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Dedicated to the memory
of Igor Vladimirovich Skrypnik

Positive solutions to singular non-linear Schrödinger-type equations

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Abstract. We study the existence and nonexistence of positive (super) solutions to a singular quasilinear second-order elliptic equations with structural coefficients from non-linear Kato-type classes. Under certain general assumptions on the behaviour of the coefficient at infinity we construct an entire positive solution in $\mathbb{R}^N$ which is bounded above and below by positive constants. An application is given to a non-existence problem in an exterior domain.

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1. Introduction and main results

In the last decades there has been growing interest in studying positive solutions to quasilinear variational equations in unbounded domains of $\mathbb{R}^N$, mostly due to numerous applications in mathematical physics. The entire positive solutions are of special interest, partly due to their applications to Liouville type theorems. For linear elliptic second-order divergence type equations the problem of existence of such solutions arises in the presence of lower order terms. For instance, for the stationary Schrödinger equation $\Delta u + Vu = 0$ with nonnegative smooth V decaying at infinity slower than $c|x|^{-2}$ it is well known (and one can easily show) that there are no positive solutions in $\mathbb{R}^N$. The optimal conditions at

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infinity for the entire solution to exist is called Green-bondedness of the potential (or Kato-class at infinity) [8, 9]. When the potential is sign-changing the existence of a positive entire solution bounded below and above by constants is of particular interest. In this paper we address this problem for a general quasilinear second-order elliptic equation in divergence form. Namely, we study the quasilinear elliptic equation

$$-\operatorname{div}\mathbf{A}(x,u,\nabla u)+a_0(x,u,\nabla u)+g(x,u)=0,\quad x\in\mathbb{R}^N. \tag{1.1}$$

Throughout the paper we suppose that the functions $\mathbf{A}:\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}^N$, $a_0:\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ and $g:\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ are such that $\mathbf{A}(\cdot,\xi,u),a_0(\cdot,u,\xi),g(\cdot,u)$ are Lebesgue measurable for all $\xi\in\mathbb{R}^N$ and all $u\in\mathbb{R}$, and $\mathbf{A}(x,u,\cdot),a_0(x,u,\cdot)$ are continuous for almost all $x\in\mathbb{R}^N,u\in\mathbb{R}$. $\mathbf{A}=(a_1,a_2,\cdots,a_N)$.

Let $1<m<N$. We also assume the following conditions

(a) there exists $\nu_0,\nu_1>0$ such that

$$\mathbf{A}(x,u,\xi)\xi\geq\nu_0|\xi|^m,$$
$$|\mathbf{A}(x,u,\xi)|\leq\nu_1|\xi|^{m-1}+h_1(x)|u|^{m-1}, \tag{1.2}$$
$$|a_0(x,u,\xi)|\leq h_2(x)|\xi|^{m-1}+h_3(x)|u|^{m-1}, \tag{1.3}$$
$$(\mathbf{A}(x,\xi)-\mathbf{A}(x,\eta))(\xi-\eta)>0. \tag{1.4}$$

(b) there exist two non-negative functions $g_1,g_2\in L^1_{loc}(\mathbb{R}^N)$ two measurable functions $g_1:\mathbb{R}^N\to\mathbb{R}$ and $g_2:\mathbb{R}^N\to\mathbb{R}$ such that

$$-g_1(x)|u|^{m-1}\leq g(x,u)\operatorname{sign}u\leq g_2(x)|u|^{m-1}. \tag{1.5}$$

The model equation for (1.1), which is of independent interest, is

$$-\Delta_m u+Vu|u|^{m-2}=0, \tag{1.6}$$

where $\Delta_m u=\operatorname{div}(|\nabla u|^{m-2}\nabla u)$ is the m-Laplacian, V is a potential.

We make the following local assumptions on the structural coefficients of (1.1)

$$\lim_{\rho\to0}\sup_{x\in\mathbb{R}^N}\int_0^\rho\left\{\frac{1}{r^{N-m}}\int_{B_r(x)}[h_2^m(y)+h_3(y)+g_1(y)+g_2(y)]\,dy\right\}^{\frac{1}{m-1}}\frac{dr}{r}=0,$$

$$\lim_{\rho\to 0}\sup_{x\in\mathbb{R}^N}\int_0^{\rho}\left\{\frac{1}{r^{N-m}}\int_{B_r(x)} h_1^{\frac{m}{m-1}}(y)\,dy\right\}^{\frac{1}{m-1}}\frac{dr}{r}=0, \tag{1.7}$$

which guarantee the validity of the local Harnack inequality and continuity of the solutions to equation (1.1) [19]. It is straightforward to see that for $m=2$ the above requirement coincides with the well known Kato class condition.

Solutions to equation (1.1) (and to all the equations treated in this paper) are understood in the weak sense. Namely, we say that $u\in W^{1,m}_{loc}(\mathbb{R}^N)$ is a weak solution to (1.1) if for any $\varphi\in W^{1,m}(\mathbb{R}^N)$ with compact support, the following integral equality holds

$$\int_{\mathbb{R}^N}\mathbf{A}(x,u,\nabla u)\cdot\nabla\varphi+a_0(x,u,\nabla u)\varphi+g(x,u)\varphi\,dx=0. \tag{1.8}$$

Remark 1.1. Condition (1.7) in the context of the equation related to the p-Laplacian was studied in [3], where amongst other things it was shown that for $p=2$ condition (1.7) coincides with the well known Kato class. In the theory of linear equations the Kato class is known to be the optimal class for a number of qualitative properties of elliptic and parabolic equations, such as standard estimates of fundamental solutions, continuity of weak solutions, the Harnack inequality [6, 18]. Recently in [19] the continuity of solutions and the Harnack inequality was proved for the general quasilinear equation (1.1) under the assumption (1.7) thus extending classical results of Serrin [17]. Namely, if $u\ge 0$ is a solution to (1.1) in the ball $B_{2r}(x_0)$ then there exists a constant C_H depending only on N,ν_1,m such that

$$\inf_{x\in B_r(x_0)}u(x)\ge C_H\sup_{x\in B_r(x_0)}u(x), \tag{1.9}$$

for $x_0\in\mathbb{R}^N,\ r\le 1$.

In order to study the global behaviour of solutions to (1.1) we need to control the global behaviour of the functions h_1,h_2,h_3,g_1,g_2. This is the reason we introduce the following quantity. Let $0\le g\in L^1_{loc}(\mathbb{R}^N)$. Set

$$\mathcal{K}(g)=\sup_{x\in\mathbb{R}^N}\int_0^{\infty}\left(\frac{1}{t^{N-m}}\int_{B_t(x)}g(y)\,dy\right)^{\frac{1}{m-1}}\frac{dt}{t}, \tag{1.10}$$

$$\tilde{\mathcal{K}}(g)=\sup_{x\in\mathbb{R}^N}\int_0^{\infty}\left(\frac{1}{t^{N-m}}\int_{B_t(x)}g(y)\,dy\right)^{\frac{1}{m}}\frac{dt}{t}, \tag{1.11}$$

where $\mathcal{K}(g)$, $\tilde{\mathcal{K}}(g)$ need not be finite. For $m = 2$ the finiteness of $\mathcal{K}(g)$ is equivalent to the condition that g is Green bounded, as one can readily verify.

We start the construction a global entire solution to (1.1) with the following auxiliary boundary value problem

$$-\operatorname{div} \mathbf{A}(x, \nabla v) + a_0(x, v, \nabla v) + g(x, v) = 0, \quad x \in B_R, \tag{1.12}$$

$$v(x) = 1, \quad x \in \partial B_R. \tag{1.13}$$

The solutions to (1.12) will provide approximations of the global solution which will be obtained passing $R \to \infty$.

First, we prove lower and upper bounds on the approximate solutions.

Theorem 1.1. *Let conditions* (a), (b) *be satisfied. Then there exists* $\tau > 0$ *such that the conditions*

$$\mathcal{K}(g_1 + h_2^m + h_3) < \tau, \quad \mathcal{K}(g_2) < \infty, \quad \tilde{\mathcal{K}}(h_1^{\frac{m}{m-1}}) < \infty$$

imply that for any $R \geq 1$ *there exists a positive solution* v *to the problem* (1.12), (1.13) *satisfying the inequality*

$$v(x) \geq \exp\big(- M_1\{\mathcal{K}(g_2) + \mathcal{K}(h_2^m + h_3) + \tilde{\mathcal{K}}(h_1^{\frac{m}{m-1}})\}\big) \tag{1.14}$$

with some positive M_1 *depending on* N, ν_0, ν_1, m *only.*

Theorem 1.2. *Let conditions* (a), (b) *be satisfied. Then there exists* $\tau > 0$ *such that the condition*

$$\mathcal{K}(g_1 + h_2^m + h_3) + \tilde{\mathcal{K}}(h_1^{\frac{m}{m-1}}) < \tau \tag{1.15}$$

implies that there exists $M_2 > 0$ *depending* N, ν_0, ν_1, m, τ *such that the solution* v *to the problem* (1.12), (1.13) *satisfies the estimate*

$$v(x) \leq M_2, \quad x \in B_R. \tag{1.16}$$

The proofs of Theorems 1.1, 1.2 are given in the main body of the paper. They are based on a modification of the Kilpeläinen–Maly method of obtaining upper bounds [10, 15].

An immediate consequence of Theorems 1.1, 1.2 is the following fundamental result, which is one of the main results of this paper.

Theorem 1.3. *Let conditions* (a), (b) *be satisfied. Let* $\mathcal{K}(g_2) < \infty$, $\mathcal{K}(g_1) < \tau$, $\mathcal{K}(h_2^m + h_3) + \tilde{\mathcal{K}}(h_1^{\frac{m}{m-1}}) < \tau$ *with* τ *defined in Theorem* 1.2. *Then there exist constants* M_1, $M_2 > 0$ *and a solution* u *to equation* (1.1) *in* $\mathbb{R}^N$ *such that*

$$\exp\big(-M_1\{\mathcal{K}(g_2^+) + \mathcal{K}(h_2^m + h_3) + \tilde{\mathcal{K}}(h_1^{\frac{m}{m-1}})\}\big) \le u(x) \le M_2, \quad x \in \mathbb{R}^N. \tag{1.17}$$

Remark 1.2. The above result extends to the quasilinear case the respective results from [8, 9, 11, 12]. It is not difficult to construct an example showing the sharpness os the conditions of Theorem 1.3. For instance, one can look at positive radial solutions of the equation $-\Delta_m u + V u^{m-1} = 0$ with the potential V behaving at infinity like $|x|^{-m}(\log|x|)^{-(m-1)}$.

Next, we study behavior of nonnegative solutions to the equation with the right hand side

$$-\operatorname{div}\mathbf{A}(x,u,\nabla u) + a_0(x,\nabla u,u) + g(x,u) = f(x), \quad x \in B_1^c, \tag{1.18}$$

where $B_1^c = \mathbb{R}^N \setminus \overline{B_1}$. We assume that

$$\lim_{\rho\to 0}\sup_{x\in\mathbb{R}^N}\int_0^\rho\left\{\frac{1}{r^{N-m}}\int_{B_r(x)}|f(y)|\,dy\right\}^{\frac{1}{m-1}}\frac{dr}{r} = 0. \tag{1.19}$$

It is known by now (see [19]) that under conditions (a), (b), (1.19) solutions to (1.18) are continuous and satisfy the weak Harnack inequality

$$\inf_{x\in A_{R,2R}} u(x) \ge H_2\left(\frac{1}{R^N}\int_{A_{R,2R}} u^q\,dx\right)^{\frac{1}{q}}, \tag{1.20}$$

where $0 < q < \frac{N(m-1)}{N-m}$ and the constant H_2 depends on q and the structural constants only.

In order to formulate the result on the behaviour of super-solutions to equation (1.18) we introduce the following two quantities:

$$G(R) = \sup_{y\in A_{R,2R}}\int_0^{R/4}\left(\frac{1}{r^{N-m}}\int_{B_r(y)}(g_2 + h_3)\,dx\right)^{\frac{1}{m-1}}\frac{dr}{r}$$

$$+\sup_{y\in A_{R,2R}}\int_0^{R/4}\left(\frac{1}{r^{N-m}}\int_{B_r(y)} h_1^{\frac{m}{m-1}}\,dx\right)^{\frac{1}{m}}\frac{dr}{r}. \qquad (1.21)$$

$$F(r)=\left(\frac{1}{r^{N-m}}\int_{A_{\frac{5r}{4},\frac{7r}{4}}} f(x)\,dx\right)^{\frac{1}{m-1}}. \qquad (1.22)$$

We make the following assumption that controls the global behaviour of the coefficients of (1.1)

(g) there exists a constant $\nu_3>0$ such that

$$\sum_{j=1}^{\infty} G(2^j)\le \nu_3 \quad\text{and}\quad \mathcal{K}(h_2^m)<\tau, \qquad (1.23)$$

where τ is defined in Theorem 1.1. Set

$$H^-(R,R_0)=\min_{R_0<\rho<R}\left\{\sum_{j=j(\rho)+1}^{j(R)} F(2^j)+\frac{F(\rho)}{G(\rho)}\right\}, \qquad (1.24)$$

$$H^+(R,R^*)=\min_{R<\rho<R^*}\left\{\sum_{j=j(R)}^{j(\rho)-1} F(2^j)+\frac{F(\rho)}{G(\rho)}\right\}, \qquad (1.25)$$

and $j(R)$ is fixed by the condition $1\le 2^{-j(R)}R<2$.

