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Estimates of the maximum of a solution to the Neumann problem for degenerate parabolic equations in unbounded domains narrowing at infinity. The fast diffusion case

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Abstract. We present sharp bilateral bounds of the norm L_∞ of a solution to the Neumann problem of doubly degenerate parabolic equations in unbounded domains narrowing at infinity.

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1. Introduction

In this paper, we consider the Neumann problem

$$u_t - \operatorname{div}(u^{m-1}|Du|^{\lambda-1}Du) = 0, \qquad \text{in } Q_T = \Omega \times (0,T), \tag{1.1}$$

$$u^{m-1}|Du|^{\lambda-1}\frac{\partial u}{\partial \overrightarrow{n}} = 0, \qquad \text{on } \partial\Omega \times (0,T), \tag{1.2}$$

$$u(x,0) = u_0(x), \qquad x \in \Omega, \tag{1.3}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is an unbounded domain, $\operatorname{mes}_N \Omega = |\Omega|_N = \infty$, with a noncompact and sufficiently smooth boundary $\partial\Omega$ of Ω, and $\overrightarrow{n}$ means the outward normal to $\partial\Omega \times (0,T)$, $T > 0$. Throughout of the paper, we assume that $m+\lambda-2 < 0$, $\lambda > 0$, $m+\lambda-1 > \max\{0, 1-\frac{\lambda+1}{N}\}$; $u_0(x) \geq 0$ for a.e. $x \in \Omega$ и $u_0 \in L_{1,loc}(\Omega)$. It is known [10] that the assumption $m + \lambda - 2 < 0$ corresponds to the equations which describe

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the processes with a fast diffusion. Next, we specify the classes of domains under consideration.

Define the function

$$l(v,\rho) = \inf\{|\partial Q \cap \Omega_\rho|_{N-1} : Q \subset \Omega_\rho, |Q|_N = v, \partial Q - \text{Lipschitz}\},$$

for any $\rho > 0$ and $0 < v \le |\Omega_\rho|_N/2$, where $\Omega_\rho = \{x \in \Omega : |x| < \rho\}$. We assume that Ω_ρ is a nonempty set.

Let the volume $V(\rho) = |\Omega_\rho\,|_N$ satisfy the conditions: for any $\delta > 0$,

$$\nu_0(\delta)V(\rho) \le V(\delta\rho) \le \nu_1(\delta)V(\rho), \qquad \text{for all } \rho \ge \max\Big(1, \frac{1}{\delta}\Big), \tag{1.4}$$

where ν_0, ν_1 are two given nondecreasing positive functions satisfying $\nu_1(\delta) < 1$ for $\delta < 1$. We assume also that

$$l(v,\rho) \ge c_0 \min\Big(v^{\frac{N-1}{N}}, \frac{V(\rho)}{\rho}\Big) := g(v,\rho) \tag{1.5}$$

for any $\rho \ge 1$, $0 < v \le V(\rho)/2$, and with a suitable $c_0 > 0$. Here, it is supposed that

$$\rho \mapsto \frac{\rho^{1-\beta}}{V(\rho)} \qquad \text{is nondecreasing for any } \rho \ge 1, \tag{1.6}$$

where $\beta > \frac{2-m-\lambda}{\lambda+1}$.

Definition 1.1. *We say that an unbounded domain $\Omega \subset \mathbb{R}^N$, $N \ge 2$, belongs to the class $\mathbf{N_0}(g)$, if $\partial\Omega$ is Lipschitz continuous, and* (1.4)–(1.6) *hold true.*

Note that (1.5) for $v < 1$ characterizes the smoothness of $\partial\Omega$, while the second component of this inequality has a meaning of the surface size $\Omega \cap \partial\Omega_\rho$ for ρ large enough. Note also that (1.4) yields $|\Omega|_N = \infty$. The class $\mathbf{N_0}(g)$ describes domains "narrowing at infinity", that is, $\underline{\lim}_{\rho\to\infty} V(\rho)/\rho = 0$ [5]. For such a class of domains, $\underline{\lim}_{v\to\infty} l(v,\rho) = 0$. Classes $\mathbf{N_0}(g)$ have been introduced in [1, 2] (see also [5, 8, 12].)

