t_2,t_3}(n; 1), \tag{3.36}$$

where $Q^*_{2;r} = a_{19} 7^{(\frac{\ln 1,8}{\ln 3}-\gamma)n}(1 + \frac{2}{7}n^{-3\varepsilon(1+\frac{z}{\ln n})})^{(\frac{\ln 1,8}{\ln 3}-\gamma)n}$.

We now show that

$$\max_{1\leq t_1<t_2<t_3\leq 4} S_{2;2;t_1,t_2,t_3}(n; 1) \leq \exp\{\sigma_{12}(\varepsilon)n\}. \tag{3.37}$$

Indeed, the inequalities $\Gamma^{(t_1)} < \varepsilon n$, $\Gamma^{(t_2)} < \varepsilon n$, $\Gamma^{(t_3)} < \varepsilon n$, $1 \leq t_1 < t_2 < t_3 \leq 4$, relations (2.9)–(2.12), and the polynomial formula allow us, by analogy with (3.33), to obtain (3.37).

Taking (3.36) and (3.37) into account for $r = 3$, we get the estimate

$$S_{2;r}(n; 1) \leq a_{20} \exp\{\sigma_{13}(\varepsilon)n\}. \tag{3.38}$$

Relations (3.36) and (3.38) prove (3.35) for $r = 3$.

Finally, we will convince ourselves that, for $r = 4$,

$$S_{2;r}(n; Q^*) \leq 9^{(\frac{\ln 1,8}{\ln 3}-\gamma)n} \exp\{\sigma_{14}(\varepsilon)n\} n^{-a_{21}(1+\frac{z}{\ln n})}. \tag{3.39}$$

From (2.6), (2.7), and (2.10)–(2.12), we get $\Gamma^{(l)} \geq 1$, $l = 2, 3, 4$. Whence, by using (1.2), (2.5), (2.22), and (3.23) for $r = 4$, we find

$$S_{2;r}(n; Q^*) \leq Q^*_{2;r} S_{2;r}(n; 1), \tag{3.40}$$

where $Q^*_{2;r} = 9^{(\frac{\ln 1,8}{\ln 3}-\gamma)n} n^{-a_{21}(1+\frac{z}{\ln n})}$.

In view of the inequalities $\Gamma^{(l)} < \varepsilon n$, $l = 1, 2, 3, 4$, and relations (2.9)–(2.12), by analogy with (3.33) for $r = 4$, we find the estimate

$$S_{2;r}(n; 1) \leq \exp\{\sigma_{14}(\varepsilon)n\}. \tag{3.41}$$

Relations (3.40) and (3.41) prove (3.39).

For $S_2(n; Q^*)$, using (3.22), (3.24), (3.31), (3.35), and (3.39), we get

$$\begin{aligned} S_{2;r}(n; 1) \leq\ & 9^{(\frac{\ln 1,8}{\ln 3}-\gamma)n} n^{-a_{21}(1+\frac{z}{\ln n})} \\ & + a_{22} \exp\{\sigma_{15}(\varepsilon)n\}(1 + a_{23} n^{-3\varepsilon(1+\frac{z}{\ln n})})^{(\frac{\ln 1,8}{\ln 3}-\gamma)n} \\ & \times (7^{(\frac{\ln 1,8}{\ln 3}-\gamma)n} + 5^{(\frac{\ln 1,8}{\ln 3}-\gamma)n} + 3^{(\frac{\ln 1,8}{\ln 3}-\gamma)n}(5^n + 3^n + 1)). \end{aligned} \tag{3.42}$$

With the help of (3.15), (3.17), (3.21), and (3.42), we find (3.14). Inequalities (3.12) and (3.14) prove (3.7).

Thus, using (2.22), we get (3.6) and (3.7), which, together with estimate (3.8), allow us to make conclusion that relation (3.5) is valid. This, in turn, contradicts the assertion that, with probability 1, there exists a unique solution $\bar{x}^{(0)}$ of system (1.1) for $n \to \infty$. □

Proof of Theorem 1.2. *Sufficiency.* We will show that (1.7) yields (3.1). Indeed, using conditions (1.6) and (1.7), we verify, by analogy with Lemmas 2.6 and 2.7, that the relations

$$3^{-T} \sum_{t=1}^{[\varepsilon_1 \frac{n}{E_n \ln n}]} \binom{n}{t} 2^t Q = o(1), \quad n \to \infty, \tag{3.43}$$

$$3^{-T} \sum_{t=[\varepsilon_1 \frac{n}{E_n \ln n}]+1}^{[\varepsilon_2 n]} \binom{n}{t} 2^t Q = o(1), \quad n \to \infty, \tag{3.44}$$

are satisfied. In view of Lemma 2.9, equalities (2.35), (3.2), (3.43), and (3.44), we get (3.1). The Chebyshev inequalities and (3.1) complete the proof of the sufficiency.

Necessity. From (1.6), we get the necessity of condition (1.4), because relation (1.6) is a separate case of (1.2). □

References

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