Theorem 1.4. *Let conditions* (a), (b), (g) *be satisfied. Assume that* $0\le f\in L^1_{loc}(\mathbb{R}^N)$. *Let u be a nonnegative super-solution to* (1.18) *in* A_{1,R^*}. *Then there exist* R_0, $M_3>0$ *such that the estimate*

$$\inf_{x\in A_{R,2R}} u(x)\ge M_3\min\{H^-(R,R_0),H^+(R,R^*)\} \qquad (1.26)$$

holds for $2R_0<R<\frac{1}{2}R^*$.

The next theorem establishes an important feature of the behaviour of super-solutions to (1.1) at infinity. This and subsequent results will require an additional structural condition on the functions $\mathbf{A}$, the implication of which is that $m\ge 2$.

$$(\mathbf{A}(x,u,\xi)-\mathbf{A}(x,u,\eta))\,(\xi-\eta)\ge|\xi-\eta|^m,\qquad u\in\mathbb{R},\quad \xi,\eta\in\mathbb{R}^N. \qquad (1.27)$$

Theorem 1.5. *Let conditions* (a), (b), (g), (1.27) *be satisfied. Suppose that* $\mathcal{K}(g_2) < \infty$ *and* $\mathcal{K}(g_1) < \tau$ *with* τ *defined in Theorem* 1.2. *Let* $R_* < \infty$ *and* u *be a nonnegative super-solution to* (1.1) *in* $B^c_{R_*}$. *Then*

$$\liminf_{|x|\to\infty} u(x) < \infty. \tag{1.28}$$

Now we are able to formulate a non-existence result for nonnegative super-solutions to the equation

$$-\operatorname{div} \mathbf{A}(x, u, \nabla u) + a_0(x, u, \nabla u) = f(x)u^p, \quad x \in \mathbb{R}^N \setminus B_{R_*}. \tag{1.29}$$

We assume the following condition (f_2') to be fulfilled

$$\sum_{j=j_*}^{\infty} \bar{F}(2^j) = \infty, \qquad \lim_{j\to\infty} \frac{\bar{F}(2^j)}{G(2^j)} = \infty, \tag{1.30}$$

where

$$\begin{gathered} \bar{F}(2^j) = \Big\{ 2^{jm} \inf_{x\in A(j)} f(x) \Big\}^{\frac{m-1}{m-p-1}}, \\ A(j) = \{x \in \mathbb{R}^N;\ 2^j \le |x| < 2^{j+1}\}. \end{gathered} \tag{1.31}$$

Theorem 1.6. *Let* $0 \le f \in L^1_{loc}(\mathbb{R}^N \setminus \overline{B_1})$. *Let conditions* (a), (g), (1.27), (1.30) *be fulfilled. Suppose that* $\mathcal{K}(g_2) < \infty$ *and* $\mathcal{K}(g_1) < \tau$ *with* τ *defined in Theorem* 1.2. *Let* $0 < p < m-1$. *Then there are no non-trivial nonnegative super-solutions to* (1.29) *in* $B^c_{R_*}$ *for any* $R_* < \infty$.

As an example, let us consider the following model equation

$$-\Delta_m u + \frac{V(x)}{|x|^m}|u|^{m-2}u = \frac{f(x)}{|x|^m}|u|^{p-1}u. \tag{1.32}$$

The next result is a direct consequence of Theorem 1.6.

Theorem 1.7. *Let* $V : \mathbb{R}^N \to \mathbb{R}^1,\ f : \mathbb{R}^N \to \mathbb{R}^1$ *be measurable functions. Assume that there exist* $\bar{V} : \mathbb{R}^1 \to \mathbb{R}^1,\ \bar{f} : \mathbb{R}^1 \to \mathbb{R}^1$ *positive non-increasing functions such that*

$$V(x) \le \bar{V}(|x|), \qquad f(x) \ge \bar{f}(|x|),$$

and

$$\int_1^\infty \bar{V}(t)\frac{dt}{t} < \infty, \qquad \int_1^\infty \bar{f}(t)\frac{dt}{t} = \infty.$$

Let $0 \le p < m-1,\ R_* < \infty$. *Then there are no nonnegative nontrvial super-solution to* (1.32) *in* $B^c_{R_*}$.

Before passing to the proofs we will remind the reader of two known results, which will be used later on.

Proposition 1.1. *Let $g \geq 0$ be a measurable function such that $\mathcal{K}(g) < \infty$. Then there exists a unique nonnegative $h \in W_0^{1,p}(B_R)$ such that*

$$-\Delta_m h = g, \quad x \in B_R, \tag{1.33}$$

and the constant $K > 0$ such that

$$h(x) \leq K\,\mathcal{K}(g), \quad x \in B_R. \tag{1.34}$$

For the proof see [10, 15].

Proposition 1.2. *Let $g \geq 0$ be a measurable function such that $\mathcal{K}(g) < \infty$. Then for any $\varepsilon > 0$ there exists $\tau_1 > 0$ such that the inequality*

$$\mathcal{K}(g) < \tau_1 \tag{1.35}$$

implies the inequality

$$\int\limits_{\mathbb{R}^N} g(x)|\varphi(x)|^m\,dx \leq \varepsilon \int\limits_{R^N} |\nabla\varphi(x)|^m\,dx \tag{1.36}$$

for any $\varphi \in W^{1,m}(\mathbb{R}^N)$ with compact support.

Proof. It suffices to prove (1.36) for $\varphi \in C_0^\infty(\mathbb{R}^N)$. Let $\operatorname{supp}\varphi \subset B_R$. By the definition of weak solutions to (1.33) we have

$$\int\limits_{B_R} g|\varphi|^m\,dx = \int_{B_R} |\nabla h|^{m-2}\nabla h \cdot \nabla|\varphi|^m\,dx \leq \frac{\varepsilon}{2m}\int\limits_{B_R} |\nabla\varphi|^m\,dx$$
$$+\, m^{m-1}(m-1)\left(\frac{\varepsilon}{2}\right)^{1-m}\int\limits_{B_R} |\nabla h|^m|\varphi|^m\,dx. \tag{1.37}$$

Testing (1.33) by $h|\varphi|^m$ we derive

$$\int\limits_{B_R} |\nabla h|^m|\varphi|^m\,dx$$
$$= \int\limits_{B_R} |\nabla h|^{m-2}\nabla h\left(\nabla(h|\varphi|^m) - h\nabla|\varphi|^m\right)\,dx$$
$$\leq \int\limits_{B_R} gh|\varphi|^m\,dx + \frac{m-1}{m}\int\limits_{B_R} |\nabla h|^m|\varphi|^m\,dx$$

$$+ m^{m-1} \int_{B_R} h^m |\varphi|^m \, dx. \quad (1.38)$$

Inequalities (1.37), (1.38),(1.34) imply the estimate

$$\Big[1 - K\,\mathcal{K}(g)(m-1)m^m \left(\frac{\varepsilon}{2}\right)^{1-m}\Big] \int_{B_R} g|\varphi|^m \, dx$$
$$\leq \left\{\frac{\varepsilon}{2m} + (m-1)m^{2m-1}\left(\frac{\varepsilon}{2}\right)^{1-m} [K\,\mathcal{K}(g)]^m\right\} \int_{B_R} |\nabla\varphi|^m \, dx.$$

Now (1.36) follows from (1.35) with an appropriate choice of τ_1. □

Remark 1.3. In the proof of Proposition 1.2 we followed M. Biroli [4].

2. Existence of bounded entire solutions to equation (1.1)

Lemma 2.1. *Let the conditions of Theorem* 1.1 *be satisfied. Then for any* $R \geq 1$ *there exists a positive solution to the problem* (1.12), (1.13).

Proof. We let $\mathcal{A} : W_0^{1,m}(B_R) \to (W_0^{1,m}(B_R))^*$ denote the operator defined by

$$\langle \mathcal{A}\psi, \varphi\rangle = \int_{B_R} \left(\mathbf{A}(x, \nabla\psi)\nabla\varphi + a_0(x, \psi+1)\varphi\right) dx \quad (2.1)$$

for $\varphi, \psi \in W_0^{1,m}(B_R)$. By (1.5), (1.7) we obtain that for any $\varepsilon > 0$ there exists $C_1(\varepsilon)$ such that

$$\int_{B_R} a_0(x, \psi+1)\psi \, dx \geq -(1+\varepsilon) \int_{B_R} g_1(x)|\psi(x)|^m \, dx$$
$$+ -C_1(\varepsilon) \int_{B_R} (g_1(x) + g_2(x)) \, dx - C_2(R).$$

This together with condition (a) implies that

$$\langle \mathcal{A}\psi, \varphi\rangle \geq (1-\theta) \int_{B_R} |\nabla\psi|^m \, dx - C(R), \quad (2.2)$$

for some $\theta \in (0,1)$, $C(R) > 0$, which yields the coerciveness of the operator $\mathcal{A}$. It follows also from the conditions of the lemma that the operator $\mathcal{A}$ is continuous and satisfies the so-called (S_+)-conditions (see [20, Chapter 1]). Then the solvability of the problem

$$\mathcal{A}\psi = 0 \text{ in } B_R, \qquad \psi = 0 \text{ on } \partial B_R$$

follows (see e.g. [20]). Therefore $v = 1 + \psi$ solves the problem (1.12), (1.13). In order to verify that $v \geq 0$ we test equation (1.1) with $\varphi = v_- := v \wedge 0$. Using the structural conditions (a) we obtain

$$\int\limits_{B_R} |\nabla v_-|^m \, dx \leq c \int\limits_{B_R} [g_1(x) + h_2^m(x) + h_3(x)](v_-)^m \, dx.$$

Now Proposition 1.2 implies that $v \geq 0$. □

Proof of Theorem 1.1. By the Harnack inequality (see (1.9)) the solution v is positive.

Lemma 2.2. *Let the conditions of Theorem* 1.1 *be fulfilled. Let v be a nonnegative solution to* (1.12)–(1.13). *Let $y \in B_R$, $0 < r < R/2$. Define the cut-off function $\eta : \mathbb{R}^N \to R$ by*

$$\eta(x) := \begin{cases} 1, & x \in B_r(y), \\ 2 - \frac{|x-y|}{r}, & x \in B_{2r}(y) \setminus B_r(y), \\ 0, & x \notin B_{2r}(y). \end{cases}$$

Set $w(x) := -\log v(x)$ for $x \in B_R$. Then

$$\begin{aligned} \frac{\alpha}{2\delta^m} \int\limits_L \Big(1 + \frac{w-l}{\delta}\Big)^{-\alpha-1} |\nabla w|^m \eta^m \, dx \\ \leq c\frac{\alpha^{1-m}}{r^m} \int\limits_L \Big(1 + \frac{w-l}{\delta}\Big)^{(\alpha+1)(m-1)} dx \\ + \frac{1}{\delta^{m-1}} \int\limits_L (g_2 + h_2^m + h_3)\eta^m \, dx \\ + \frac{1}{\delta^m} \int\limits_L h_1^{\frac{m}{m-1}} \eta^m \, dx, \end{aligned} \tag{2.3}$$

where

$$L = \{w > l\} \cap B_R \cap B_{2r}(y).$$

Testing the equality

$$\int\limits_{B_R} (\mathbf{A}(x, v, \nabla v)\nabla\varphi + a_0(x, v, \nabla v)\varphi + g(x, v)\varphi) \, dx = 0 \tag{2.4}$$

with φ, where

$$\varphi(x) = v(x)^{1-m}\Phi_1\Big(\Big[\frac{w(x)-l}{\delta}\Big]_+\Big)\eta(x)^m, \qquad \Phi_1(z) = \big(1-(1+z)^{-\alpha}\big)_+, \quad z>0 \tag{2.5}$$

and l, δ, α are some positive numbers, using condition (a) we obtain

$$\int_L v^{-m}\Phi_1\Big(\frac{w-l}{\delta}\Big)|\nabla v|^m\eta^m\,dx + \frac{1}{\delta}\int_L v^{-m}\Big(1+\frac{w-l}{\delta}\Big)^{-\alpha-1}|\nabla v|^m\eta^m\,dx \le \frac{c}{r}\int_L v^{1-m}|\nabla v|^{m-1}\eta^{m-1}\,dx + \frac{c}{r}\int_L h_1\eta^{m-1}\,dx + c\int_L v^{1-m}|\nabla v|^{m-1}\Phi_1\Big(\frac{w-l}{\delta}\Big)h_2\eta^m\,dx + c\int_L (g_2+h_3)\eta^m\,dx.$$

By the Young inequality we have

$$\frac{c}{r}\int_L v^{1-m}|\nabla v|^{m-1}\eta^{m-1}\,dx \le \frac{1}{2\delta}\int_L v^{-m}\Big(1+\frac{w-l}{\delta}\Big)^{-\alpha-1}|\nabla v|^m\eta^m\,dx + \frac{c\delta^{m-1}}{r^m}\int_L \Big(1+\frac{w-l}{\delta}\Big)^{(\alpha+1)(m-1)}\,dx.$$

Similarly,

$$c\int_L v^{1-m}|\nabla v|^{m-1}\Phi_1\Big(\frac{w-l}{\delta}\Big)h_2\eta^m\,dx \le \frac{1}{2}\int_L v^{-m}\Phi_1\Big(\frac{w-l}{\delta}\Big)|\nabla v|^m\eta^m\,dx + c\int_L h_2^m\eta^m\,dx,$$

$$\frac{c}{r}\int_L h_1\eta^{m-1}\,dx \le \frac{c\delta^{m-1}}{r^m}\,\mathrm{meas}(L) + \frac{c}{\delta}\int_L h_1^{\frac{m}{m-1}}\eta^m\,dx.$$

This yields the assertion. □

Set $r_j = 2^{-j-1}R$, $\eta_j(x) = \eta(x)$ for $r = r_j$. Define recursively the sequence (l_j) by: $l_0 = 0$,

$$l_{j+1} = l_j + \left(\frac{\varkappa}{r_j^N} \int\limits_{L'_j} (w - l_j)^{(\alpha+1)(m-1)}\, dx \right)^{\frac{1}{(\alpha+1)(m-1)}}, \tag{2.6}$$

where we used the notation

$$L_j = \{w > l_j\} \cap B_R \cap B_{2r_j}(y), \qquad L'_j = L_j \cap B_{r_j}(y),$$

and $\varkappa > 0$ is to be chosen later. Denote $\delta_j = l_{j+1} - l_j$.