The typical example of $\mathbf{N_0}(g)$ is (see [2])

$$\Omega^\epsilon = \{x = (x', x_N) \in \mathbb{R}^N : |x'| < x_N^{-\epsilon},\ x_N > d\} \subset \mathbb{R}^N, d > 0,$$

for $0 < \epsilon < \frac{1}{N-1}$. Here, $V(\rho) = c\rho^{1-\epsilon(N-1)}$, $\rho = x_N > 2d$. Evidently, $|\Omega^\epsilon|_N = \infty$ and, for all $v > 0$, $l(v,\infty) = 0$. Other examples can be found in [1] and [2].

Out purpose is the investigation of the temporal behavior of the solution of problem (1.1)–(1.3) in Q_∞ in dependence on the geometry of

Ω. We will obtain sharp bilateral bounds of the supremum norm of the solution.

The qualitative behavior of solutions of the Neumann problem for linear uniformly parabolic equations with measurable coefficients in domains with non-compact boundaries was first studied in [6, 7]. It was established in those papers for expanding domains which satisfy the global isoperimetric property that the bilateral estimate

$$\|u(t)\|_{L_\infty,\Omega} \sim \|u_0\|_{L_1,\Omega} V(\sqrt{t})^{-1} \tag{1.7}$$

is valid for all $t > 1$. Here and hereafter, $\|u(t)\|_{p,\Omega} = \|u(x,t)\|_{L_p,\Omega}$. Note that this result was proven, by assuming the initial datum from $L_1(\Omega)$ only. To have similar estimates for domains narrowing at infinity, the authors of [5, 12] have assumed an extra assumption on the initial data at infinity: namely $u_0\,|x| \in L_1(\Omega)$. That is, the moment of the initial datum must be finite. Concerning the further results for linear parabolic equations in domains with noncompact boundaries, we quote works [17] (the third boundary value problem) and [13] (the Dirichlet problem). In [15] similar to (1.7), the estimates for a solution of (1.1)–(1.3) with $m = 1$ and $\lambda > 1$ in domains expanding at infinity were obtained. These estimates have form

$$\|u(t)\|_{L_\infty,\Omega} \sim \|u_0\|_{L_1,\Omega} V(R(t))^{-1}, \tag{1.8}$$

where $R(t)$ is the inverse function to $s^{\lambda+1}V(s)^{\lambda-1}$. In [1–3], such a sort of results was extended for solutions of (1.1)–(1.3) with $m + \lambda - 2 > 0$ (i.e., in the slow diffusion case) in both expanding and narrowing domains.

Definition 1.2. *We say that $u(x,t)$ is a solution of* (1.1)–(1.3) *in Q_T, if $u(x,t) \geq 0$ and $u(x,t) \in C(0,T; L_{2,loc}(\overline{\Omega})) \cap L_{\infty,loc}(\overline{\Omega} \times (0,T))$, $u^{m-1}|Du|^{\lambda+1} \in L_{1,loc}(\overline{\Omega} \times (0,T))$; and u satisfies* (1.1)–(1.3) *in the integral sense*

$$\int_0^T \int_\Omega (-u\xi_t + u^{m-1}\ |Du|^{\lambda-1}\ Du\ D\xi)\, dx\, dt = -\int_\Omega u_0(x)\ \xi(x,0)\, dx,$$

for all $\xi \in C_1(R^N \times [0,T])$ such that $\xi \equiv 0$ out of $\{|x| \leq K < \infty\}$, for a suitable $K > 0$, and $\xi(x,T) = 0$.

