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Characteristic subgroups of the infinitely iterated wreath product of elementary Abelian groups

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Abstract. We consider the infinitely iterated wreath product of elementary Abelian groups of rank n. The main result of the paper is the statement of necessary and sufficient conditions, according to which subgroups of the mentioned wreath product are characteristic.

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1. Introduction

In work [1] which is devoted to the study of the Sylow structure of a finitary symmetric group, the construction of a generalized wreath product of cyclic groups of prime order naturally arises. Such generalizations are also can be found in the study of other inductive limits of finite symmetric groups (see [3,6]). In work [6], the Sylow p-subgroup (we denote it by U_p^∞) of the inductive limit of symmetric groups of degrees p^n ($n = 1, 2, \ldots$) with strictly diagonal embeddings was distinguished. The group U_p^∞ can be considered as some modification of an infinitely iterated wreath product of cyclic groups of prime order p. Subsequently in [4], using the method of L. A. Kaloujnine [2], the structure of U_p^∞ (a class of the so-called “parallelotopic subgroups”, normal and characteristic subgroups) was investigated. On the other hand, the results of L. A. Kaloujnine were generalized in [7] by V. I. Sushchanskii to the case of a finite wreath product of elementary Abelian groups.

This article is the continuation of paper [5], where the structure of the infinitely iterated (the so-called *left-truncated*) wreath product $U_{p,n}^\infty$

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of elementary Abelian groups of rank n was studied. In article [5], the concept of the *weighted degree* of a polynomial was modified, and the class of *parallelotopic subgroups* (and *homogeneous parallelotopic subgroups*) was distinguished. In addition, the criterion for normal subgroups and the necessary condition for characteristic subgroups were proved.

In Sections 2 and 3 of the present paper, we introduce the necessary definitions and notations and present the main results of [5]. The main result is the following criterion for characteristic subgroups of the group $U_{p,n}^{\infty}$.

Theorem 1.1. *If $p \neq 2$, then a subgroup of $U_{p,n}^{\infty}$ is characteristic if and only if it is a normal and homogeneous parallelotopic subgroup.*

2. Wreath product

Let $\mathbb{F}_p^n$ be an elementary Abelian p-group of rank n considered as an additive group of the n-dimensional vector space over a finite field $\mathbb{F}_p$ ($|\mathbb{F}_p| = p$, p is a prime, and $p \neq 2$). One can define the right group action of $\mathbb{F}_p^n$ on itself. In [5], we examined the group $U_{p,n}^{\infty}$. This group can be considered as a *left-truncated infinitely iterated wreath product* of copies of the group $\mathbb{F}_p^n$. Elements of $U_{p,n}^{\infty}$ are infinite almost zero sequences (or *a table*)

$$u = [\overline{a}_1(\overline{v}_2, \ldots, \overline{v}_k), \ldots, \overline{a}_m(\overline{v}_{m+1}, \ldots, \overline{v}_k), \overline{0}, \overline{0}, \ldots], \quad k > m \ (k, m \in \mathbb{N}), \tag{2.1}$$

where $\overline{a}_j(\overline{v}_{j+1}, \ldots, \overline{v}_k)$ is a map from $\mathbb{F}_p^n \times \ldots \times \mathbb{F}_p^n$ ($k-j$ factors) into $\mathbb{F}_p^n$, $\overline{0}$ is a zero-vector. The action of $U_{p,n}^{\infty}$ on the Cartesian product $\prod_{i=1}^{\infty} \mathbb{F}_p^n$ is defined by the rule

$$(\overline{v}_1, \ldots, \overline{v}_m, \ldots)^u = (\overline{v}_1 + \overline{a}_1(\overline{v}_2, \ldots, \overline{v}_k), \ldots, \overline{v}_m + \overline{a}_m(\overline{v}_{m+1}, \ldots, \overline{v}_k), \ldots), \tag{2.2}$$

where $(\overline{v}_1, \ldots, \overline{v}_m, \ldots) \in \prod_{i=1}^{\infty} \mathbb{F}_p^n$, and u is Table (2.1).

Let $\overline{v}_i = (x_{1i}, \ldots, x_{ni})^T$ denote a column vector in $\mathbb{F}_p^n$; and let $X_{i,j} = (\overline{v}_i, \ldots, \overline{v}_j)$ be a sequence of column vectors (in fact, $X_{i,j} = (x_{st})_{n \times (j-i+1)}$ is a matrix over $\mathbb{F}_p$, where $s \in \{1, \ldots, n\}$ and $t \in \{i, \ldots, j\}$). Each map $\overline{a}_j$, $j \in \mathbb{N}$ in Table (2.1) can be replaced by an array of *reduced* (with degree of at most $p-1$ in each variable) polynomials over $\mathbb{F}_p$. As a result, Table (2.1) can be expressed as

$$u = \begin{bmatrix} a_{11}(X_{2,k}), & \ldots, & a_{1m}(X_{m+1,k}), & 0, & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n1}(X_{2,k}), & \ldots, & a_{nm}(X_{m+1,k}), & 0, & \ldots \end{bmatrix}, \quad k > m \ (k, m \in \mathbb{N}), \tag{2.3}$$

where $a_{ij}(X_{j+1,k})$ is a reduced polynomial ($i \in \{1, \dots, n\}$, $j \in \mathbb{N}$) which is called the (i, j)-*coordinate* of Table (2.3). The *depth* of the table is the maximum number of its non-zero columns, and the *rank* of the table is the maximum second index of variables that appear in its expression.