Lemma 2.3. *Under the above assumptions there exist positive constants $\varkappa, K_1$ such that for every $j = 1, 2, \ldots$ one has*

$$\delta_j < \frac{1}{2}\delta_{j-1} + K_1 \left(\frac{1}{r^{N-m}} \int\limits_{L_j} (g_2 + h_2^m + h_3)\eta_j^m\, dx \right)^{\frac{1}{m-1}} + K_1 \left(\frac{1}{r_j^{N-m}} \int\limits_{L_j} h_1^{\frac{m}{m-1}} \eta_j^m\, dx \right)^{\frac{1}{m}}. \tag{2.7}$$

Proof. We suppose that

$$\delta_j \geq \frac{1}{2}\delta_{j-1} \tag{2.8}$$

since otherwise inequality (2.7) is obvious. In (2.3) set $\delta = \delta_j$, $\eta = \eta_j$, $l = l_j$, $r = r_j$, $L = L_j$. Then we have

$$\frac{1}{r_j^N} \operatorname{meas} L_j \leq \frac{1}{r_j^N} \int\limits_{L_j} \Big(\frac{w - l_{j-1}}{\delta_{j-1}} \Big)^{(\alpha+1)(m-1)} dx \leq \frac{2^N}{r_{j-1}^N} \int\limits_{L'_{j-1}} \Big(\frac{w - l_{j-1}}{\delta_{j-1}} \Big)^{(\alpha+1)(m-1)} dx = \frac{2^N}{\varkappa}. \tag{2.9}$$

Analogously using (2.8) we obtain

$$\frac{1}{r_j^N} \int\limits_{L_j} \Big(\frac{w - l_j}{\delta_j} \Big)^{(\alpha+1)(m-1)} dx \leq \frac{2^{N+(\alpha+1)(m-1)}}{r_{j-1}^N} \int\limits_{L'_{j-1}} \Big(\frac{w - l_{j-1}}{\delta_{j-1}} \Big)^{(\alpha+1)(m-1)} dx \leq \frac{C(\alpha)}{\varkappa}. \tag{2.10}$$

Set $\Phi_2(z) = \big(\int_0^z (1+s)^{-\frac{\alpha+1}{m}}\,ds\big)_+$ and $m^* = \frac{mN}{N-m}$. Using the Sobolev inequality to estimate the left hand side of (2.3) we have

$$\left(\frac{1}{r_j^N}\int\limits_{L_j}\Big[\Phi_2\Big(\frac{w-l_j}{\delta_j}\Big)\eta_j\Big]^{m^*}dx\right)^{\frac{m}{m^*}} \le \frac{C_5}{r_j^{N-m}\delta_j^m}\int\limits_{L_j}\Big(1+\frac{w-l_j}{\delta_j}\Big)^{-\alpha-1}|\nabla w|^m\eta^m\,dx + \frac{C_6(\alpha)}{r_j^N}\int\limits_{L_j}\Big(1+\frac{w-l_j}{\delta_j}\Big)^{m-\alpha-1}dx. \quad (2.11)$$

Let $\gamma\in(0,1)$. Set $L'_j(\gamma) = L'_j\cap\{w > l_j+\gamma\delta_j\}$. Then we have

$$\frac{1}{r_j^N}\int\limits_{L_j}\Big[\Phi_2\Big(\frac{w-l_j}{\delta_j}\Big)\eta_j\Big]^{m^*}dx \ge \frac{C_7(\alpha)}{r_j^N}\int\limits_{L'_j(\gamma)}\Big(\frac{w-l_j}{\delta_j}\Big)^{(m-\alpha-1)\frac{m^*}{m}}dx. \quad (2.12)$$

Choose $\alpha = \frac{m-1}{N-m+1}$ so that

$$(\alpha+1)(m-1) = (m-\alpha-1)\frac{m^*}{m}. \quad (2.13)$$

Then by (2.6) and (2.9) we have

$$\frac{1}{\varkappa} = \frac{1}{r_j^N}\int\limits_{L'_j(\gamma)}\Big(\frac{w-j}{\delta_j}\Big)^{(\alpha+1)(m-1)} + \frac{1}{r_j^N}\int\limits_{L'_j\setminus L'_j(\gamma)}\Big(\frac{w-l_j}{\delta_j}\Big)^{(\alpha+1)(m-1)} \le \frac{1}{r_j^N}\int\limits_{L'_j(\gamma)}\Big(\frac{w-l_j}{\delta_j}\Big)^{(\alpha+1)(m-1)} + \frac{2^N}{\varkappa}\gamma^{(\alpha+1)(m-1)}. \quad (2.14)$$

Now choosing γ such that $\gamma^{(\alpha+1)(m-1)} = 2^{-N-1}$ from (2.3), (2.9)–(2.12), (2.14) we deduce that

$$\varkappa^{-\frac{m^*}{m}} \le C_8\Bigg(\varkappa^{-1} + \frac{1}{r_j^{N-m}\delta_j^{m-1}}\int\limits_{L_j}(g_2+h_2^m+h_3)\eta_j^m\,dx$$

$$+ \frac{1}{r_j^{N-m}\delta_j^m} \int\limits_{L_j} h_1^{\frac{m}{m-1}} \eta_j^m \, dx \Bigg). \quad (2.15)$$

Now we fix $\varkappa$ by choosing $\varkappa^{1-\frac{m}{m^*}} = 2C_8$. Then (2.15) yields (2.7). Summing up inequalities (2.7) over $j = 1, 2, \dots, J$ we obtain

$$l_{J+1} - l_1 \le \frac{1}{2} l_J + K_1 \sum_{j=1}^{J} \left(\frac{1}{r_j^{N-m}} \int\limits_{L_j} (g_2 + h_2^m + h_3) \eta_j^m \, dx \right)^{\frac{1}{m-1}}$$
$$+ K_1 \sum_{j=1}^{J} \left(\frac{1}{r_j^{N-m}} \int\limits_{L_j} h_1^{\frac{m}{m-1}} \eta_j^m \, dx \right)^{\frac{1}{m}},$$

which implies that

$$\frac{1}{2} l_{J+1} \le \left(\frac{2^N \varkappa}{R^N} \int\limits_{B_R} [w^+]^q \, dx \right)^{\frac{1}{q}}$$
$$+ C_9 \int\limits_0^R \left(\frac{1}{r^{N-m}} \int\limits_{B_r(y)\cap B_R} (g_2 + h_2^m + h_3) \, dx \right)^{\frac{1}{m-1}} \frac{dr}{r}$$
$$+ C_9 \int\limits_0^R \left(\frac{1}{r^{N-m}} \int\limits_{B_r(y)\cap B_R} h_1^{\frac{m}{m-1}} \, dx \right)^{\frac{1}{m}} \frac{dr}{r} \quad (2.16)$$

with $q = (\alpha + 1)(m - 1)$. It follows that the sequence (l_j) is bounded above, and therefore due to the continuity of w we have

$$w(y) = \lim_{j\to\infty} l_j.$$

From (2.16) we conclude that there exists a constant $C > 0$ such that

$$w(y) \le C_{10} \left(\frac{1}{R^N} \int\limits_{B_R} [w^+]^q \, dx \right)^{\frac{1}{q}} + C\mathcal{K}(g_2 + h_2^m + h_3) + C\tilde{\mathcal{K}}(h_1^{\frac{m}{m-1}}). \quad (2.17)$$

It remains to estimate the first term in the right hand side of (2.17).

Claim. For every $\varepsilon > 0$ and $\varkappa_1 \in (1, m^*)$ there exists $K_2(\varepsilon, \varkappa_1)$ such that

$$\left(\frac{1}{R^N} \int\limits_{B_R} [w^+]^{\varkappa_1} \, dx \right)^{\frac{1}{\varkappa_1}} \le \varepsilon \max_{x\in B_R} w^+(x) + K_2(\varepsilon, \varkappa_1)\, \mathcal{K}(g_2 + h_2^m + h_3). \quad (2.18)$$

Indeed, let $\varepsilon_1 > 0$, and $\beta \in (0,1)$. Taking $\varphi = v^{1-m}\big([w^+ + \varepsilon_1]^\beta - \varepsilon_1^\beta\big)$ as a test function in (2.4) and using conditions (a) and (b) we obtain

$$\int\limits_{B_R^+} (w+\varepsilon_1)^{\beta-1}|\nabla w|^m\,dx \le C(\beta)\int\limits_{B_R^+}(g_2 + h_2^m + h_3)(w+\varepsilon_1)^\beta\,dx,$$
$$B_R^+ = B_R \cap \{w > 0\}. \quad (2.19)$$

By the Sobolev inequality we have

$$\left(\frac{1}{R^N}\int\limits_{B_R^+}\big[(w+\varepsilon_1)^{1+\frac{\beta-1}{m}} - \varepsilon_1^{1+\frac{\beta-1}{m}}\big]^{m^*}\right)^{\frac{m}{m^*}}$$
$$\le C\,\frac{1}{R^{N-m}}\int\limits_{B_R^+}(w+\varepsilon_1)^{\beta-1}|\nabla w|^m\,dx. \quad (2.20)$$

Combining this with (2.19) and passing to the limit $\varepsilon_1 \to 0$ we obtain

$$\left(\frac{1}{R^N}\int\limits_{B_R^+} w^{(m-1+\beta)\frac{N}{N-m}}\,dx\right)^{\frac{m}{m^*}}$$
$$\le C(\beta)\frac{1}{R^{N-m}}\int\limits_{B_R^+}(g_2+h_2^m+h_3)[w^+]^\beta\,dx. \quad (2.21)$$

The right hand side of the last inequality is estimated using the Young inequality and the definition of $\mathcal{K}(g)$ as follows

$$\left(\frac{1}{R^{N-m}}\int\limits_{B_R^+}(g_2+h_2^m+h_3)[w^+]^\beta\,dx\right)^{\frac{1}{m-1+\beta}}$$
$$\le \varepsilon\max_{x\in B_R} w^+(x) + C(\varepsilon,\beta)\left(\frac{1}{R^{N-m}}\int\limits_{B_R}(g_2+h_2^m+h_3)\,dx\right)^{\frac{1}{m-1}}$$
$$\le \varepsilon\max_{x\in B_R} w^+(x) + C(\varepsilon,\beta)\,\mathcal{K}(g_2+h_2^m+h_3). \quad (2.22)$$

Hence (2.18) follows. This completes the proof of the claim.

To complete the proof of Theorem 1.1 we choose the point $y \in B_R$ so that $w(y) = \max_{x\in B_R} w(x)$. Then (2.17),(2.18) with $\varepsilon = \frac{1}{2C_{10}}$ and $\varkappa_1 = q$ yield

$$w(x) \le C\,\mathcal{K}(g_2 + h_2^m + h_3) + \tilde{\mathcal{K}}(h_1^{\frac{m}{m-1}}), \tag{2.23}$$

which is (1.14). □

Next, we prove Theorem 1.2. We start with the following lemma.

Lemma 2.4. *Let the conditions of Theorem* 1.2 *be fulfilled. Let v be the solution to* (1.12), (1.13). *Then there exist positive constants τ_2, K_4 such that the inequality*

$$\mathcal{K}(g_1 + h_2^m + h_3) < \tau_2 \tag{2.24}$$

implies that

$$\frac{1}{R^N}\int_{B_R} v(x)^{m^*}\,dx \le K_4. \tag{2.25}$$

Proof. Take $\varphi = v-1$ as a test function in (2.4). Using conditions (a),(b) we obtain in a standard way

$$\int_{B_R} |\nabla v|^m\,dx \le C\Bigg(\int_{B_R\cap\{v>1\}} (g_1 + h_2^m + h_3)\,[v-1]^m\,dx + \int_{B_R} (g_1 + h_3)\,dx\Bigg). \tag{2.26}$$

Using Proposition 1.2 in the first term of the right hand side of (2.26) and the definition of $\mathcal{K}(g_1 + h_2^m + h_3)$ in the second term and choosing τ_2 appropriately we obtain, for some $C > 0$,

$$R^{m-N}\int_{B_R} |\nabla v|^m\,dx \le C.$$

Now the assertion follows from the Sobolev inequality. □

Proof of Theorem 1.2. In the proof we follow Kilpeläinen–Maly method [10] with modifications from [13]. Testing (1.8) by

$$\varphi(x) = v(x)^\lambda \Phi_2(x)\eta^m(x), \qquad \Phi_2(x) := \left[\int_l^{v(x)^\beta} \Big(1 + \frac{s-l}{\delta}\Big)^{-1-\alpha}\,ds\right]_+,$$

with η as in Lemma 2.2 and a sufficiently small $\lambda > 0$. Using conditions (a) and the Young inequality we have

$$I_1 := \int_L v^{\lambda-1}\Phi_2|\nabla v|^m\eta^m\,dx + \int_L v^{\lambda+\beta-1}\Big(1+\frac{s-l}{\delta}\Big)^{-1-\alpha}|\nabla v|^m\eta^m\,dx \le cr^{-m}\delta^m\int_L v^{\lambda-(\beta-1)(m-1)}\Big(1+\frac{v^\beta-l}{\delta}\Big)^{(1+\alpha)(m-1)}dx + c\int_L v^{m-1+\lambda}\Phi_2 H_1\eta^m\,dx + c\int_L v^{m-1+\lambda+\beta}\Big(1+\frac{v^\beta-l}{\delta}\Big)^{-1-\alpha}h_1^{\frac{m}{m-1}}\eta^m\,dx, \quad (2.27)$$

where $H_1 = g_1 + h_2^m + h_3$, $L = \{v^\beta > l\}\cap B_R \cap B_{2r}(y)$.