In what follows, we use the notation

$$\mu(t) = \int_{\overline{\Omega}} u(x,t)\ \frac{|x|}{V(|x|)}\, dx.$$

The main result of the paper is

Theorem 1.1. *Let $\Omega \in \mathbf{N_0}(g)$, $u_0 \geq 0$, $u_0 \in L_1(\Omega)$, $\mu(0) < \infty$. Then problem* (1.1)–(1.3) *has a solution global in time, and the following estimates hold true:*

$$\|u(t)\|_{\infty,\Omega} \leq \gamma \max \Big\{ t^{-\frac{N}{K}} \|u_0\|_{1,\Omega}^{(\lambda+1)/K}, \quad t^{-1/(2\lambda+m-1)} \mu(0)^{(\lambda+1)/(2\lambda+m-1)}, \quad t^{-1/(2\lambda+m-1)} \|u_0\|_{1,\Omega}^{(\lambda+1)/(2\lambda+m-1)} \left[\frac{P(\tau)}{V(P(\tau))}\right]^{(\lambda+1)/(2\lambda+m-1)} \Big\} \tag{1.9}$$

for all $t > 0$ where $K = N(m+\lambda-2)+\lambda+1$. Here, $P(\tau) \geq 1$ with $\tau = t\|u_0\|_{1,\Omega}^{m+\lambda-2}$ is defined as a maximum solution of the equation

$$\rho\,(\rho/V(\rho))^{-(m+\lambda-2)/(2\lambda+m-1)} = \max\Big\{\tau^{1/(2\lambda+m-1)}, 1\Big\}. \tag{1.10}$$

Moreover, for t large enough,

$$\gamma_1 \|u_0\|_{1,\Omega} / V(P(\tau)) \leq \|u(t)\|_{\infty,\Omega} \leq \gamma_2 \|u_0\|_{1,\Omega} / V(P(\tau)). \tag{1.11}$$

In the proof of Theorem 1.1, we combine ideas of [1, 2, 4].

Throughout the paper, we use the symbols γ, γ_i for positive constants depending on data of the problem only.

Remark 1.1. Notice that, for t large enough, the third term on the right-hand side of (1.9) dominates; thus, relation (1.10) yields

$$t^{-\frac{1}{2\lambda+m-1}} \|u_0\|_{1,\Omega}^{\frac{\lambda+1}{2\lambda+m-1}} \left[\frac{P(\tau)}{V(P(\tau))}\right]^{\frac{\lambda+1}{2\lambda+m-1}} = \frac{\|u_0\|_{1,\Omega}}{V(P(\tau))},$$

while the first term is the largest one for t small enough. The latter means that the estimate has a local structure and, for small times, coincides with the corresponding estimate of solutions of the Cauchy problem.

Remark 1.2. The results of Theorem 1.1 are still valid in the case $m+\lambda-2=0$ and therefore in the linear case $m=\lambda=1$ with minor changes in the proof. In the latter case, our results follows from [5, 12].

The structure of the paper is as follows. In Section 2, we formulate auxiliary results. In Section 3, we prove the local maximum estimates of solutions. Section 4 is devoted to the local estimate of the mass of the solution. The global bounds of the moment of the solution is given in Section 5. Finally in Section 6, we prove our main result.

2. Auxiliary results

Set

$$\omega(z,\rho) = \frac{z^{\frac{N-1}{N}}}{g(z,\rho)} = \gamma \max\left(1, z^{\frac{N-1}{N}} \frac{\rho}{V(\rho)}\right), \ z \geq 0, \rho \geq 1.$$

We need the following parabolic multiplicative inequality.

Lemma 2.1 ([2]). *Let* $\Omega \in \mathbf{N_0}(g)$*,* $\rho \geq 1$*, and* $v \in L_\infty((0,T); L_r(\Omega_\rho))$*,* $Dv \in (L_p(\Omega_\rho \times (0,T)))^N$*, with* $p > 1$*,* $r \geq 1$*. Assume that, for* $\theta_0 \in (0,1)$*,*

$$\sup_{(0,T)} |\operatorname{supp} v(\cdot,t)|_N \leq \theta_0 V(\rho). \tag{2.1}$$