We denote Table (2.3) by $[a_{ij}(X_{j+1,k})]_{j=1}^{\infty}$. Let $f(X_{s,t}^{u})$ be a polynomial which is obtained from the polynomial

$$f\begin{pmatrix} x_{1s} + a_{1s}(X_{s+1,k}) & \dots & x_{1t} + a_{1t}(X_{t+1,k}) \\ \dots & \dots & \dots \\ x_{ns} + a_{ns}(X_{s+1,k}) & \dots & x_{nt} + a_{nt}(X_{t+1,k}) \end{pmatrix}$$

after its reduction. Then the product of $u = [a_{ij}(X_{j+1,k})]_{j=1}^{\infty}$ and $v = [b_{ij}(X_{j+1,k})]_{j=1}^{\infty}$ is the table $[a_{ij}(X_{j+1,k}) + b_{ij}(X_{j+1,k}^{u})]_{j=1}^{\infty}$.

3. Necessary condition

Let $[u]_{ij}$ denote the (i, j)-coordinate of a table u, and let $\mathtt{co}_{ij}(f(X))$ be a table, whose (i, j)-coordinate is $f(X)$, while all other coordinates are equal to zero.

Difinition 3.1. *The weighted degree of the monomial* $x_{11}^{i_{11}} x_{21}^{i_{21}} \cdots x_{n1}^{i_{n1}} \cdots \times x_{1k}^{i_{1k}} x_{2k}^{i_{2k}} \cdots x_{nk}^{i_{nk}}$ *is the positive integer*

$$h = \sum_{t=1}^{k} \left[d^{-t} \sum_{s=1}^{n} i_{st} \right] + 1,$$

where $d = n(p-1) + 1$. *The weighted degree* $h[f]$ *of the polynomial* f *is the maximum weighted degree of its monomials. The highest term of a polynomial is the sum of those monomials, whose weighted degrees are equal to the weighted degree of the polynomial. Let also* $h[0] = 0$.

Consequently, $h[a_{ij}(X_{j+1,k})] \in \{0\} \cup [1; 1 + d^{-j})$ for all polynomials $a_{ij}(X_{j+1,k})$ in the jth column of Table (2.3).

Lemma 3.1 ([5]). 1. *If* $u \in U_{p,n}^{\infty}$, *then*

1) $[u^{-1}]_{ij} = -a_{ij}(X_{j+1,k}^{u^{-1}})$;
2) $[vuv^{-1}]_{ij} = a_{ij}(X_{j+1,k}^{v}) + b_{ij}(X_{j+1,k}) - b_{ij}(X_{j+1,k}^{vuv^{-1}})$;
3) $[(uv)]_{ij} = a_{ij}(X_{j+1,k}) - a_{ij}(X_{j+1,k}^{uvu^{-1}}) + b_{ij}(X_{j+1,k}^{u})] - b_{ij}(X_{j+1,k}^{uvu^{-1}v^{-1}})$.

2. *The set* $\{\mathtt{co}_{ij}(a_{ij}(X_{j+1,k}))\}$, *where* $i \in \{1, \dots, n\}$, $j \in \mathbb{N}$ *and* $a_{ij}(X_{j+1,k})$ *is an arbitrary reduced polynomial, is the generating system of the group* $U_{p,n}^{\infty}$.

3. *The set* $\{\mathtt{co}_{ij}(x_{1,j+1}^{i_{1,j+1}}\dots x_{nk}^{i_{nk}})\}$*, where* $i \in \{1,\dots,n\}$*,* $j \in \mathbb{N}$*,* $i_{st} \in \mathbb{F}_p\backslash\{0\}$*,* $s \in \{1,\dots,n\}$*,* $t \in \{j+1,\dots,k\}$*, is the generating system of the group* $U_{p,n}^{\infty}$*.*

4. *If polynomials* $f(X_{1,k})$ *and* $g(X_{1,k})$ *are not equal to zero simultaneously, then*

   1) $h[f(X_{1,k}) + g(X_{1,k})] \leq \max\{h[f(X_{1,k})], g(X_{1,k})\}$,

   2) $h[f(X_{1,k}) \cdot g(X_{1,k})] \leq h[f(X_{1,k})] + h[g(X_{1,k})] - 1$.

Lemma 3.2 ([5]). 1. *If* $f(X_{1,k})$ *is a polynomial* ($k \in \mathbb{N}$) *and* $u \in U_{p,n}^{\infty}$ *then*

$$h[f(X_{1,k}^{u})] = h[f(X_{1,k})].$$

2. *If* $f(X_{1,k})$ *is a polynomial* ($k \in \mathbb{N}$) *and* $u \in U_{p,n}^{\infty}$ *then*

$$h[f(X_{1,k}) - f(X_{1,k}^{u})] < h[f(X_{1,k})].$$

3. *If* $f(X_{1,k})$ *is a polynomial* ($h[f(X_{1,k})] = h > 1$, $k \in \mathbb{N}$)*, then, for any* $m \in \mathbb{N}$ ($m \geq k$)*, there exists a table* $u \in U_{p,n}^{\infty}$ *such that*

$$h[f(X_{1,k}) - f(X_{1,k}^{u})] = h - d^{-m}.$$

4. *If* $f(X_{s,t})$ *is a polynomial and* u *is a table of depth* r*, then*

$$h[f(X_{s,t}) - f(X_{s,t}^{u})] < 1 + d^{1-s} - d^{-r}.$$

5. *If* u *is a table of depth* r *and rank* k*, then, for any* $t \in \mathbb{N}$ ($t > k$)*, there exists a polynomial* $f(X_{s,t})$ ($s \leq r \leq t$) *such that*

$$h[f(X_{s,t}) - f(X_{s,t}^{u})] = 1 + d^{1-s} - d^{-r} - d^{-t}.$$

Given $u \in U_{p,n}^{\infty}$, we denote the weighted degree of a table u by $|u|_{ij}$. The matrix $|u| = (|u|_{ij})_{n\times\infty}$, ($i = 1,\dots,n$, $j \in \mathbb{N}$) is called the *multidegree* of the table u. The set of all multidegrees can be ordered by the rule: $|u| \preceq |v|$ if and only if $|u|_{ij} \leq |v|_{ij}$ for all $i = 1,\dots,n$, $j \in \mathbb{N}$.