Set

$$\Psi(x) = \frac{1}{\delta}\Bigg[\int_l^{v(x)^\beta} s^{\frac{\lambda-(\beta-1)(m-1)}{\beta m}}\Big(1+\frac{s-l}{\delta}\Big)^{-\frac{1+\alpha}{m}}ds\Bigg]_+. \quad (2.28)$$

Using conditions (a) and direct calculations we obtain

$$\int_L |\nabla\Psi|^m\eta^m\,dx \le cr^{-m}\int_L v^{\lambda-(\beta-1)(m-1)}\Big(1+\frac{v^\beta-l}{\delta}\Big)^{(1+\alpha)(m-1)}dx + c\delta^{1-m}l^{\frac{m-1+\lambda}{\beta}}\int_L H_1\eta^m\,dx + c\delta^{-m}l^{\frac{m-1+\lambda+\beta}{\beta}}\int_L h_1^{\frac{m}{m-1}}\eta^m\,dx.$$

In order to separate the term $\int_L H_1\eta^m dx$ we introduced w as the solution to $-\Delta_p w = H_1$, $w \in W_0^{1,m}(B_R)$, substituted $-\Delta_p w$ in place of H_1, integrated by parts and applied the Young inequality. For the details of this routine calculation we refer the reader to [13].

Define the sequence (l_j) inductively by setting $l_0 = 0$ and

$$\varkappa = r_j^{-N}\int_{L_j}\Big(\frac{v^\beta}{l_{j+1}}\Big)^{\frac{\lambda-(\beta-1)(m-1)}{\beta}}\Big(\frac{v^\beta-l_j}{l_{j+1}-l_j}\Big)^{(1+\alpha)(m-1)}dx.$$

Much in the same way as in Lemma 2.3 , for $\delta_j = l_{j+1} - l_j$, we obtain

$$\delta_j \le \frac{1}{2}\delta_{j-1} + \gamma l_j \Bigg(r_j^{m-N} \int\limits_{B_j} H_1\, dx \Bigg)^{\frac{1}{m-1}} + \gamma l_j \Bigg(r_j^{m-N} \int\limits_{B_j} h_1^{\frac{m}{m-1}}\, dx \Bigg)^{\frac{1}{m}}. \tag{2.29}$$

Now in the same way as in the proof of Claim in Theorem 1.1 we arrive at

$$\max_{y\in B_R} v(y) \le C \Bigg(\frac{1}{R^N} \int\limits_{B_R} v^q\, dx \Bigg)^{\frac{1}{q}}. \tag{2.30}$$

Now the assertion follows from Lemma 2.4. □

Proof of Theorem 1.3. Let $R_j = 2^j$, $j = 1, 2, \dots$ Let v_j be the solution to (1.12), (1.13) with $R = R_j$. Extend v_j to the whole of $\mathbb{R}^N$ by setting $v_j(x) = 1$ for $|x| > R_j$. Then inequality (1.15) yields estimates (2.25), (2.26) for each j. Therefore the sequence (v_j) is bounded in $W^{1,m}(B_R)$ for each $R > 1$. Passing to a subsequence we can assume that the sequence (v_j) converges weakly in $W^{1,m}_{loc}(\mathbb{R}^N)$ to $u \in W^{1,m}_{loc}(\mathbb{R}^N)$, and hence strongly in $L^m_{loc}(\mathbb{R}^N)$ and poitwise almost every where. By Theorems 1.1, 1.2, this implies that the limit u enjoys estimate (1.17). Using monotone operator arguments one can readily verify that u solves equation (1.1). □

3. Lower bound for positive solutions to (1.18)

Let u be a weak solution to equation (1.18). By definition

$$\int\limits_{B_1^c} \left(\mathbf{A}(x,u,\nabla u)\nabla\varphi + a_0(x,u,\nabla u)\varphi + g(x,u)\varphi \right)\, dx = \int\limits_{B_1^c} f(x)\varphi(x)\, dx, \tag{3.1}$$

Further on we assume that conditions (a), (b), (f) are fulfilled. Consequently $u > 0$ in B_1^c. For $R > 2$ set

$$m(R) = \inf_{x\in A_{R/2,4R}} u(x), \tag{3.2}$$

and define the function $\psi : \mathbb{R}^N \to [0,1]$ by

$$\psi(x) = \mathcal{P} \min\left\{ \left[2\frac{|x|}{R} - 1 \right]^+, \left[5 - 2\frac{|x|}{R} \right]^+, 1 \right\}, \quad x \in \mathbb{R}^N. \tag{3.3}$$

Theorem 3.1. *Let u be a nonnegative non-trivial solution to* (1.18) *in* $A_{R/2,4R}$. *Then for any $\gamma > 0$ there exist $B(\gamma)$, $M(\gamma) > 0$ such that the inequality*

$$\sup_{x\in A_{R,2R}} \frac{1}{u(x) - m(R) + k(R)m(R)} \le M(\gamma)\left(\frac{1}{R^N}\int\limits_{A_{\frac{R}{2},\frac{5R}{2}}} \frac{dx}{[u(x) - m(R) + m(R)k(R)]^\gamma}\right)^{\frac{1}{\gamma}} \tag{3.4}$$

holds with

$$k(R) = BG_2(R), \qquad B \ge B(\gamma) \ge 1, \tag{3.5}$$

and G_2 defined in (1.21).

Proof. Test (3.1) with

$$\varphi = \Phi_1\Big(\frac{\psi v - l}{\delta}\Big) v^{2m-2}\psi^{m-1}\eta^m,$$

where

$$v(x) = \frac{1}{u(x) - m(R) + k(R)m(R)}, \tag{3.6}$$

and Φ_1, η, l, δ are the same as in the proof of Theorem 1.1, α defined by (2.13). Using conditions (a), (b), (f) we obtain from (3.1)

$$\begin{aligned}
&\frac{1}{\delta}\int\limits_L \Big(1 + \frac{\psi v - l}{\delta}\Big)^{-\alpha-1} |\nabla v|^m \psi^m \eta^m \, dx \\
&\qquad + \int\limits_L \Phi_1\Big(\frac{\psi v - l}{\delta}\Big) v^{-1} |\nabla v|^m \psi^{m-1}\eta^m \, dx \\
&\le \frac{C}{\delta}\int\limits_L \Big(1 + \frac{\psi v - l}{\delta}\Big)^{-\alpha-1} v^{2m-1}(|\nabla u|^{m-1} + h_1 u^{m-1})|\nabla\psi|\psi^{m-1}\eta^m \, dx \\
&\quad + C\int\limits_L \Phi_1\Big(\frac{\psi v - l}{\delta}\Big) v^{2m-2}(|\nabla u|^{m-1} + h_1 u^{m-1})|\nabla\psi|\psi^{m-1}\eta^m \, dx \\
&\quad + C\int\limits_L \Phi_1\Big(\frac{\psi v - l}{\delta}\Big) v^{2m-2}(|\nabla u|^{m-1} + h_1 u^{m-1})|\nabla\eta|\psi^m\eta^{m-1} \, dx \\
&+ C\int\limits_L \Phi_1\Big(\frac{\psi v - l}{\delta}\Big) v^{2m-2}(h_2|\nabla u|^{m-1} + g_2 u^{m-1} + h_3 u^{m-1})\psi^m\eta^m \, dx,
\end{aligned} \tag{3.7}$$

where $L = \{\psi v > l\} \cap A_{\frac{R}{2},\frac{5R}{2}} \cap B_{2r}(y)$. Next, using the Young inequality we have

$$\frac{1}{\delta^m}\int\limits_L \Big(1+\frac{\psi v - l}{\delta}\Big)^{-\alpha-1}|\nabla(\psi v)|^m \eta^m\,dx \le C\sum_{i=1}^{7} I_i, \tag{3.8}$$

where

$$\begin{aligned}
I_1 &= \frac{1}{\delta^m}\int\limits_L \Big(1+\frac{\psi v - l}{\delta}\Big)^{-\alpha-1} v^m|\nabla\psi|^m\eta^m\,dx,\\
I_2 &= \frac{1}{\delta^{m-1}}\int\limits_L v^{m-1}|\nabla\psi|^m\eta^m\,dx,\\
I_3 &= \int\limits_L \Big(1+\frac{\psi v - l}{\delta}\Big)^{(\alpha+1)(m-1)}|\nabla\eta|^m\,dx,\\
I_4 &= \frac{1}{\delta^{m-1}}\int\limits_L g_2^+ v^{2m-2}u^{m-1}\psi^m\eta^m\,dx,\\
I_5 &= \frac{1}{\delta^{m-1}}\int\limits_L h_2^m v^{m-1}\psi^m\eta^m\,dx,\\
I_6 &= \frac{1}{\delta^m}\int\limits_L h_1^{\frac{m}{m-1}} v^{2m}u^m\psi^m\eta^m\,dx,\\
I_7 &= \frac{1}{\delta^{m-1}}\int\limits_L h_1^{\frac{m}{m-1}} v^{2m-1}u^m\psi^m\eta^m\,dx.
\end{aligned}$$

Notice that $\psi v > 0$ on L. We have, for $x \in L,\ 0 < \mu < m$,

$$\begin{aligned}
\frac{1}{\delta^\mu}v^\mu = \psi^{-\mu}\Big(\frac{\psi v}{\delta}\Big)^\mu &\le C\Big[\Big(\frac{\psi v - l}{\delta}\Big)^\mu + \Big(\frac{l}{\delta}\Big)^\mu\Big]\psi^{-\mu}\\
&\le C\Big[\Big(1+\frac{\psi v - l}{\delta}\Big)^\mu + \Big(\frac{l}{\delta}\Big)^\mu\Big]\psi^{-\mu}.
\end{aligned} \tag{3.9}$$

By means of the Young inequality this implies, for $\mu_1 > \alpha + 1$, that

$$\begin{aligned}
I_1 &\le \frac{C}{\delta^{m-\mu_1}}\int\limits_L \Big[\Big(1+\frac{\psi v - l}{\delta}\Big)^{\mu_1-\alpha-1} + \Big(\frac{l}{\delta}\Big)^{\mu_1}\Big]\psi^{-\mu_1}v^{m-\mu_1}|\nabla\psi|^m\eta^m\,dx\\
&\le C\int\limits_L \Big(1+\frac{\psi v - l}{\delta}\Big)^{(\alpha+1)(m-1)}\psi^{-\mu_1}|\nabla\psi|^m\eta^m\,dx\\
&\quad + \int\limits_L \Big[\Big(\frac{v}{\delta}\Big)^{\gamma_1} + \Big(\frac{l}{\delta}\Big)^{\mu_1}\Big(\frac{v}{\delta}\Big)^{m-\mu_1}\Big]\psi^{-\mu_1}|\nabla\psi|^m\eta^m\,dx,
\end{aligned} \tag{3.10}$$

where $\gamma_1 = \frac{(\alpha+1)(m-1)}{(\alpha+1)m-\mu_1}(m-\mu_1)$. Analogously we obtain

$$I_2 \le \frac{C}{\delta^{m-\mu_2-1}} \int\limits_L \Big[\Big(1+\frac{\psi v - l}{\delta}\Big)^{\mu_2} + \Big(\frac{l}{\delta}\Big)^{\mu_2}\Big]\psi^{-\mu_2} v^{m-1-\mu_2} |\nabla\psi|^m \eta^m \, dx$$
$$\le C \int\limits_L \Big(1+\frac{\psi v - l}{\delta}\Big)^{(\alpha+1)(m-1)} \psi^{-\mu_2} |\nabla\psi|^m \eta^m \, dx$$
$$+ \int\limits_L \Big[\Big(\frac{v}{\delta}\Big)^{\gamma_2} + \Big(\frac{l}{\delta}\Big)^{\mu_2}\Big(\frac{v}{\delta}\Big)^{m-1-\mu_2}\Big]\psi^{-\mu_2}|\nabla\psi|^m \eta^m \, dx, \quad (3.11)$$

where $0 < \mu_2 < m-1$, $\gamma_2 = \frac{(\alpha+1)(m-1)}{(\alpha+1)(m-1)-\mu_2}(m-1-\mu_2)$.