Then

$$\int_0^T \int_{\Omega_\rho} |v|^{p+\frac{pr}{N}}\, dx\, dt \leq \gamma \sup_{0<t<T} \left[\omega(|\operatorname{supp} v(\cdot,t)|_N, \rho)^p \left(\int_{\Omega_\rho} |v(x,t)|^r\, dx \right)^{\frac{p}{N}} \right] \times \int_0^T \int_{\Omega_\rho} |Dv|^p\, dx\, dt,$$

where $\gamma = \gamma(p,r,N)$*.*

Without loss of generality, we may assume

$$\frac{\rho}{V(\rho)} = 1, \qquad \text{for } \rho = 1.$$

Define the function

$$f(\rho) = \begin{cases} 1, & 0 \leq \rho \leq 1, \\ \frac{\rho}{V(\rho)}, & \rho > 1, \end{cases}$$

and the function

$$F(x) = \frac{1}{|x|} \int_0^{|x|} f(s)\, ds, \qquad x \in \Omega.$$

Lemma 2.2. *The following holds: $F(x) \equiv 1$ in Ω_1, and, for $\gamma_0 \in (0,1)$,*

$$\gamma_0 \frac{|x|}{V(|x|)} \le F(x) \le \frac{|x|}{V(|x|)}, \qquad x \in \Omega \setminus \Omega_1,$$

$$|DF(x)| \le \frac{1}{\gamma_0} \frac{1}{V(|x|)}, \qquad x \in \Omega \setminus \Omega_1.$$

The proof of the lemma is based on (1.4) and (1.6).

3. The local estimate of the maximum of the solution

For simplicity, we suppose that the solution of (1.1)–(1.3) is smooth enough.

Proposition 3.1. *Let u be the bounded solution of (1.1)–(1.3) in $\Omega_{2\rho} \times (0,t)$. Then, for any $\theta > 0$, we have*

$$\|u\|_{\infty,\Omega_\rho \times (t/2,t)} \le \gamma \max\Big(t^{-\frac{N}{K_\theta}} G_\theta(t, \rho(1+\sigma))^{\frac{\lambda+1}{K_\theta}}, \\ t^{-\frac{1}{H_\theta}} G_\theta(t, \rho(1+\sigma))^{\frac{\lambda+1}{H_\theta}} \Big[\frac{\rho}{V(\rho)}\Big]^{\frac{\lambda+1}{H_\theta}}, \Big[\frac{t}{\rho^{\lambda+1}}\Big]^{\frac{1}{2-m-\lambda}} \Big), \quad (3.1)$$

where

$$G_\theta(t,\rho) = \sup_{0<\tau<t} \int_{\Omega_\rho} u(x,\tau)^\theta \, dx, \qquad t > 0, \ \rho \ge 1,$$

$0 < \sigma < 1$, $K_\theta = N(m+\lambda-2) + \theta(\lambda+1)$, $H_\theta = m + \lambda - 2 + \theta(\lambda+1)$.

Proof. Set $Q_\infty = \Omega_\rho \times (t/2, t)$. We estimate the norm $\|u\|_{\infty,Q_\infty}$. Consider the sequences

$$\rho_n = \rho(1+\sigma 2^{-n}) \qquad t_n = \frac{t}{2}(1 - \frac{\sigma}{2^n}).$$

Let

$$Q_n = \Omega_{\rho_n} \times (t_n, t),$$

and

$$k_n = k\Big(1 - \frac{1}{2^{n+1}}\Big), \qquad n = 0, 1, \dots,$$

where $k > 0$ will be chosen later on.

Let $(x,\tau) \to \zeta_n(x,\tau)$ for any $n = 0, 1, \dots$ be a nonnegative smooth cutoff function in Q_n, i.e.,

$$\zeta_n(x,t) = \begin{cases} 1, & \text{on } Q_{n+1}, \\ 0, & \text{out of } Q_n, \end{cases}$$

and such that $0 \le \zeta_{nt} \le \frac{2^{n+2}}{\sigma t}$, $|D\zeta_n| \le \frac{2^{n+1}}{\sigma\rho}$.