Difinition 3.2. *A subgroup* R *of the group* $U_{p,n}^{\infty}$ *is called a parallelotopic subgroup if* $u \in R$ *and* $|v| \preceq |u|$ *yield* $v \in R$*.*

We put a parallelotopic subgroup R in correspondence to the infinite matrix $|R| = (k_{ij}^{\varepsilon})_{n\times\infty}$, ($i \in \{1,\dots,n\}$, $j \in \mathbb{N}$, $\varepsilon \in \{+,-\}$), such that

1) $k_{ij} = \sup_{u\in R} |u|_{ij}$;

2) if R contains such a table u that $|u|_{ij} = k_{ij}$, then $\varepsilon =$"$+$";

3) otherwise, $\varepsilon =$"$-$".

This matrix is called the *indicatrix* of the subgroup R. The *depth* of a parallelotopic subgroup is the maximum number of its non-zero column.

The set $\mathbb{R}_j^\varepsilon$ $(j \in \mathbb{N})$ of elements k^ε, where $k \in \{0\} \cup [1; 1 + d^{-j}]$, $\varepsilon \in \{+, -\}$, can be ordered by the rule:

1) $k^- \preceq k^+$ for all k;

2) $k^\varepsilon \preceq l^\eta$ for all $\varepsilon, \eta \in \{+, -\}$, if $k < l$.

Theorem 3.1 ([5]). *Let R be a parallelotopic subgroup which has depth r and the indicatrix $(k_{ij}^{\pm})_{n\times\infty}$. Then R is a normal subgroup of the group $U_{p,n}^\infty$ if and only if $k_{ij} \geq 1 + d^{-j} - d^{-r}$ $(i = 1, \ldots, n;\ j = 1, \ldots, r)$. Particularly, all proper normal parallelotopic subgroups of $U_{p,n}^\infty$ have finite depth.*

Difinition 3.3. *A parallelotopic subgroup R is called homogeneous if all rows of its indicatrix are identical.*

A homogeneous parallelotopic subgroup R is uniquely determined by a sequence $|R| = (k_j^\varepsilon)$, $k_j \in \{0\} \cup [1; 1 + d^{-j}]$, $j \in \mathbb{N}$. This sequence (we denote it by $|R|$) is also called the *indicatrix* of the homogeneous parallelotopic subgroup R.

Theorem 3.2 ([5]). *If R is a characteristic (fully invariant or verbal) subgroup of the group $U_{p,n}^\infty$, then R is a homogeneous parallelotopic subgroup.*

4. Sufficient condition

Lemma 4.1. *A normal homogeneous parallelotopic subgroup U_r, which has depth r, $r \in \mathbb{N}$, and the indicatrix $\langle [1+d^{-1}]^-, \ldots, [1+d^{-r}]^-, 0, 0, \ldots\rangle$, is a characteristic subgroup of the group $U_{p,n}^\infty$.*

Proof. Obviously, U_r is a subgroup of the group $U_{p,n}^\infty$ which contains all tables, whose ranks are at most r. By calculating vuv^{-1}, where $u \in U_r$, $v \in U_{p,n}^\infty$, we can see that U_r is a normal subgroup of $U_{p,n}^\infty$. Then we apply the method of mathematical induction.

1) We now show that U_1 is a characteristic subgroup. The subgroup U_1 has the indicatrix $\langle [1 + d^{-1}]^-, 0, 0, \ldots\rangle$ and is a normal Abelian subgroup of $U_{p,n}^\infty$. Hence, the image $\varphi(U_1)$ under the action of any automorphism $\varphi \in \mathtt{Aut}\, U_{p,n}^\infty$ is a normal Abelian subgroup of $U_{p,n}^\infty$.

Suppose U_1 is not a characteristic subgroup of $U^\infty_{p,n}$, i.e. there exist an automorphism $\varphi \in \mathtt{Aut}\, U^\infty_{p,n}$ and a table $w \in U_1$ such that $\varphi(w) \notin U_1$. Moreover, we assume (without loss of generality) that $u = \varphi(w) = [\overline{a}_1(X_{2,k}), \overline{a}_2(X_{3,k}), 0, 0, \ldots]$, where $a_{12}(X_{3,k}) \neq 0$.

Let us consider the table $v = \mathtt{co}_{11}(x^2_{12}) \in U^\infty_{p,n}$. Then elements $uvuv^{-1}$ and $vuv^{-1}u$ have the following $(1,2)$-, $(1,1)$-coordinates:

$$[uvuv^{-1}]_{12} = a_{12}(X_{3,k}) + 0 + a_{12}(X^{uv}_{3,k}) - 0 = 2a_{12}(X_{3,k});$$
$$[vuv^{-1}u]_{12} = 0 + a_{12}(X^{v}_{3,k}) - 0 + a_{12}(X^{vuv^{-1}}_{3,k}) = 2a_{12}(X_{3,k});$$

$$\begin{aligned}[uvuv^{-1}]_{11} &= a_{11}(X_{2,k}) + (x_{12} + [u]_{12})^2 \\ &\quad + a_{11}(X^{uv}_{2,k}) - (x_{12} + [uvuv^{-1}]_{12})^2 \\ &= a_{11}(X_{2,k}) + (x_{12} + a_{12}(X_{3,k}))^2 \\ &\quad + a_{11}(X^{u}_{2,k}) - (x_{12} + 2a_{12}(X_{3,k}))^2 \\ &= a_{11}(X_{2,k}) + a_{11}(X^{u}_{2,k}) \\ &\quad - 2x_{12}a_{12}(X_{3,k}) - 3\left(a_{12}(X_{3,k})\right)^2;\end{aligned}$$