Further we assume that

$$\mu_1 = m - \theta, \qquad \mu_2 = m - 1 - \theta - 1 \quad (3.12)$$

with some $\theta \in (0,1)$ to be determined later. ¿From the definition of the function ψ it follows that

$$|\nabla\psi| \le \frac{2\mathcal{P}}{R}\Big(\frac{2|x|}{R} - 1\Big)^{\mathcal{P}-1} = \frac{2\mathcal{P}}{R}[\psi(x)]^{\frac{\mathcal{P}-1}{\mathcal{P}}} \quad \text{for } \frac{R}{2}|x| < R,$$
$$|\nabla\psi| \le \frac{2\mathcal{P}}{R}\Big(5 - \frac{2|x|}{R}\Big)^{\mathcal{P}-1} = \frac{2\mathcal{P}}{R}[\psi(x)]^{\frac{\mathcal{P}-1}{\mathcal{P}}} \quad \text{for } 2R|x| < \frac{5R}{2}.$$

Set $\mathcal{P} = \frac{m}{\theta}$. Then

$$\psi^{-\mu_1}|\nabla\psi|^m \le \Big(\frac{2m}{\theta R}\Big)^m, \qquad \psi^{-\mu_2}|\nabla\psi|^m \le \Big(\frac{2m}{\theta R}\Big)^m. \quad (3.13)$$

It follows from (3.6) that

$$u(x) = u(x) - m(R) + m(R) \le v(x)^{-1} + k(R)^{-1} v(x)^{-1}. \quad (3.14)$$

Hence we obtain

$$I_4 \le \Big[\frac{k(R)+1}{\delta k(R)}\Big]^{m-1} \int\limits_L (g_2 + h_3)(\psi v)^{m-1}\eta^m \, dx. \quad (3.15)$$

Analogously by (3.14) we have

$$I_6 \le C\Big[\frac{k(R)+1}{\delta k(R)}\Big]^{m} \int\limits_L h_1^{\frac{m}{m-1}}(\psi v)^m \eta^m \, dx. \quad (3.16)$$

$$I_7 \le C\delta\Big[\frac{k(R)+1}{\delta k(R)}\Big]^{m} \int\limits_L h_1^{\frac{m}{m-1}}(\psi v)^{m-1} \eta^m \, dx. \quad (3.17)$$

Set $w=\psi v$. Then using (3.10), (3.11), (3.13), (3.15), (3.16), (3.17) we obtain from (3.8)

$$\frac{1}{\delta^m}\int_L\Big(1+\frac{w-l}{\delta}\Big)^{-\alpha-1}|\nabla w|^m\eta^m\,dx\le\frac{C}{r^m}\int_L\Big(1+\frac{w-l}{\delta}\Big)^{(\alpha+1)(m-1)}dx$$
$$\le\frac{C}{R^m}\int_L\Big[\Big(\frac{v}{\delta}\Big)^{\gamma_1}+\Big(\frac{v}{\delta}\Big)^{\gamma_2}+\Big(\frac{l}{\delta}\Big)^{m-\theta}\Big(\frac{v}{\delta}\Big)^{\theta}+\Big(\frac{l}{\delta}\Big)^{m-1-\theta}\Big(\frac{v}{\delta}\Big)^{\theta}\Big]\eta^m\,dx$$
$$+\frac{C}{(\delta k(R))^{m-1}}\int_L(g_2+h_3)w^{m-1}\eta^m\,dx+\frac{C}{(\delta k(R))^m}\int_L h_1^{\frac{m}{m-1}}w^m\eta^m\,dx$$
$$+\frac{C}{\delta^{m-1}k(R)^m}\int_L h_1^{\frac{m}{m-1}}w^{m-1}\eta^m\,dx+\frac{1}{\delta^{m-1}}\int_L h_2^m w^{m-1}\eta^m\,dx \quad (3.18)$$

where

$$\gamma_1=\frac{(\alpha+1)(m-1)}{\alpha m+\theta}\theta,\qquad \gamma_2=\frac{(\alpha+1)(m-1)}{\alpha(m-1)+\theta}\theta. \quad (3.19)$$

Let (l_j), (r_j), $(y_j(x))$, (δ_j), (L_j), (L'_j) denote the same sequences as in the proof of Theorem 1.1.

Claim. There exist positive constants $A(\theta), K_5(\theta)$ depending only on known parameters and θ such that

$$\delta_j\le\frac14\delta_{j-1}+\frac14\Big(1-2^{-\frac{\theta}{m-\theta}}\Big)l_j\Big(\frac{r_j}{R}\Big)^{\theta}+\frac14\Big(1-2^{-\frac{\theta}{m-1-\theta}}\Big)l_j\Big(\frac{r_j}{R}\Big)^{\theta}$$
$$+K_5(\theta)\Bigg\{\Big[\frac{1}{R^m r_j^{N-m}}\int_{L_j}v^{\gamma_1}(x)\Big]^{\frac{1}{\gamma_1}}+\Big[\frac{1}{R^m r_j^{N-m}}\int_{L_j}v^{\gamma_2}(x)\,dx\Big]^{\frac{1}{\gamma_2}}$$
$$+\Big[\Big(1+\frac{r_j}{R}\Big)\frac{1}{R^{m-1-\theta}r^{N-m+1+\theta}}\int_{L_j}v^{\theta}(x)\,dx\Big]^{\frac{1}{\theta}}$$
$$+\Big[\frac{1}{(k(R))^{m-1}r_j^{N-m}}\int_{L+_j}(g_2+h_3)w^{m-1}\eta_j^m\,dx\Big]^{\frac{1}{m-1}}$$
$$+\Big[\frac{1}{r_j^{N-m}}\int_{L+_j}h_2^m w^{m-1}\eta_j^m\,dx\Big]^{\frac{1}{m-1}}$$
$$+\Big[\frac{1}{(k(R))^m r_j^{N-m}}\int_{L+_j}h_1^{\frac{m}{m-1}}w^m\eta_j^m\,dx\Big]^{\frac{1}{m}}$$

$$+\left[\frac{1}{(k(R))^m r_j^{N-m}}\int\limits_{L+j} h_1^{\frac{m}{m-1}} w^{m-1}\eta_j^m\,dx\right]^{\frac{1}{m-1}}\Bigg\}. \tag{3.20}$$

Proof of the Claim. We can assume that

$$\delta_j \geq \frac{1}{4}\delta_{j-1}, \qquad \delta_j \geq \frac{1}{4}\big(1-2^{-\frac{\theta}{m-1-\theta}}\big)l_j\Big(\frac{r_j}{R}\Big)^{\frac{\theta}{m-\theta}}, \tag{3.21}$$

since otherwise inequality (3.20) is evident.

In (3.18) set $\delta=\delta_j,\ l=l_j,\ r=r_j,\ \eta=\eta_j,\ L=L_j$. Then similarly to the proof of (2.15) we obtain

$$\begin{aligned}\varkappa^{-\frac{m}{m^*}} \leq C_{3.22}(\theta)\Bigg\{\varkappa^{-1} &+ \frac{1}{r_j^{N-m}[\delta_j k(R)]^{m-1}}\int\limits_{L_j}(g_2+h_3)w^{m-1}\eta_j^m\,dx\\
&+\frac{1}{r_j^{N-m}[\delta_j k(R)]^m}\int\limits_{L_j} h_1^{\frac{m}{m-1}}w^{m-1}\eta_j^m\,dx\\
&+\frac{1}{r_j^{N-m}\delta_j^{m-1}}\int\limits_{L_j} h_1^{\frac{m}{m-1}}w^{m-1}\eta_j^m\,dx\\
&+\frac{1}{R^m r_j^{N-m}}\int\limits_{L_j}\Big[\Big(\frac{v}{\delta_j}\Big)^{\gamma_1}+\Big(\frac{v}{\delta_j}\Big)^{\gamma_2}\\
&+\Big(\Big(\frac{l_j}{\delta_j}\Big)^{m-\theta}+\Big(\frac{l_j}{\delta_j}\Big)^{m-1-\theta}\Big)\Big(\frac{v}{\delta_j}\Big)^{\theta}\Big]\eta_j^m\,dx\Bigg\}.\end{aligned} \tag{3.22}$$

The second and the third inequalities in (3.20) imply that

$$\begin{aligned}\frac{1}{R^m r_j^{N-m}}\Big(\Big(\frac{l_j}{\delta_j}\Big)^{m-\theta}+\Big(\frac{l_j}{\delta_j}\Big)^{m-1-\theta}\Big)\int\limits_{L_j}\Big(\frac{v}{\delta_j}\Big)^{\theta}\eta_j^m\,dx&\\
\leq \frac{C_{3.23}(\theta)}{R^{m-1-\theta}r_j^{N-m+1+\theta}}\int\limits_{L_j}\Big(\frac{v}{\delta_j}\Big)^{\theta}\eta_j^m\,dx.&\end{aligned} \tag{3.23}$$

Fix $\varkappa$ by the equality

$$\varkappa^{1-\frac{m}{m^*}} = 2C_{3.22}.$$

Then (3.22) and (3.23) yield (3.20), which proves the claim.

Fix $\gamma\in(0,1)$ and choose θ such that

$$\min\Big\{\frac{m}{\gamma_2},\frac{m-1}{\theta}-1\Big\} = \gamma+\frac{N}{\gamma}. \tag{3.24}$$

By the Hölder inequality we obtain, for $k=1,2$,

$$\frac{1}{R^m r_j^{N-m}}\int\limits_{L_j} v^{\gamma_k}\,dx \le C\bigg(\frac{1}{R^N}\int\limits_{L_j} v^{\gamma}\,dx\bigg)^{\frac{\gamma_k}{\gamma}}\Big(\frac{r_j}{R}\Big)^{m-N\frac{\gamma_k}{\gamma}}$$

$$\le C\bigg(\frac{1}{R^N}\int\limits_{A_{\frac{R}{2},\frac{5R}{2}}} v^{\gamma}\,dx\bigg)^{\frac{\gamma_k}{\gamma}}\Big(\frac{r_j}{R}\Big)^{\gamma_k\gamma}.$$

Now we have

$$\sum_{k=1}^{2}\bigg(\frac{1}{R^m r_j^{N-m}}\int\limits_{L_j} v^{\gamma_k}\,dx\bigg)^{\frac{1}{\gamma_k}}+\bigg(\frac{1}{R^{m-\theta} r_j^{N-m+\theta}}\int\limits_{L_j} v^{\theta}\,dx\bigg)^{\frac{1}{\theta}}$$

$$\le C\bigg(\frac{1}{R^N}\int\limits_{A_{\frac{R}{2},\frac{5R}{2}}} v^{\gamma}\,dx\bigg)^{\frac{1}{\gamma}}\Big(\frac{r_j}{R}\Big)^{\gamma}. \quad (3.25)$$

Summing inequalities (3.20) over $j=1,2,\dots,J$ and taking into account (3.25) we obtain

$$l_{J+1}-l_1 \le \frac{1}{4}l_J+\frac{1}{4}l_J+C(\gamma)\bigg(\frac{1}{R^N}\int\limits_{A_{\frac{R}{2},\frac{5R}{2}}} v^{\gamma}\,dx\bigg)^{\frac{1}{\gamma}}$$

$$+C(\gamma)\sum_{j=1}^{J}\bigg(\frac{1}{r_j^{N-m}[k(R)]^m}\int\limits_{L_j} h_1^{\frac{m}{m-1}}w^{m-1}\eta_j^m\,dx\bigg)^{\frac{1}{m-1}}$$

$$+C(\gamma)\sum_{j=1}^{J}\bigg(\frac{1}{r_j^{N-m}}\int\limits_{L_j} h_2^{m}w^{m-1}\eta_j^m\,dx\bigg)^{\frac{1}{m-1}}$$

$$+C(\gamma)\sum_{j=1}^{J}\bigg(\frac{1}{r_j^{N-m}[k(R)]^m}\int\limits_{L_j} h_1^{\frac{m}{m-1}}w^{m}\eta_j^m\,dx\bigg)^{\frac{1}{m}}$$

$$+C(\gamma)\sum_{j=1}^{J}\bigg(\frac{1}{r_j^{N-m}[k(R)]^{m-1}}\int\limits_{L_j} (g_2+h_3)w^{m-1}\eta_j^m\,dx\bigg)^{\frac{1}{m-1}}. \quad (3.26)$$

It follows that

$$\frac{1}{2}l_{J+1} \le \left(\frac{2^N \varkappa}{R^N} \int\limits_{A_{\frac{R}{2},\frac{5R}{2}}} [w^+]^{(\alpha+1)(m-1)}\,dx\right)^{\frac{1}{(\alpha+1)(m-1)}}$$

$$+ C(\gamma)\left(\frac{1}{R^N}\int\limits_{A_{\frac{R}{2},\frac{5R}{2}}} v^\gamma\,dx\right)^{\frac{1}{\gamma}}$$

$$+ \frac{1}{k(R)}\int\limits_0^{R/2}\left(\frac{1}{r^{N-m}}\int\limits_{B_r(y)} h_1^{\frac{m}{m-1}} w^m\,dx\right)^{\frac{1}{m}}\frac{dr}{r}$$

$$+ \frac{1}{[k(R)]^{\frac{m}{m-1}}}\int\limits_0^{R/2}\left(\frac{1}{r^{N-m}}\int\limits_{B_r(y)} h_1^{\frac{m}{m-1}} w^{m-1}\,dx\right)^{\frac{1}{m-1}}\frac{dr}{r}$$

$$+ \int\limits_0^{R/2}\left(\frac{1}{r^{N-m}}\int\limits_{B_r(y)} h_2^m w^{m-1}\,dx\right)^{\frac{1}{m-1}}\frac{dr}{r}$$

$$+ \int\limits_0^{R/2}\left(\frac{1}{r^{N-m}[k(R)]^{m-1}}\int\limits_{B_r(y)} (g_2+h_3) w^{m-1}\,dx\right)^{\frac{1}{m-1}}\frac{dr}{r}. \quad (3.27)$$