Multiplying (1.1) by $(u-k_n)_+^q \zeta_n^{\lambda+1}$ and integrating by parts over Q_n, we obtain

$$\sup_{t_n<\tau<t} \int\limits_{\Omega_{\rho_n}(\tau)} (u-k_n)_+^{q+1}\ \zeta_n^{\lambda+1}\,dx + \iint\limits_{Q_n} |D(u-k_n)_+^{\frac{m+q+\lambda-1}{\lambda+1}}\zeta_n|^{\lambda+1}\,dx\,d\tau \le \gamma\Bigg(\iint\limits_{Q_n}(u-k_n)_+^{q+1}\ \zeta_{n\tau}\,dx\,d\tau + \iint\limits_{Q_n}(u-k_n)_+^{q+m-1}\ |Du|^{\lambda}\ \zeta_n^{\lambda}\ |D\zeta_n|\,dx\,d\tau\Bigg). \quad (3.2)$$

We have

$$\int\limits_{\Omega_{\rho_n}} (u-k_n)_+^{q+1}\ \zeta_n^{\lambda+1}\,dx \ge \int\limits_{\Omega_{\rho_n}\cap\{u>k_{n+1}\}} (u-k_n)_+^{q+1}\ \zeta_n^{\lambda+1}\,dx \ge (k_{n+1}-k_n)^{2-m-\lambda}\int\limits_{\Omega_{\rho_n}} (u-k_n)_+^{q+m+\lambda-1}\ \zeta_n^{\lambda+1}\,dx \ge (k/2^{n+2})^{2-m-\lambda}\int\limits_{\Omega_{\rho_n}} (u-k_n)_+^{q+m+\lambda-1}\ \zeta_n^{\lambda+1}\,dx.$$

Thus, using the last estimate, we get

$$(k/2^n)^{2-m-\lambda}\sup_{t_n<\tau<t}\int\limits_{\Omega_{\rho_n}(\tau)} (u-k_n)_+^{q+m+\lambda-1}\ \zeta_n^{\lambda+1}\,dx + \iint\limits_{Q_n}\Big|D((u-k_n)_+^{\frac{m+q+\lambda-1}{\lambda+1}}\zeta_n)\Big|^{\lambda+1}\,dx\,d\tau \le \gamma\Bigg(\frac{2^n}{\sigma t}\|u\|_{\infty,Q_0}^{2-m-\lambda}\iint\limits_{Q_n}(u-k_n)_+^{q+m+\lambda-1}\,dx\,d\tau + \frac{2^{n(\lambda+1)}}{(\sigma\rho)^{\lambda+1}}\iint\limits_{Q_n}(u-k_n)_+^{q+m+\lambda-1}\,dx\,d\tau\Bigg). \quad (3.3)$$

Next, we assume that

$$\frac{t}{\rho^{\lambda+1}}\|u\|_{\infty,Q_0}^{m+\lambda-2} < 1, \tag{3.4}$$

otherwise nothing is to prove. Denote $M = \frac{\|u\|_{\infty,Q_0}^{2-m-\lambda}}{\sigma^{\lambda+1}t}$. Then (3.3) implies

$$(k/2^n)^{2-m-\lambda} \sup_{t_n<\tau<t} \int\limits_{\Omega_{\rho n}(\tau)} (u-k_{n+1})_+^{q+m+\lambda-1}\,\zeta_n^{\lambda+1}\,dx + \iint\limits_{Q_n} \left|D((u-k_{n+1})_+^{\frac{m+q+\lambda-1}{\lambda+1}}\zeta_n)\right|^{\lambda+1} dx\,d\tau \le \gamma 2^{n(\lambda+1)} M \iint\limits_{Q_n} (u-k_n)_+^{q+m+\lambda-1}\,dx\,d\tau. \tag{3.5}$$

Denote $v = (u-k_{n+1})_+^{\frac{m+q+\lambda-1}{\lambda+1}}\zeta_n$. Applying the Hölder inequality, we have

$$I_{n+1} \equiv \iint\limits_{Q_{n+1}} (u-k_{n+1})_+^{q+m+\lambda-1}\,dx\,d\tau \le \iint\limits_{Q_n} v^{\lambda+1}\,dx\,d\tau \le \left[\iint\limits_{Q_n} v^{\overline{q}}\,dx\,d\tau\right]^{\frac{\lambda+1}{\overline{q}}} |A_{n+1}|_{N+1}^{1-\frac{\lambda+1}{\overline{q}}}, \tag{3.6}$$

where $A_{n+1} = \{(x,\tau)\in Q_n : u(x,\tau) > k_{n+1}\} \subset R^{N+1}$.