$$\begin{aligned}[vuv^{-1}u]_{11} &= x^2_{12} + a_{11}(X^{v}_{2,k}) - (x_{12} + [vuv^{-1}]_{12})^2 + a_{11}(X^{vuv^{-1}}_{2,k}) \\ &= x^2_{12} + a_{11}(X_{2,k}) - (x_{12} + a_{12}(X_{3,k}))^2 + a_{11}(X^{u}_{2,k}) \\ &= a_{11}(X_{2,k}) + a_{11}(X^{u}_{2,k}) - 2x_{12}a_{12}(X_{3,k}) - \left(a_{12}(X_{3,k})\right)^2.\end{aligned}$$

Since $a_{12}(X_{3,k}) \neq 0$ and $1 \neq 3 \pmod p$ (where $p > 2$), we have $[uvuv^{-1}]_{11} \neq [vuv^{-1}u]_{11}$, i.e. $uvuv^{-1} \neq vuv^{-1}u$ or $u\cdot u^v \neq u^v \cdot u$. The last inequality contradicts the commutativity of the subgroup $\varphi(U_1)$. Hence, the assumption is not correct, and U_1 is a characteristic subgroup of the group $U^\infty_{p,n}$.

2) Let $r \in \mathbb{N}$, $r > 1$, and let $\phi_r : U_r \to U_r$ be a homomorphism such that the table $[\overline{a}_1(X_{2,k}), \ldots, \overline{a}_r(X_{r+1,k}), 0, 0, \ldots]$ maps to $[\overline{a}_1(X_{2,k}), 0, 0, \ldots]$. Obviously, $\phi_r(U_r) = U_1$, and the kernel of ϕ_r is a subgroup that is isomorphic to U_{r-1}. Then $U_r/U_{r-1} \simeq U_1$. Similarly, one can show that $U^\infty_{p,n}/U_{r-1} \simeq U^\infty_{p,n}$.

3) Suppose that U_t is a characteristic subgroup of the group $U^\infty_{p,n}$ for $t \in \{1, 2, \ldots, r-1\}$, but U_r is not a characteristic subgroup. That is, there exist an automorphism $\psi \in \mathtt{Aut}\, U^\infty_{p,n}$ and a table $w \in U_r$ such that $\psi(w) \notin U_r$. The automorphism ψ induces the automorphism ψ' in the factor group $U^\infty_{p,n}/U_{r-1}$.

Let $w' \in U_r/U_{r-1}$ be the image of the element $w \in U_r$ under the action of the homomorphism $U^\infty_{p,n} \to U^\infty_{p,n}/U_{r-1}$. Then, on the one hand, $\psi(w) \notin U_r$, and therefore $\psi'(w') \notin U_r/U_{r-1}$. On the other hand, since $U_r/U_{r-1} \simeq U_1$, U_r/U_{r-1} is a characteristic subgroup of the group $U_{p,n}^{\infty}/U_{r-1}$. Thus, $\psi'(w') \in U_r/U_{r-1}$.

Consequently, the assumption is not correct, and U_r is a characteristic subgroup of the group $U_{p,n}^{\infty}$. □

The *mutual commutant* of subgroups X and Y is such a subgroup $[X, Y]$ that is generated by commutators $(x, y) = xyx^{-1}y^{-1}$, $x \in X$, $y \in Y$. Obviously, a mutual commutant of a normal (characteristic) subgroup is a normal (characteristic) subgroup.

Lemma 4.2. *A normal homogeneous parallelotopic subgroup $U_{r_1}^{r_2}$ which has depth r_1, $r_1 \in \mathbb{N}$, and the indicatrix*

$$\langle [1 + d^{-1} - d^{-r_2}]^-, \ldots, [1 + d^{-r_1} - d^{-r_2}]^-, 0, 0, \ldots \rangle,$$

$r_2 \in \mathbb{N}$, $r_2 > r_1$, is a characteristic subgroup of the group $U_{p,n}^{\infty}$.

Proof. According to Lemma 4.1, subgroups U_{r_1} and U_{r_2} are characteristic. Therefore, the mutual commutant $U' = [U_{r_1}; U_{r_2}]$ is a characteristic subgroup as well. We now show that the group U' has the required indicatrix.

Given $u = [a_{ij}(X_{i+1,k})]_{j=1}^{\infty} \in U_{r_2}$ and $v = [b_{ij}(X_{i+1,k})]_{j=1}^{\infty} \in U_{r_1}$, it is easy to show that each inner automorphism of the group $U_{p,n}^{\infty}$ preserves the depth of any table (moreover, it preserves the last non-zero column). Hence, v^u and $(u^{-1})^v$ have depths r_1 and r_2, respectively.