Analogously to the proof of Theorem 1.1 we obtain from (3.27)

$$w(y) \le C_{3.28}(\gamma)\Bigg\{\left(\frac{1}{R^N}\int\limits_{A_{\frac{R}{2},\frac{5R}{2}}} [w^+]^{(\alpha+1)(m-1)}\,dx\right)^{\frac{1}{(\alpha+1)(m-1)}}$$

$$+ \left(\frac{1}{R^N}\int\limits_{A_{\frac{R}{2},\frac{5R}{2}}} v^\gamma\,dx\right)^{\frac{1}{\gamma}}$$

$$+ \int\limits_0^{R/2}\left(\frac{1}{r^{N-m}}\int\limits_{B_r(y)} h_2^m w^{m-1}\,dx\right)^{\frac{1}{m-1}}\frac{dr}{r}$$

$$+ \frac{1}{k(R)}\int\limits_0^{R/2}\left(\frac{1}{r^{N-m}}\int\limits_{B_r(y)} h_1^{\frac{m}{m-1}} w^m\,dx\right)^{\frac{1}{m}}\frac{dr}{r}$$

$$+ \frac{1}{[k(R)]^{\frac{m}{m-1}}}\int\limits_0^{R/2}\left(\frac{1}{r^{N-m}}\int\limits_{B_r(y)} h_1^{\frac{m}{m-1}} w^{m-1}\,dx\right)^{\frac{1}{m-1}}\frac{dr}{r}$$

$$+\frac{1}{k(R)}\int_0^{R/2}\left(\frac{1}{r^{N-m}}\int_{B_r(y)}(g_2+h_3)w^{m-1}\,dx\right)^{\frac{1}{m-1}}\frac{dr}{r}\Bigg\}. \quad (3.28)$$

By the Young inequality we have

$$C_{3.28}(\gamma)\left(\frac{1}{R^N}\int_{A_{\frac{R}{2},\frac{5R}{2}}}[w^+]^{(\alpha+1)(m-1)}\,dx\right)^{\frac{1}{(\alpha+1)(m-1)}}$$
$$\leq \frac{1}{4}\sup_{x\in A_{\frac{R}{2},\frac{5R}{2}}} w(x)+C\left(\frac{1}{R^N}\int_{A_{\frac{R}{2},\frac{5R}{2}}} v^\gamma\,dx\right)^{\frac{1}{\gamma}}. \quad (3.29)$$

Now the choice of $k(R)$ (cf. (3.5)) implies that

$$\frac{1}{k(R)}\int_0^{R/2}\left(\frac{1}{r^{N-m}}\int_{B_r(y)}(g_2+h_3)w^{m-1}\,dx\right)^{\frac{1}{m-1}}\frac{dr}{r}$$
$$+\frac{1}{k(R)}\int_0^{R/2}\left(\frac{1}{r^{N-m}}\int_{B_r(y)}h_1^{\frac{m}{m-1}}w^m\,dx\right)^{\frac{1}{m}}\frac{dr}{r}$$
$$+\frac{1}{[k(R)]^{\frac{m}{m-1}}}\int_0^{R/2}\left(\frac{1}{r^{N-m}}\int_{B_r(y)}h_1^{\frac{m}{m-1}}w^{m-1}\,dx\right)^{\frac{1}{m-1}}\frac{dr}{r}$$
$$\leq \frac{1}{B}\sup_{x\in A_{\frac{R}{2},\frac{5R}{2}}} w(x). \quad (3.30)$$

Taking into account that

$$\int_0^{R/2}\left(\frac{1}{r^{N-m}}\int_{B_r(y)}h_2^m w^{m-1}\,dx\right)^{\frac{1}{m-1}}\frac{dr}{r}\leq \tau\sup_{x\in A_{\frac{R}{2},\frac{5R}{2}}} w(x)$$

and choosing $B=B(\gamma)=4C_{3.28}$ we obtain from (3.28)–(3.30)

$$\sup_{x\in A_{\frac{R}{2},\frac{5R}{2}}} w(x)\leq C\left(\frac{1}{R^N}\int_{A_{\frac{R}{2},\frac{5R}{2}}} v^\gamma\,dx\right)^{\frac{1}{\gamma}},$$

which completes the proof of Theorem 3.1. □

Next, using Theorem 3.1 we prove a variant of the weak Harnack inequality.

Theorem 3.2. *Let the conditions of Theorem* 3.1 *be fulfilled. Then for any* $q \in (0, \frac{N(m-1)}{N-m})$ *there exist* $\bar{B}(q), M(q)$ *such that*

$$\inf_{A_{R,2R}} [u(x) - m(R) + k(R)m(R)]$$
$$\geq M(q)\left(\frac{1}{R^N}\int\limits_{A_{R,2R}} [u(x) - m(R) + k(R)m(R)]^q \, dx\right)^{\frac{1}{q}}, \quad (3.31)$$

where $k(R) = BG(R)$ *with* $B \geq \bar{B}(q)$.

Proof. Test (3.1) with $\varphi = v^t\eta^s$, $t > 0$, $s \geq m$, where v and η are the same as in the proof of Theorem 3.1. Using conditions (a), (b), (f) we obtain

$$\int\limits_{B_{2r}(y)} v^{t+1-2m}|\nabla v|^m \eta^s \, dx$$
$$\leq C_{3.32}(t,s)\Bigg(\int\limits_{B_{2r}(y)} v^{t-2m+2}|\nabla v|^{m-1}|\nabla\eta|\eta^{s-1}\, dx$$
$$+ \int\limits_{B_{2r}(y)} (g_2 + h_3)u^{m-1}v^t\eta^s \, dx + \int\limits_{B_{2r}(y)} h_2 v^{t-2m+2}|\nabla v|^{m-1}\eta^s \, dx$$
$$+ \int\limits_{B_{2r}(y)} h_1 v^t u^{m-1}\eta^{s-1}|\nabla\eta| \, dx\Bigg). \quad (3.32)$$

By (3.14) and the Young inequality (3.32) implies that

$$\int\limits_{B_{2r}(y)} v^{t+1-2m}|\nabla v|^m \eta^s \, dx$$
$$\leq C_{3.33}(t,s)\Bigg(\frac{1}{r^m}\int\limits_{B_{2r}(y)} v^{t-m+1}\eta^{s-m}\, dx$$
$$+ \frac{1}{k(R)^m}\int\limits_{B_{2r}(y)} h_1^{\frac{m}{m-1}} v^{t-m+1}\eta^s \, dx$$
$$+ \frac{1}{k(R)^{m-1}}\int\limits_{B_{2r}(y)} (g_2 + h_2^m + h_3)v^{t-m+1}\eta^s \, dx\Bigg). \quad (3.33)$$

For $r < \frac{R}{16}$, $y \in A_{\frac{R}{2},\frac{5R}{2}}$, we have

$$G(R) \geq \int_{2r}^{4r} \left(\frac{1}{\rho^{N-m}} \int_{B_\rho(y)} (g_2 + h_3)\, dx \right)^{\frac{1}{m-1}} \frac{d\rho}{\rho}$$
$$\geq C \left(\frac{1}{r^{N-m}} \int_{B_{2r}(y)} (g_2 + h_3)\, dx \right)^{\frac{1}{m-1}},$$

$$G(R) \geq \int_{2r}^{4r} \left(\frac{1}{\rho^{N-m}} \int_{B_\rho(y)} h_1^{\frac{m}{m-1}}\, dx \right)^{\frac{1}{m-1}} \frac{d\rho}{\rho}$$
$$\geq C \left(\frac{1}{r^{N-m}} \int_{B_{2r}(y)} h_1^{\frac{m}{m-1}}\, dx \right)^{\frac{1}{m-1}}. \quad (3.34)$$

Therefore it follows from the definition of $k(R)$ that

$$\frac{1}{[k(R)]^m} \int_{B_{2r}(y)} h_1^{\frac{m}{m-1}}\, dx + \frac{1}{[k(R)]^{m-1}} \int_{B_{2r}(y)} (g_2 + h_3)\eta^m\, dx \leq C r^{N-m}. \quad (3.35)$$

Now (3.33) with $t = m - 1$ and $s = m$ together with (3.35) imply that

$$\int_{B_{2r}(y)} |\nabla \log v|^m\, dx \leq C r^{N-m} \quad (3.36)$$

for $r < \frac{R}{16}$, $y \in A_{\frac{R}{2},\frac{5R}{2}}$.

It follows from (3.36) and the John–Nirenberg inequality that there exist $\gamma_*, \mathcal{C}_* > 0$ such that, for $0 < \gamma < \gamma_*$,

$$\left(\frac{1}{R^N} \int_{A_{\frac{R}{2},\frac{5R}{2}}} v^\gamma\, dx \right)^{\frac{1}{\gamma}} \leq \mathcal{C}_* \left(\frac{1}{R^N} \int_{A_{\frac{R}{2},\frac{5R}{2}}} v^{-\gamma}\, dx \right)^{-\frac{1}{\gamma}}. \quad (3.37)$$

Them the last inequality and Theorem 3.1, with $\gamma = \gamma_*$, imply that

$$\inf_{A_{R,2R}} [u(x) - m(R) + k(R) m(R)] \geq C \left(\frac{1}{R^N} \int_{A_{\frac{R}{2},\frac{5R}{2}}} v^{-\gamma_*}\, dx \right)^{-\frac{1}{\gamma_*}}. \quad (3.38)$$

with $k(R) = BG(R) + R^{-1},\ B \geq B(\gamma_*)$.

Now we need to prove inequality (3.31) for $q > \gamma_*$.

In the same way as in the proof of Proposition 1.2 one can verify that for any $\varepsilon > 0$ and any $\varphi \in W_0^{1,m}(A_{\frac{R}{2},\frac{5R}{2}})$

$$\frac{1}{[k(R)]^m}\int\limits_{\mathbb{R}^N} h_1^{\frac{m}{m-1}}|\varphi|^m\,dx+\frac{1}{[k(R)]^{m-1}}\int\limits_{\mathbb{R}^N}(g_2+h_3)|\varphi|^m\,dx\leq\varepsilon\int\limits_{\mathbb{R}^N}|\nabla\varphi|^m\,dx \tag{3.39}$$

with $k(R) = BG(R) + R^{-1}$ and $B > \bar{B}_*(\varepsilon)$.

From (3.33) with $r = \frac{R}{4}$ using (3.39) we obtain, for $y \in A_{R,2R}$,

$$\int\limits_{B_{\frac{R}{2}}} v^{t+1-2m}|\nabla v|^m\eta^s\,dx \leq C(t,s)\frac{1}{R^m}\int\limits_{B_{\frac{R}{2}}} v^{t+1-m}\eta^{s-m}\,dx \tag{3.40}$$

if ε in (3.39) is fixed by the equality

$$\varepsilon\Big(\frac{t+1-m}{m}\Big)^m C_{3.33} = \frac{1}{2}. \tag{3.41}$$

From (3.40) by the Sobolev inequality we have

$$\left(\frac{1}{R^N}\int\limits_{B_{\frac{R}{2}}(y)}\left[v^{t+1-m}\eta^s\right]^{\frac{m^*}{m}}\,dx\right)^{\frac{m}{m^*}} \leq C(t,s)\frac{1}{R^N}\int\limits_{B_{\frac{R}{2}}(y)} v^{t+1-m}\eta^{s-m}\,dx. \tag{3.42}$$

Define the finite sequences (t_j) and (s_j) by

$$t_j+1-m = (t_1+1-m)\Big(\frac{m^*}{m}\Big)^{j-1}, \qquad t_J+1-m = -q\frac{m^*}{m},$$
$$\frac{m}{m^*}\gamma_* < t_1+1-m \leq \gamma_*, \qquad s_j+N = N\Big(\frac{m^*}{m}\Big)^{j-1}.$$

The finite sequence (ε_j), $j = 1, 2, \dots, J$ is defined according to equality (3.41). As a result we obtain the following inequality

$$\left(\frac{1}{R^N}\int\limits_{B_{\frac{R}{2}}(y)} v^{-q}\eta^{s_J}\,dx\right)^{\frac{1}{q}} \leq C(q)\left(\frac{1}{R^N}\int\limits_{B_{\frac{R}{2}}(y)} v^{-\gamma_*}\eta^m\,dx\right)^{\frac{1}{\gamma_*}} \tag{3.43}$$

if $B > B^*(q) = \max\{B_*(\varepsilon_1), \dots, B_*(\varepsilon_J)\}$. Choosing a finite cover of the annulus $A_{R,2R}$ by the balls $B_{\frac{R}{4}}(y_1), \dots, B_{\frac{R}{4}}(y_I)$ we deduce from (3.43)

$$\left(\frac{1}{R^N}\int\limits_{A_{R,2R}}[u(x)-m(R)+k(R)m(R)]^q\,dx\right)^{\frac{1}{q}}$$

$$\leq C(q)\left(\frac{1}{R^N}\int\limits_{A_{\frac{R}{2},\frac{5R}{2}}}v(x)^{-\gamma_*}\,dx\right)^{\frac{1}{\gamma_*}} \tag{3.44}$$

if $k(R)=BG(R)$ with $B\geq\tilde{B}=B(\gamma_*)+B^*(q)$. Now (3.38) and (3.44) imply the assertion of the theorem. □

Lemma 3.1. *Let the conditions of Theorem* 3.1 *be satisfied. Then there exist constants* $\tilde{B}$ *and* K *such that the inequality*

$$\inf_{A_{R,2R}}[u(x)-m(R)+k(R)m(R)]\geq K\left(\frac{1}{R^{N-m}}\int\limits_{A_{\frac{5R}{4},\frac{7R}{4}}}f(x)\,dx\right)^{\frac{1}{m-1}} \tag{3.45}$$

holds with $k(R)=BG(R)$ *if* $B\geq\tilde{B}$.