Set $\overline{q} = (\lambda+1)(1+\frac{\lambda+1}{N})$. Let $p = r = \lambda+1$ in Lemma 2.1. For all $n \ge 0$, $\tau > t_n$ and θ_0, as in (2.1), we have

$$|\operatorname{supp}(u-k_{n+1})_+\zeta_n(\tau)| \le V(2\rho) \le \theta_0 V(C\rho),$$

provided $C = C(\theta_0) > 0$ is chosen large enough. Thus, (2.1) is satisfied in this context by substituting the spatial domain $\Omega_{(1+\sigma)\rho}$ with a larger domain $\Omega_{C\rho}$, in which we apply the embedding inequality (note that v vanishes outside of $\Omega_{(1+\sigma)\rho}$). Then, from (3.6), we obtain

$$I_{n+1} \le \gamma\left(\sup_{t_n<\tau<t}\left[\omega(|A_{n+1}(\tau)|_N, C\rho)^{\lambda+1}\left(\int\limits_{\Omega_{\rho n}(\tau)} v(x,t)^{\lambda+1}\,dx\right)^{\frac{\lambda+1}{N}}\right] \times \iint\limits_{Q_n} |Dv|^{\lambda+1}\,dx\,d\tau\right)^{\frac{N}{N+\lambda+1}} |A_{n+1}|_{N+1}^{\frac{\lambda+1}{N+\lambda+1}}, \tag{3.7}$$

where $A_{n+1}(\tau) = \{x \in \Omega_{\rho_n} : u(x,\tau) > k_{n+1}\} \subset R^N$. To estimate $|A_{n+1}|_{N+1}$, $|A_{n+1}(\tau)|_N$ we proceed as follows:

$$I_n = \iint\limits_{Q_n} (u - k_n)_+^{q+m+\lambda-1}\, dx\, d\tau \geq \iint\limits_{Q_n \cap \{u > k_{n+1}\}} (k_{n+1} - k_n)^{q+m+\lambda-1}\, dx\, d\tau = \left[\frac{k}{2^{n+1}}\right]^{q+m+\lambda-1} |A_{n+1}|_{N+1}.$$

Hence,

$$|A_{n+1}|_{N+1} \leq 2^{(n+1)(q+m+\lambda-1)} k^{-(q+m+\lambda-1)} I_n, \tag{3.8}$$

$$|A_{n+1}(\tau)|_N \leq 2^{(n+1)(q+m+\lambda-1)} k^{-(q+m+\lambda-1)} \int\limits_{\Omega_{\rho_n}} (u - k_n)_+^{q+m+\lambda-1}\, dx \leq \gamma \frac{2^{n(q+m+\lambda-1)} k^{-(q+m+\lambda-1)}}{t\sigma^{\lambda+1}} I_0. \tag{3.9}$$

By (1.6), we get

$$\omega(|A_{n+1}(\tau)|_N, C\rho) \leq \gamma 2^{n(q+m+\lambda-1)} \omega\left(\frac{k^{-(q+m+\lambda-1)}}{t\sigma^{\lambda+1}} I_0, \rho\right). \tag{3.10}$$

We estimate integrals on the right-hand side of (3.7) by (3.5) and (3.8)–(3.10). We have

$$I_{n+1} \leq \gamma b^n \left[\omega\left(\frac{k^{-(q+m+\lambda-1)}}{t\sigma^{\lambda+1}} I_0, \rho\right)\right]^{\frac{(\lambda+1)N}{N+\lambda+1}} M k^{\frac{-(q+1)(\lambda+1)}{N+\lambda+1}} I_n^{1+\frac{\lambda+1}{N+\lambda+1}}, \tag{3.11}$$

where $b = 2^l > 1$, l is a constant depending on data.