Let us to fix the index $i \in \{1, \ldots, n\}$. By Lemma 3.1 (items 1.3 and 4.1) and Lemma 3.2 (item 4), if $j \in \{1, \ldots, r_1\}$, then

$$\begin{aligned} &|(u, v)|_{ij} \\ &\le \max\{h[a_{ij}(X_{j+1,k}) - a_{ij}(X_{j+1,k}^{uvu^{-1}})], h[b_{ij}(X_{j+1,k}) - b_{ij}(X_{j+1,k}^{vu^{-1}v^{-1}})]\} < \\ &\qquad < \max\{1 + d^{-j} - d^{-r_1}, 1 + d^{-j} - d^{-r_2}\} = 1 + d^{-j} - d^{-r_2}; \end{aligned}$$

and if $j > r_1$, then

$$\begin{aligned} [(u, v)]_{ij} &= 0 - 0 + b_{ij}(X_{j+1,k}^{u}) - b_{ij}(X_{j+1,k}^{uvu^{-1}v^{-1}}) \\ &= b_{ij}(X_{j+1,k}) - b_{ij}(X_{j+1,k}) = 0. \end{aligned}$$

We now show that $1+d^{-j}-d^{-r_2} = \sup_{u \in U'} |u|_{ij}$ for all $j \in \{1, \ldots, r_1\}$. Let $u = \mathsf{co}_{ir_2}(1) \in U_{r_2}$, $v = \mathsf{co}_{ij}(x_{1,j+1}^{p-1} \ldots x_{n,j+1}^{p-1} \ldots x_{1m}^{p-1} \ldots x_{nm}^{p-1}) \in U_{r_1}$, where $m > r_2$. Then

$$[(u, v)]_{ij} = x_{1,j+1}^{p-1} \cdots x_{i-1,r_2}^{p-1}(x_{ir_2} + 1)^{p-1} x_{i+1,r_2}^{p-1} \cdots x_{nm}^{p-1} - [v]_{ij}.$$

The highest term of $[(u,v)]_{ij}$ is the monomial $f = C^1_{p-1} x^{p-1}_{1,j+1} \cdots x^{p-1}_{i-1,r_2} \times x^{p-2}_{ir_2} x^{p-1}_{i+1,r_2} \cdots x^{p-1}_{nm}$, where

$$|(u,v)|_{ij} = h[f] = 1 + (d-1)d^{-(j+1)} + \cdots + (d-1)d^{-m} \\ + \left[(d-2)d^{-r_2} - (d-1)d^{-r_2}\right] = 1 + d^{-j} - d^{-r_2} - d^{m}.$$

Consequently,

$$\lim_{m\to\infty} |(u,v)|_{ij} = 1 + d^{-j} - d^{-r_2}.$$

□

Lemma 4.3. *If $u \in U^{\infty}_{p,n}$ has depth r and order p, then $|u|_{ij} < 1 + d^{-j} - d^{-r}$, $i = 1,\dots,n$, $j = 1,\dots,r-1$.*

Proof. If u has depth r, then it has the non-zero rth column. Without loss of generality, we can assume that $[u]_{nr} \neq 0$.

Let $\delta(X_{j+1,r}) = x^{p-1}_{1,j+1} \dots x^{p-1}_{n-1,r}$. It is clear that the (i,j)-coordinate of the table u ($i \in \{1,\dots,n\}$, $j \in \{1,\dots,r-1\}$) can be always expressed as

$$[u]_{ij} = a_{ij}(X_{j+1,k}) = \sum_{t=0}^{p-1} \delta(X_{j+1,r}) x^t_{nr} f_t(X_{r+1,k}) + f(X_{j+1,k}),$$

where the polynomial $f(X_{j+1,k})$ does not contain monomials that are divisible by $\delta(X_{j+1,r})$, and so $h[f(X_{j+1,k})] < 1 + d^{-i} - d^{-r}$. Suppose $f_{p-1}(X_{r+1,k}) \neq 0$. Then

$$\begin{aligned}[u^p]_{ij} &= \sum_{m=0}^{p-1} a_{ij}(X^{u^m}_{j+1,k}) \\ &= \sum_{t=0}^{p-2} \left[\sum_{m=0}^{p-1} \delta(X^{u^m}_{j+1,r})(x_{nr} + m \cdot [u]_{nr})^t f_t(X_{r+1,k}) \right] \\ &\quad + \sum_{m=0}^{p-1} \delta(X^{u^m}_{j+1,r})(x_{nr} + m \cdot [u]_{nr})^{p-1} f_{p-1}(X_{r+1,k}) \\ &\quad + \sum_{m=0}^{p-1} f(X^{u^m}_{j+1,k}). \qquad (4.4)\end{aligned}$$

Let us consider each summand from the right-hand side of the previous equation. Monomials that are divisible by $\delta(X_{j+1,r})$ can be regrouped in each inner sum in the first summand as

$$\delta(X_{j+1,r}) \sum_{m=0}^{p-1} (x_{nr} + m \cdot [u]_{nr})^t f_t(X_{r+1,k})$$

(the rest of the sum has the weighted degree less than that of this polynomial). However,

$$\sum_{m=0}^{p-1} (x_{nr} + m \cdot [u]_{nr})^t = 0 \ (\bmod p) \quad \text{if } t \in \{0, \dots, p-2\}$$

Consequently, the first summand does not contain monomials that are divisible by $\delta(X_{j+1,r})$. Obviously, the third summand in (4.4) does not contain such monomials as well. One the other hand, all monomials in the second summand that contain $\delta(X_{j+1,r})$ can be regrouped as

$$\delta(X_{j+1,r}) \sum_{m=0}^{p-1} (x_{nr} + m \cdot [u]_{nr})^{p-1} f_t(X_{r+1,k}),$$

where

$$\sum_{m=0}^{p-1} (x_{nr} + m \cdot [u]_{nr})^{p-1} \neq 0 \ (\bmod p).$$

Thus, $[u^p]_{ij} \neq 0$, since we assume that $f_{p-1}(X_{r+1,k}) \neq 0$.

Hence, the (i,j)-coordinate of the table u does not contain monomials that are divisible by $\delta(X_{j+1,r})$. Then (one can verify it directly) $|u|_{ij} < 1 + d^{-j} - d^{-r}$. □

Lemma 4.4. *A normal homogeneous parallelotopic subgroup L_r^- which has depth r, $r \in \mathbb{N}$, and the indicatrix*

$$\langle [1 + d^{-1} - d^{-r}]^-, \dots, [1 + d^{-(r-1)} - d^{-r}]^-, 1^+, 0, 0, \dots \rangle,$$

is a characteristic subgroup of the group $U_{p,n}^\infty$.