Proof. Test (3.1) with $\varphi=\eta^m$, where η is the same as in the proof of Theorem 1.1. Let $y\in A_{\frac{5R}{4},\frac{7R}{4}}$, $r=\frac{R}{8}$. From condition (a) by the Hölder inequality we obtain

$$\int\limits_{B_{\frac{R}{4}}(y)}f\eta^m\,dx\leq\frac{C}{R}\int\limits_{B_{\frac{R}{4}}(y)}(|\nabla u|^{m-1}+h_1u^{m-1})\eta^{m-1}\,dx$$

$$+\int\limits_{B_{\frac{R}{4}}(y)}(g_2+h_3)u^{m-1}\eta^m\,dx+C\int\limits_{B_{\frac{R}{4}}(y)}h_2|\nabla u|^{m-1}\eta^m\,dx$$

$$\leq\frac{C}{R}\left(\int\limits_{B_{\frac{R}{4}}(y)}v^{\alpha-2m}|\nabla v|^m\eta^m\,dx\right)^{\frac{m-1}{m}}\cdot\left(\int\limits_{B_{\frac{R}{4}}(y)}v^{-\alpha(m-1)}\,dx\right)^{\frac{1}{m}}$$

$$+\frac{C}{Rk(R)^{m-1}}\left(\int\limits_{B_{\frac{R}{4}}(y)}h_1^{\frac{m}{m-1}}v^{\alpha-m}\eta^m\,dx\right)^{\frac{m-1}{m}}\cdot\left(\int\limits_{B_{\frac{R}{4}}(y)}v^{-\alpha(m-1)}\,dx\right)^{\frac{1}{m}}$$

$$\leq\frac{C}{R}\left(\int\limits_{B_{\frac{R}{4}}(y)}v^{\alpha-2m}|\nabla v|^m\eta^m\,dx\right)^{\frac{m-1}{m}}\cdot\left(\int\limits_{B_{\frac{R}{4}}(y)}h_2^m v^{-\alpha(m-1)}\,dx\right)^{\frac{1}{m}}$$

$$+\frac{C_1}{k(R)^{m-1}}\Big(\int\limits_{B_{\frac{R}{4}}(y)}(g_2+h_3)v^{\alpha-m}\eta^m\,dx\Big)^{\frac{m-1}{m}}$$
$$\times\Big(\int\limits_{B_{\frac{R}{4}}(y)}(g_2+h_3)v^{-\alpha(m-1)}\eta^m\,dx\Big)^{\frac{1}{m}} \quad (3.46)$$

where C, C_1 some positive constants and $\alpha > 0$ will be chosen later. Inequalities (3.39),(3.40) imply that

$$\frac{1}{k(R)^{m-1}}\int\limits_{B_{\frac{R}{4}}(y)}(g_2+h_3)v^{\alpha-m}\eta^m\,dx+\frac{1}{k(R)^m}\int\limits_{B_{\frac{R}{4}}(y)}h_1^{\frac{m}{m-1}}v^{\alpha-m}\eta^m\,dx$$
$$\le C(\alpha)\int\limits_{B_{\frac{R}{4}}(y)}v^{\alpha-2m}|\nabla v|^m\eta^m\,dx+\frac{C(\alpha)}{R^m}\int\limits_{B_{\frac{R}{4}}(y)}v^{\alpha-m}\,dx$$
$$\le\frac{C(\alpha)}{R^m}\int\limits_{B_{\frac{R}{4}}(y)}v^{\alpha-m}\,dx, \quad (3.47)$$

$$\frac{1}{k(R)^{m-1}}\int\limits_{B_{\frac{R}{4}}(y)}(g_2+h_3)v^{-\alpha(m-1)}\eta^m\,dx+\int\limits_{B_{\frac{R}{4}}(y)}h_2^m v^{-\alpha(m-1)}\,dx$$
$$\le C(\alpha)\int_{B_{\frac{R}{4}}(y)}v^{-\alpha(m-1)-m}|\nabla v|^m\eta^m\,dx+\frac{C(\alpha)}{R^m}\int\limits_{B_{\frac{R}{4}}(y)}v^{-\alpha(m-1)}\,dx$$
$$\le\frac{C(\alpha)}{R^m}\int\limits_{B_{\frac{R}{4}}(y)}v^{-\alpha(m-1)}\,dx \quad (3.48)$$

if $B \ge B'(\alpha)$ for some $B'(\alpha)$. Hence from (3.46)–(3.48) we obtain

$$\int\limits_{B_{\frac{R}{4}}(y)}f\eta^m\,dx\le\frac{C(\alpha)}{R^m}\Big(\int\limits_{B_{\frac{R}{4}}(y)}v^{-\alpha(m-1)}\,dx\Big)^{\frac{1}{m}}\cdot\Big(\int\limits_{B_{\frac{R}{4}}(y)}v^{\alpha-m}\,dx\Big)^{\frac{m-1}{m}} \quad (3.49)$$

Choose α as follows

$$\alpha=\frac{1}{2}\Big(\Big[\frac{N-m^2}{N-m}\Big]^+ +\frac{N}{N-m}\Big).$$

Since $\alpha(m-1) < \frac{(m-1)N}{N-m}$ and $m-\alpha < \frac{(m-1)N}{N-m}$ we can apply (3.31) to both integrals in the right hand side of (3.49). Therefore

$$\int\limits_{B_{\frac{R}{4}}(y)} f\eta^m\,dx \le CR^{N-m} \inf_{A_{R,2R}} [u(x) - m(R) + k(R)m(R)]^{m-1}. \quad (3.50)$$

Covering the annulus $A_{\frac{5R}{4},\frac{7R}{4}}$ by the balls of radius $R/4$ we complete the proof. □

Proof of Theorem 1.4. It follows from Lemma 3.1 and (1.20) that

$$\inf_{x\in A_{R,2R}} [u(x) - m(R) + k(R)m(R)] \ge KF(R). \quad (3.51)$$

Recall that $m(R), k(R)$ are defined by (3.3),(3.4) with a suitable choice of B.

Fix R_0 by the condition that $k(R) < \frac{1}{2}$ for all $R > R_0$. Set $\mu(R) = \inf_{x\in A_{R,2R}} u(x)$. We distinguish the following cases:

i(R) $m(R) = \mu\Big(\dfrac{R}{2}\Big)$,

ii(R) $m(R) = \mu(R),\ \mu(R) < \mu\Big(\dfrac{R}{2}\Big)$,

iii(R) $m(R) = \mu(2R),\ \mu(2R) < \mu\Big(\dfrac{R}{2}\Big),\ \mu(2R) < \mu(R)$.

For the case ii(R) we immediately obtain that

$$\mu(R) \ge \frac{KF(R)}{BG(R)}, \quad (3.52)$$

and hence inequality (1.26) follows.

For the case i(R) we consider inequality (3.51) with $\frac{R}{2}$ in place of R, and the corresponding three cases: i$(\frac{R}{2})$, ii$(\frac{R}{2})$, iii$(\frac{R}{2})$. Now for the case ii$(\frac{R}{2})$ we obtain the following two inequalities

$$\begin{gathered} \mu(R) - \mu\Big(\frac{R}{2}\Big) + k(R)\mu\Big(\frac{R}{2}\Big) \ge KF(R), \\ k\Big(\frac{R}{2}\Big)\mu\Big(\frac{R}{2}\Big) \ge KF\Big(\frac{R}{2}\Big), \end{gathered} \quad (3.53)$$

which imply that

$$\mu(R) \geq K \frac{F\left(\frac{R}{2}\right)}{k\left(\frac{R}{2}\right)}(1-k(R)) + KF(R) \geq \frac{KF\left(\frac{R}{2}\right)}{2BG\left(\frac{R}{2}\right)}, \tag{3.54}$$

and hence inequality (1.26) follows.

For the case iii($\frac{R}{2}$) we have

$$\mu\Big(\frac{R}{2}\Big) > \mu(R).$$

Then this inequality and the first inequality in (3.53) imply that

$$\mu(R) \geq \frac{KF(R)}{k(R)} = \frac{KF(R)}{BG(R)}, \tag{3.55}$$

and (1.26) follows.

For the case i($\frac{R}{2}$) we proceed to the three cases arising for inequality (3.51) with $\frac{R}{4}$ in place of R. After repeating the above procedure two options are possible. First is that the cases i(R),i($\frac{R}{2}$), ..., i($2^{-j}R$) are realized for some j such that $2^{-(j+1)}R \geq R_0$, and then one of the two cases ii($2^{-(j+1)}R$) or iii($2^{-(j+1)}R$) is realized. The second option is that the cases i($2^{-j}R$), $j = 1, \dots, J(R_0)$ are realized for $J(R_0)$ determined by the condition $1 \leq 2^{-J(R_0)}R_0 < 2$.

For the first option from (3.51), (3.52), (3.54), (3.55) we obtain

$$\begin{gathered}\mu(2^{-i}R) - \mu(2^{-(i+1)}R)\left[1 - k(2^{-i}R)\right] \geq KF(2^{-i}R),\\ \mu(2^{-j}R) \geq \frac{K}{2B}\min\left\{\frac{F(2^{-j}R)}{G(2^{-j}R)}, \frac{F(2^{-(j+1)}R)}{G(2^{-(j+1)}R)}\right\}\end{gathered} \tag{3.56}$$

for $i = 0, 1, \dots, j$. This implies that

$$\mu(R) \geq \mu(2^{-j}R)\prod_{i=0}^{j-1}\big(1 - k(2^{-i}R)\big) + K\sum_{i=0}^{j-1}F(2^{-i}R)\prod_{l=0}^{i-1}\big(1 - k(2^{-l}R)\big). \tag{3.57}$$

taking into account condition (g") and the choice of $k(R)$ we deduce that there exists a constant $C > 0$ such that

$$\prod_{i=0}^{J(R_0)}\big(1 - k(2^{-i}R)\big) \geq C. \tag{3.58}$$

Therefore we conclude that

$$\mu(R) \geq C\Bigg\{\sum_{i=0}^{j-1}F(2^{-i}R) + \min\left[\frac{F(2^{-j}R)}{G(2^{-j}R)}, \frac{F(2^{-(j+1)}R)}{G(2^{-(j+1)}R)}\right]\Bigg\}$$

$$\geq C \min_{0\leq j\leq J(R_0)} \left\{ \sum_{i=0}^{j-2} F(2^{-i}R) + \frac{F(2^{-j}R)}{G(2^{-j}R)} \right\}. \quad (3.59)$$

Clearly the same estimate holds if the cases i$(2^{-j}R)$ are realized for $j = 1, \dots, J(R_0)$.

Inequality (3.59) implies estimate (1.26) for the case i(R). Consideration of the case iii(R) is analogous. This completes the proof of Theorem 1.4. □

4. Behaviour at infinity of super-solutions to equation (1.1)

Proof of Theorem 1.5. For a contradiction, assume that (1.28) does not hold. Let u be a nonnegative super-solution to (1.28) in $B^c_{R_*}$ such that

$$\liminf_{|x|\to\infty} u(x) = \infty. \quad (4.1)$$

Let Q be an arbitrary positive number and choose $R(Q)$ such that

$$u(x) \geq Q \quad \text{for} \quad |x| \geq \frac{R(Q)}{2} > R_*. \quad (4.2)$$

Let v be the solution to the auxiliary problem

$$- \quad \operatorname{div} \mathbf{A}(x, u, \nabla v) + a_0(x, v, \nabla v) + g'(x, v) = 0, \quad x \in B_{R^*}, \quad (4.3)$$

$$v(x) = \frac{Q}{2} \quad \text{for} \quad |x| = R^*, \quad (4.4)$$

with

$$g'(x, u) = g_2(x)|v|^{m-2}v. \quad (4.5)$$

The existence of the solution to (4.3),(4.4) and the estimate

$$0 < C_1 Q \leq v(x) \leq C_2 Q, \quad x \in B_{R^*}, \quad (4.6)$$

with some positive C_1, C_2, follow from Lemma 2.1 and Theorems 1.1, 1.2.

Define the function $\xi : \mathbb{R}^N \to [0, 1]$ by

$$\xi(x) = \min\left\{ \left[\frac{|x|}{R_*} - 1\right]_+, 1 \right\}. \quad (4.7)$$

In equation (4.3) for v and inequality (1.1) for u (as u is the super-solution) use

$$\varphi(x) = [v(x) - u(x)]^+ \xi^m(x)$$

as a test function. Subtracting one from the other we obtain

$$I_0 := \int\limits_{A_{R_*,R^*}} [\mathbf{A}(x,v,\nabla v) - \mathbf{A}(x,v,\nabla u)]\nabla(v-u)^+\xi^m\,dx$$
$$= \int\limits_{A_{R_*,R^*}} [\mathbf{A}(x,v,\nabla u) - \mathbf{A}(x,u,\nabla u)]\nabla(v-u)^+\xi^m\,dx$$
$$- \int\limits_{A_{R_*,R^*}} [\mathbf{A}(x,v,\nabla v) - \mathbf{A}(x,u,\nabla u)](v-u)^+\nabla\xi^m\,dx$$
$$+ \int\limits_{A_{R_*,R^*}} [a_0(x,u,\nabla u) - a_0(x,v,\nabla v)](v-u)^+\xi^m\,dx$$
$$+ \int\limits_{A_{R_*,R^*}} [g'(x,u) - a_0'(x,v)](v-u)^+\xi^m\,dx = \sum_{i=1}^{4} I_i. \quad (4.8)$$

First, for I_2 and I_3 we have

$$I_2 \le \gamma \int\limits_{A_{R_*,R^*}} (|\nabla v|^{m-1} + |\nabla u|^{m-1} + h_1 v^{m-1})(v-u)^+\xi^{m-1}|\nabla\xi|\,dx$$
$$\le \gamma \int\limits_{A_{R_*,R^*}} |\nabla v|^m\xi^m\,dx + \gamma \int\limits_{A_{R_*,R^*}} |\nabla u|^m\xi^m\,dx$$
$$+ \gamma \int\limits_{A_{R_*,R^*}} h_1^{\frac{m}{m-1}} v^m\xi^m\,dx + \gamma \int_{A_{R_*,R^*}} [(v-u)^+]^m|\nabla\xi|^m dx; \quad (4.9)$$

$$I_3 \le \gamma \int\limits_{A_{R_*,R^*}} [h_2|\nabla v|^{m-1} + h_2|\nabla u|^{m-1} + h_3 v^{m-1}](v-u)^+\xi^m\,dx$$
$$\le \gamma \int\limits_{A_{R_*,R^*}} |\nabla v|^m\xi^m\,dx + \gamma \int\limits_{A_{R_*,R^*}} |\nabla u|^m\xi^m\,dx$$
$$+ \gamma \int\limits_{A_{R_*,R^*}} h_2^m[(v-u)^+]^m\xi^m\,dx + \gamma \int\limits_{A_{R_*,R^*}} h_3 v^{m-1}(v-u)^+\xi^m\,dx. \quad (4.10)$$

Note that $I_4 \le 0$.