Proof. According to Lemma 4.2 and Lemma 3.1 (item 2), it is sufficient to show that the image of $u = \mathtt{co}_{ir}(1)$, $i \in \{1, \dots, n\}$, under the action of any automorphism from $\mathtt{Aut}\, U_{p,n}^\infty$ is contained in L_r^-.

Since $u \in U_r$ and U_r is a characteristic subgroup (by Lemma 4.1), we have

$$(u, v) = uvu^{-1}v^{-1} = u\left(u^{-1}\right)^v \in U_r$$

for any $v = [b_{ij}(X_{j+1,k})]_{j=1}^\infty \in U_{p,n}^\infty$. Moreover,

$$\begin{aligned}[(u,v)]_{ir} = 1 - 1 + b_{ir}(X_{r+1,k}^u)] - b_{ir}(X_{r+1,k}^{uvu^{-1}v^{-1}}) \\ = b_{ir}(X_{r+1,k})] - b_{ir}(X_{r+1,k}) = 0.\end{aligned}$$

If $\varphi \in \mathtt{Aut}\, U_{p,n}^\infty$, then $w = \varphi(u) \in U_r$, i.e. $w = [\overline{a}_1(X_{2,k}), \dots, \overline{a}_r(X_{r+1,k}), 0, \dots]$. In addition, $\varphi(u,v) \in U_{r-1}$ for any $v \in U_{p,n}^\infty$, i.e. $\varphi(u,v)$ has depth of at most $r-1$.

We denote $z = \varphi(v)$. Then $\varphi(u,v) = wzw^{-1}z^{-1}$. Suppose that $h[a_{ir}(X_{r+1,k})] > 1$. That is, for some index $i \in \{1,\dots,n\}$, the polynomial $a_{ir}(X_{r+1,k})$ is not a constant. So, if $[z]_{ir} = c_{ir}(X_{r+1,k})$ is the (i,r)-coordinate of z, then

$$[\varphi(u,v)]_{ir} = a_{ir}(X_{r+1,k}) - a_r(X^z_{r+1,k}).$$

By Lemma 3.2 (item 3), the group $U^\infty_{p,n}$ contains a table v such that $|\varphi(u,v)|_{ir} > 1 > 0$. In other words, $\varphi(u,v)$ has depth r, which contradicts the previous estimations.

Consequently, $h[a_{ir}(X_{r+1,k})] \leq 1$ for all $i \in \{1,\dots,n\}$. It is significant that there exists an index $i \in \{1,\dots,n\}$ such that $h[a_{ir}(X_{r+1,k})] = 1$ (in other words, $[\varphi(u)]_{ir} = \mathit{const} \neq 0$), since, otherwise, the existence of the automorphism φ^{-1} would contradict Lemma 4.1.

Finally, since $\varphi(u)$ has depth r and order p, we have, according to Lemma 4.3, $|\varphi(u)|_{ij} < 1 + d^{-j} - d^{-r}$ for all $i \in \{1,\dots,n\}$, $j \in \{1,\dots,r-1\}$.

Therefore, $\varphi(u) \in L^-_r$ and L^-_r is a characteristic subgroup of the group $U^\infty_{p,n}$. □

Corollary 4.1. *A normal homogeneous parallelotopic subgroup H_r which has depth r, $r \in \mathbb{N}$, and the indicatrix*

$$\langle [1 + d^{-1} - d^{-r}]^-, \dots, [1 + d^{-(r-1)} - d^{-r}]^-, [1 + d^{-r}]^-, 0, 0, \dots\rangle,$$

is a characteristic subgroup of the group $U^\infty_{p,n}$.

Corollary 4.2. *A normal homogeneous parallelotopic subgroup F_r which has depth r, $r \in \mathbb{N}$, and the indicatrix*

$$\langle [1 + d^{-1}]^-, \dots, [1 + d^{-(r-1)}]^-, 1^+, 0, 0, \dots\rangle,$$

is a characteristic subgroup of the group $U^\infty_{p,n}$.

Lemma 4.5. *A normal homogeneous parallelotopic subgroup L^+_r which has depth r, $r \in \mathbb{N}$, and the indicatrix*

$$\langle [1 + d^{-1} - d^{-r}]^+, \dots, [1 + d^{-(r-1)} - d^{-r}]^+, 1^+, 0, 0, \dots\rangle,$$

is a characteristic subgroup of the group $U^\infty_{p,n}$.

Proof. Let $u = \mathtt{co}_{ij}(x^{p-1}_{1,j+1} \dots x^{p-1}_{nr})$ and $\varphi \in \mathtt{Aut}\, U^\infty_{p,n}$. In view of Lemma 4.4, it is sufficient to show that the weighted degree of the (i,j)-coordinate of $\varphi(u)$ is at most $1 + d^{-j} + d^{-r}$ for all $i \in \{1,\dots,n\}$,

$j \in \{1, \ldots, r-1\}$. Suppose that this is not true, i.e. there exists an automorphism $\varphi \in \mathtt{Aut}\, U_{p,n}^{\infty}$ such that $|\varphi(u)|_{ij} = 1 + d^{-j} - d^{-r} + \varepsilon$, $\varepsilon > 0$.

If $v \in U_{p,n}^{\infty}$, then, according to Lemma 3.1 (item 1.3) and Lemma 3.2 (item 2), $|(u,v)|_{ij} < |u|_{ij} = 1 + d^{-j} - d^{-r}$. Thus, $|\varphi(u,v)|_{ij} < 1 + d^{-j} - d^{-r}$ (since (u,v) has depth of at most r, and U_{j-1} and U_j^r are characteristic subgroups of $U_{p,n}^{\infty}$).