Now we distinguish two cases.

Case 1. $m \geq 2$. For I_1 by the Young inequality we have

$$I_1 \leq \gamma \int\limits_{A_{R_*,R^*}} (|\nabla u|^{m-1} + h_1 v^{m-1} + h_1 u^{m-1})|\nabla(v-u)^+|\xi^m \, dx$$
$$\leq \frac{1}{8} \int\limits_{A_{R_*,R^*}} |\nabla(v-u)^+|^m \xi^m \, dx + \gamma \int\limits_{A_{R_*,R^*}} |\nabla u|^m \xi^m \, dx$$
$$+ \gamma \int\limits_{A_{R_*,R^*}} h_1^{\frac{m}{m-1}} v^{m-1} \xi^m \, dx. \quad (4.11)$$

From (4.8), (1.3), (2.1) and (1.2) by condition (a) we obtain that

$$\int\limits_{A_{R_*,R^*}} |\nabla[v-u]^+|^m \xi^m \, dx \leq \gamma \int\limits_{A_{R_*,R^*}} |\nabla v|^m \xi^m \, dx + \int\limits_{A_{R_*,R^*}} |\nabla u|^m \xi^m \, dx$$
$$+ \gamma Q^m \Bigg(\int\limits_{A_{R_*,R^*}} h_1^{\frac{m}{m-1}} \, dx + \int\limits_{A_{R_*,R^*}} h_2^m \, dx + \int\limits_{A_{R_*,R^*}} h_3 \, dx + R_*^{N-m} \Bigg)$$
$$(4.12)$$

To estimate the first term in the right hand side of (2.2) we use $(v - Q/2)\xi^m$ as a test functions. It follows that

$$\int\limits_{A_{R_*,R^*}} |\nabla v|^m \xi^m \, dx + \int\limits_{A_{R_*,R^*}} g'(x,v)(v - Q/2)\xi^m \, dx$$
$$\leq \int\limits_{A_{R_*,R^*}} (|\nabla v|^{m-1} + h_1 v^{m-1})|v - Q/2||\nabla \xi^m| \, dx$$
$$+ \int\limits_{A_{R_*,R^*}} (h_2 |\nabla v|^{m-1} + h_3 v^{m-1})|v - Q/2|\xi^m \, dx. \quad (4.13)$$

Therefore by the Young inequality we deduce

$$\int\limits_{A_{R_*,R^*}} |\nabla v|^m \xi^m \, dx \leq \gamma Q \Bigg[\int\limits_{A_{R_*,R^*}} \big(h_1^{\frac{m}{m-1}} + h_2^m + h_3 + g_2\big) \, dx + R_*^{N-m} \Bigg].$$
$$(4.14)$$

$$\int\limits_{A_{R_*,R^*}} |\nabla[v-u]^+|^m \xi^m \, dx$$
$$\leq \gamma\Bigg(\int\limits_{A_{R_*,2R_*}} |\nabla u|^m \, dx + Q^m R_*^{N-m} \Bigg)$$
$$+ \gamma Q^m \int\limits_{A_{R_*,R^*}} \big(h_1^{\frac{m}{m-1}} + h_2^m + h_3 + g_2 \big) \, dx. \quad (4.15)$$

Then the Hardy inequality yields

$$\int\limits_{A_{R_*,R^*}} \frac{1}{|x|^m} ([v-u]^+)^m \xi^m \, dx$$
$$\leq C_H \gamma \Bigg(\int\limits_{A_{R_*,2R_*}} |\nabla u|^m dx + Q^m R_*^{N-m} \Bigg)$$
$$+ C_H \gamma Q^m \int\limits_{A_{R_*,R^*}} \big(h_1^{\frac{m}{m-1}} + h_2^m + h_3 + g_2 \big) \, dx. \quad (4.16)$$

Using smallness of $\tilde{\mathcal{K}}(h_1^{\frac{m}{m-1}})$ and $\mathcal{K}(h_2^m + h_3 + g_2)$ this together with (4.6) implies that

$$Q^m R^{N-m} \leq C_{4.17} \Bigg(\frac{1}{R^m} \int\limits_{A_{R,2R}} u^m \, dx + \int\limits_{A_{R_*,2R_*}} |\nabla u|^m \, dx + Q^m R_*^{N-m} \Bigg) \quad (4.17)$$

for arbitrary $R \in [2R_*, \frac{R^*}{2}]$. Define $\bar{R}$ by $\bar{R}^{N-m} = 2C_{4.17} R_*^{N-m}$. We can assume that R^* is large enough so that $R^* > 2\bar{R}$. From (4.17) we have

$$\frac{1}{2} Q^m \leq \frac{1}{R_*^{N-m} \bar{R}^m} \int\limits_{A_{\bar{R},2\bar{R}}} u^m \, dx + \frac{1}{R_*^{N-m}} \int\limits_{A_{R_*,2R_*}} |\nabla u|^m \, dx. \quad (4.18)$$

As $\bar{R}$ depends on the known parameters only, the right hand side of (4.18) is a constant, while the left hand side is an arbitrary number, which is a contradiction.

Case 2. $1 < m < 2$. For I_1 we have

$$I_1 \leq \gamma \int\limits_{A_{R_*,2R_*}} |\nabla u|^{m-1} |\nabla (v-u)^+| \xi^m \, dx$$

$$\leq \gamma \int\limits_{A_{R_*,2R_*}} (|\nabla u| + |\nabla v|)^{\frac{m}{2}-1}(|\nabla u| + |\nabla v|)^{\frac{m}{2}} |\nabla (v-u)^+| \xi^m \, dx$$

$$\leq \frac{1}{8} \int\limits_{A_{R_*,2R_*}} (|\nabla u| + |\nabla v|)^{m-2} |(v-u)^+|^2 \xi^m \, dx$$

$$+ \gamma \int\limits_{A_{R_*,2R_*}} (|\nabla u|^m + |\nabla v|^m) \xi^m \, dx. \quad (4.19)$$

Next,

$$\int\limits_{A_{R_*,2R_*}} (|\nabla u| + |\nabla v|)^{m-2} |(v-u)^+|^2 \xi^m \, dx \leq \gamma \int\limits_{A_{R_*,2R_*}} |\nabla u|^m \xi^m \, dx$$

$$+ \gamma Q^m \Bigg(\int\limits_{A_{R_*,2R_*}} (h_2^m + h_3 + g_2) \, dx + R_*^{N-m} \Bigg). \quad (4.20)$$

By the Hardy inequality

$$\int\limits_{A_{R_*,2R_*}} \frac{1}{|x|^m} [(v-u)^+]^m \xi^m \, dx$$

$$\leq \gamma \int\limits_{A_{R_*,2R_*}} |\nabla (v-u)^+|^m \xi^m \, dx + \gamma Q^m R_*^{N-m}. \quad (4.21)$$

Furthermore,

$$\int\limits_{A_{R_*,2R_*}} |\nabla (v-u)^+|^m \xi^m \, dx$$

$$\leq \gamma \int\limits_{A_{R_*,2R_*}} |\nabla (v-u)^+| (|\nabla u| + |\nabla v|)^{m-1} \xi^m \, dx$$

$$= \gamma \int\limits_{A_{R_*,2R_*}} |\nabla (v-u)^+| (|\nabla u| + |\nabla v|)^{m/2-1} (|\nabla u| + |\nabla v|)^{m/2} \xi^m \, dx$$

$$\leq \gamma \int\limits_{A_{R_*,2R_*}} |\nabla (v-u)^+|^2 (|\nabla u| + |\nabla v|)^{m-2} \xi^m \, dx$$

$$+ \int\limits_{A_{R_*,2R_*}} (|\nabla u|^m + |\nabla v|^m) \xi^m \, dx. \quad (4.22)$$

This together with (2.4) implies that

$$\int\limits_{A_{R_*,2R_*}} \frac{1}{|x|^m}[(v-u)^+]^m \xi^m\, dx \le \gamma \int\limits_{A_{R_*,2R_*}} |\nabla u|^m \xi^m\, dx + \gamma Q^m \Bigg[\int\limits_{A_{R_*,2R_*}} (h_2^m + h_3 + g_2)\xi^m\, dx + R_*^{N-m} \Bigg]. \quad (4.23)$$

This leads to a contradiction in the same way as in Case 1. □

5. Application to a non-existence result

Proof of Theorem 1.6. For a contradiction assume that there exists a nontrivial nonnegative super-solution to equation (1.29) for some $R_* < \infty$. By the weak Harnack inequality we conclude that for any $R > R_*$

$$\inf_{A_{R_*,R}} u(x) > 0.$$

Let $R > 2R^*$. Define a cut-off function $\xi : \mathbb{R}^N \to [0,1]$ such that

$$\xi = 1 \text{ for } x \in A_{R,2R}, \qquad \xi = 0 \text{ for } x \notin A_{R/2,4R}, \qquad |\nabla \xi| \le 4.$$

Testing inequality (1.29) (for super-solutions) by

$$\varphi(x) = u^{1-m}(x)\xi^m(x),$$

we obtain

$$\int\limits_{B_{R^*}^c} \left(u^{-m}|\nabla u|^m + f(x)u^{p+1-m}\right)\xi^m\, dx \le C\Bigg(R^{N-m} + \int\limits_{B_{R^*}^c} (h_1^{\frac{m}{m-1}} + h_2^m + h_3 + g_2)\xi^m\, dx\Bigg). \quad (5.1)$$

Analogous to (3.35) we obtain for the integral in the right hand side of (5.1)

$$\int\limits_{B_{R^*}^c} (h_1^{\frac{m}{m-1}} + h_2^m + h_3 + g_2)\xi^m\, dx \le CR^{N-m}. \quad (5.2)$$

The weak Harnack inequality (1.20) implies that

$$\int\limits_{B_{R^*}^c} f(x)u^{p+1-m}\xi^m\, dx \ge CR^N \min_{x\in A_{R,2R}} f(x)\Big(\min_{x\in A_{R,2R}} u(x)\Big)^{p+1-m}. \quad (5.3)$$

Therefore from (5.1)–(5.3) we obtain

$$\min_{x\in A_{R,2R}} u(x) \geq C_{5.4}\Big(R^m \min_{x\in A_{R,2R}} f(x)\Big)^{\frac{1}{m-p-1}}. \tag{5.4}$$

It follows that u is a super-solution to the equation

$$-\operatorname{div}\mathbf{A}(x,u,\nabla u) + a_0(x,\nabla u,u) = \bar{f}(x), \quad x\in\mathbb{R}^N\setminus B_{R_*}, \tag{5.5}$$

where $\bar{f}$ is defined on the annulus $A(j) = A_{2^j,2^{j+1}}$, $2^j \geq R_*$, by

$$\bar{f}(x) = \Big\{C_{5.4}^p 2^{jmp}\Big(R^m \min_{x\in A(j)} f(x)\Big)^{m-1}\Big\}^{\frac{1}{m-p-1}}. \tag{5.6}$$

By Theorem 1.4 we conclude that

$$\inf_{x\in A(j)} u(x) \geq C\min\{\bar{H}^-(j,j_0),\bar{H}^+(j)\} \tag{5.7}$$

with integer j_0 satisfying $\frac{1}{2}\max(R_0,R_*) \leq 2^{j_0} < \max(R_0,R_*)$. Here $\bar{H}^-$ and $\bar{H}^+$ are defined by

$$\begin{aligned}\bar{H}^-(j,j_0) &= \min_{j_0\leq i<j}\Big\{\sum_{k=i+1}^{j}\bar{F}(2^k) + \frac{\bar{F}(2^j)}{G(2^j)}\Big\},\\ \bar{H}^+(j) &= \min_{i>j}\Big\{\sum_{k=j}^{i-1}\Big\{\bar{F}(2^k) + \frac{\bar{F}(2^j)}{G(2^j)}\Big\}\Big\},\end{aligned} \tag{5.8}$$

and $\bar{F}(2^k)$ was introduced in (1.31).

Note that condition (1.30) implies that

$$\lim_{j\to\infty}\bar{H}^-(j,j_0) = \infty, \qquad \lim_{j\to\infty}\bar{H}^+(j) = \infty. \tag{5.9}$$

Therefore (5.7) contradicts to (1.28), which completes the proof. □

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