On the other hand, $\varphi(u,v) = \varphi(u)\varphi(v)\varphi(u)^{-1}\varphi(v)^{-1}$, and, according to Lemma 3.2 (item 3), the group $U_{p,n}^{\infty}$ contains such a table v that $|\varphi(u,v)|_{ij} > 1 + d^{-j} - d^{-r}$.

Hence, the assumption mentioned at the beginning of the proof is not true, and L_r^{+} is a characteristic subgroup of the group $U_{p,n}^{\infty}$. □

By $\ell(x)$, we denote the minimal number r such that a real $x \in [1;2)$ can be expressed as the finite sum $x = 1 + t_1 d^{-1} + \ldots + t_r d^{-r}$, where $0 \le t_1, \ldots, t_r \le d-1$, $t_r \ne 0$. If such r does not exist, then $\ell(x) = \infty$.

Lemma 4.6. *If a characteristic subgroup R which has depth s, $s \in \mathbb{N}$, and the indicatrix $\langle [1+d^{-1}]^{-}, \ldots, [1+d^{-(s-1)}]^{-}, \chi_s^{-}, 0, 0, \ldots \rangle$, where $\chi_s > 1 + d^{-r}$ and $\ell(\chi_s) \le r$, then a subgroup which has the indicatrix $\langle [1+d^{-1}]^{-}, \ldots, [1+d^{-(s-1)}]^{-}, [\chi_s - d^{-r}]^{-}, 0, 0, \ldots \rangle$, is also a characteristic subgroup of the group $U_{p,n}^{\infty}$.*

Proof. If $u = [a_{ij}(X_{j+1,k})]_{j=1}^{\infty} \in R$, $v \in L_r^{+}$, then $(u,v) = u\left(u^{-1}\right)^{v} \in U_s$, and, according to Lemma 3.1 (item 1.3), $|(u,v)|_{is} = h[a_{is}(X_{s+1,k}) - a_{is}(X_{s+1,k}^{uvu^{-1}})]$.

Let $M = x_{1,s+1}^{t_{1,s+1}} \cdots x_{nk}^{t_{nk}}$ denote a monomial of $a_{is}(X_{s+1,k})$, and $w = uvu^{-1}$.

Since L_r^{+} is a characteristic subgroup, $w \in L_r^{+}$. If $[w]_{ij} = b_{ij}(X_{j+1,k})$ (here and below, we omit symbols $X_{j+1,k}$), then M_1 can be expressed as the sum of terms of the form

$$M_J = x_{1,s+1}^{t_{1,s+1} - j_{1,s+1}} \cdots x_{nr}^{t_{nr} - j_{nr}} b_{1,s+1}^{j_{1,s+1}} \cdots b_{nr}^{j_{nr}} x_{1,r+1}^{t_{1,r+1}} \cdots x_{nk}^{t_{nk}},$$

where $j_{yz} \in \{0, \ldots, t_{yz}\}$, $y \in \{1, \ldots, n\}$, $z \in \{s+1, \ldots, r\}$ (and at least one of the powers j_{yz} is not equal to 0). Then, by Lemma 3.1 (item 4.2), $h[M_J] \le h[M] - d^{-r} < \chi_s - d^{-r}$.

Consequently, $|(u,v)|_{is} < \chi_s - d^{-r}$ for all $i \in \{1, \ldots, n\}$. Thus, the sth term of the indicatrix of the mutual commutant $[R, L_r^{+}]$ does not exceed $[\chi_s - d^{-r}]^{-}$.

Since $\ell(\chi_s) \le r$, we have $\chi_s = 1 + t_s d^{-s} + \cdots + t_m d^{-m}$, where $m \le r$ and $t_m \ne 0$. Let us consider the following two cases.

1) $m = r$ and $t_r = 1$. Let $\{f_j\}_{j \in \mathbb{N}}$ be a sequence of monomials such that

$$f_j(X_{s+1,j}) = x_{1,s+1}^{t_{1,s+1}} \cdots x_{n,r-1}^{t_{n,r-1}} x_{1,r+1}^{p-1} \cdots x_{nj}^{p-1},$$

where $t_s = \sum_{i=1}^{n} t_{is}, \ldots, t_{r-1} = \sum_{i=1}^{n} t_{i,r-1}$. Then $h[f_j] = 1 + t_s d^{-s} + \cdots + t_{r-1}d^{-(r-1)} + d^{-r} - d^{-j} = \chi_s - d^{-j}$. Without loss of generality, we assume that $t_{n,r-1} \neq 0$.

Let $v = \mathtt{co}_{n,r-1}(x_{1r}^{p-1} \cdots x_{nr}^{p-1}) \in L_r^+, u_j = \mathtt{co}_{ns}(f_j)$. Then, by Lemma 3.1 (item 1.3), $|(v, u_j)|_{ns} = h[f_j(X_{s+1,j}^v) - f_j(X_{s+1,j})] = h[f_j] - d^{-r}$.

Thus, if $j \to \infty$, then $\lim |(v, u_j)|_{ns} = \lim (h[f_j] - d^{-r}) = \chi_s - d^{-r}$, i.e. the sth term of the indicatrix of the subgroup $[R, L_r^+]$ is equal to $[\chi_s - d^{-r}]^-$. Then $R' = \langle U_{s-1}, [R, L_r^+]\rangle$ (a subgroup generated by U_{s-1} and $[R, L_r^+]$) is a characteristic subgroup with the required indicatrix.

2) $m < r$ or $t_r \geq 2$. In this case, we have the same argumentation. The only difference is that we consider a sequence of monomials

$$f_j(X_{s+1,j}) = x_{1,s+1}^{t_{1,s+1}} \cdots x_{n-1,r}^{t_{n-1,r}} x_{nr}^{t_{nr}-1} x_{1,r+1}^{p-1} \cdots x_{nj}^{p-1},$$

where $t_s = \sum_{i=1}^{n} t_{is}, \ldots, t_r = \sum_{i=1}^{n} t_{ir}$, and $v = \mathtt{co}_{nr}(1)$. □

Lemma 4.7. *Let $\varepsilon = t_{r+1}d^{-(r+1)} + \cdots + t_k d^{-k}$, where $0 \leq t_{r+1}, \ldots, t_k \leq d-1$. Then a normal homogeneous parallelotopic subgroup R_r^ε, $r \in \mathbb{N}$ which has the indicatrix $\langle [1 + d^{-1}]^-, \ldots, [1 + d^{-(r-1)}]^-, [1 + d^{-r} - \varepsilon]^-, 0, 0, \ldots\rangle$, is a characteristic subgroup of the group $U_{p,n}^\infty$.*

Proof. According to Lemmas 4.1 and 4.5, all terms of the sequence $\{R_j\}_{j\in\mathbb{N}}$, where $R_0 = U_r$ and $R_j = [R_{j-1}; L_k^+]$ $(j = 1, 2, \ldots)$, are characteristic subgroups of the group $U_{p,n}^\infty$. Let $n = t_{r+1}d^{k-r-1} + t_{r+2}d^{k-r-2} + \cdots + t_k$. The subgroups R_{j-1} and L_k^+ satisfy the conditions of Lemma 4.6 for all $j \in \{1, \ldots, n\}$. Hence, the rth term of the characteristic of R_n is equal to $1 + d^{-r} - \varepsilon$. Finally, $R_r^\varepsilon = \langle R_n, U_{r-1}\rangle$ (a subgroup generated by R_n and U_{r-1}) is a characteristic subgroup with the required indicatrix. □

Lemma 4.8. *A normal homogeneous parallelotopic subgroup S_r^χ which has depth r, $r \in \mathbb{N}$, and the indicatrix $\langle [1+d^{-1}]^-, \ldots, [1+d^{-(r-1)}]^-, \chi^-, 0, 0, \ldots\rangle$, where $\chi \in (1; 1+d^{-r}]$, is a characteristic subgroup of the group $U_{p,n}^\infty$.*

Proof. If $\ell(\chi) < \infty$, then this statement can be reduced to the previous one. Thus, let χ can be expressed as the infinite sum $\chi = \sum_{i=r+1}^{\infty} k_i d^i$ only. Then one can generate a sequence $\{\chi_j\}_{j\in\mathbb{N}}$ such that $\chi_i < \chi_{i+1}$, $\ell(\chi_i) < \infty$ for all $j \in \mathbb{N}$ and $\lim \chi_j = \chi$ if $j \to \infty$. In this case, $R = \bigcup_{j=1}^{\infty} R_r^{\chi_j}$ is a characteristic subgroup (as the union of characteristic subgroups). Finally, by Theorem 3.2, the subgroup R is a homogeneous parallelotopic subgroup, moreover, $R = S_r^\chi$. □

Since the subgroups H_r and S_r^χ are characteristic (according to Corollary 4.1 and Lemma 4.8), the intersection $H_r \cap S_r^\chi$ is a characteristic subgroup. So, we have the following statement.

Corollary 4.3. *A normal homogeneous parallelotopic subgroup T_r^χ which has depth r, $r \in \mathbb{N}$, and the indicatrix*

$$\langle [1+d^{-1}-d^{-r}]^-, \ldots, [1+d^{-(r-1)}-d^{-r}]^-, \chi^-, 0, 0, \ldots \rangle,$$

where $\chi \in (1; 1+d^{-r}]$, is a characteristic subgroup of the group $U_{p,n}^\infty$.

Theorem 4.1. *Any normal homogeneous parallelotopic subgroup is a characteristic subgroup of the group $U_{p,n}^\infty$.*

Proof. Each normal homogeneous parallelotopic subgroup of the group $U_{p,n}^\infty$ has finite depth. Let $\langle [\chi_1]^-, \ldots, [\chi_r]^-, 0, \ldots \rangle$ be the indicatrix of such a subgroup R.

By Theorem 3.1, we have $\chi_j \geq 1 + d^{-j} - d^{-r}$, $j \in \{1, \ldots, r\}$. Let us consider a characteristic subgroup $R' = \langle T_1^{\chi_1}, \ldots, T_r^{\chi_r} \rangle$ as a subgroup generated by $T_j^{\chi_j}$, $j \in \{1, \ldots, r\}$. If we denote the ith term of the indicatrix of R' by $|R'|_i$, then $|R'|_i = \max_j \{|T_j^{\chi_j}|_i\} = \chi_i$. Consequently, $R = R'$.

If the subgroup R has the indicatrix $\langle \ldots, [\chi_j]^+, \ldots \rangle$, where the jth term is marked by "+", then we consider a sequence of characteristic subgroups $\{R_n\}_{n=n_0}^\infty$ with indicatrices $\langle \ldots, [\chi_j + 1/n]^-, \ldots \rangle$. It is clear that one can choose a number n_0 such that $\chi_j + 1/n_0 < 1 + d^{-j}$. Let $R' = \bigcap_{n=n_0}^\infty R_n$. The subgroup R' is characteristic, as the intersection of characteristic subgroups. Moreover, any table $u \in R$ belongs to each of the subgroups R_n, $n \geq n_0$, i.e. $u \in R'$. On the other hand, if $v \in R'$, then the (i,j)-coordinate of its multidegree is at most χ_j, i.e. $v \in R$. Thus, $R = R'$. □

Hence, Theorem 1.1 follows from Theorems 3.2 (necessary condition) and 4.1 (sufficient condition).

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