By Lemma 5.6, [11, Ch. 2], we get

$$I_n = \iint\limits_{Q_n} (u - k_n)_+^{q+m+\lambda-1}\, dx\, d\tau \xrightarrow[n\to\infty]{} 0,$$

i.e.,

$$\|u\|_{\infty, Q_\infty} \leq k, \tag{3.12}$$

if

$$\gamma I_0^{\frac{\lambda+1}{N+\lambda+1}} \left[\omega\left(\frac{k^{-(q+m+\lambda-1)}}{t\sigma^{\lambda+1}} I_0, \rho\right)\right]^{\frac{(\lambda+1)N}{N+\lambda+1}} M k^{\frac{-(q+1)(\lambda+1)}{N+\lambda+1}} b^{\frac{N+\lambda+1}{\lambda+1}} \leq 1,$$

the more if

$$\gamma\|u\|_{\theta,Q_0}^{\theta\frac{\lambda+1}{N+\lambda+1}}\left[\omega\left(\frac{k^{-\theta}}{t\sigma^{\lambda+1}}\|u\|_{\theta,Q_0}^{\theta},\rho\right)\right]^{\frac{(\lambda+1)N}{N+\lambda+1}} M k^{\frac{-(q+1)(\lambda+1)}{N+\lambda+1}} b^{\frac{N+\lambda+1}{\lambda+1}} \leq 1,$$

where $\theta = q+m+\lambda-1$. Choose k from the relation

$$\gamma\|u\|_{\theta,Q_0}^{\theta\frac{\lambda+1}{N+\lambda+1}}\left[\omega\left(\frac{k^{-\theta}}{t\sigma^{\lambda+1}}\|u\|_{\theta,Q_0}^{\theta},\rho\right)\right]^{\frac{(\lambda+1)N}{N+\lambda+1}} M k^{\frac{-(q+1)(\lambda+1)}{N+\lambda+1}} b^{\frac{N+\lambda+1}{\lambda+1}} = 1. \tag{3.13}$$

From (3.12) and (3.13) with regard for (1.6), we get

$$\|u\|_{\infty,Q_\infty}^{q+1} \leq \gamma\|u\|_{\theta,Q_0}^{\theta} M^{\frac{N+\lambda+1}{\lambda+1}}\left[\omega\left(\frac{\|u\|_{\infty,Q_\infty}^{-\theta}}{t\sigma^{\lambda+1}}\|u\|_{\theta,Q_0}^{\theta},\rho\right)\right]^{N}. \tag{3.14}$$

Recalling the definition of M, we can rewrite (3.14) as follows:

$$\|u\|_{\infty,Q_\infty}^{q+1} \leq \gamma\|u\|_{\theta,Q_0}^{\theta}\left[\frac{\|u\|_{\infty,Q_0}^{2-m-\lambda}}{t\sigma^{\lambda+1}}\right]^{\frac{N+\lambda+1}{\lambda+1}}\left[\omega\left(\frac{\|u\|_{\infty,Q_\infty}^{-\theta}}{t\sigma^{\lambda+1}}\|u\|_{\theta,Q_0}^{\theta},\rho\right)\right]^{N}. \tag{3.15}$$

Consider the sequences

$$r_{i+1} = r_i + \sigma\rho 2^{-(i+1)}; \quad r_0 = \rho, \qquad t_{i+1} = t_i - \sigma t 2^{-(i+2)}; \quad t_0 = t/2,$$

$$Q^i = \Omega_{r_i}\times(t_i,t); \quad Q^0 = Q_\infty; \quad Q^\infty = Q_0, \quad Q^i \subset Q^{i+1}, \quad i = 0,1,\ldots$$

Denote $Y_i = \|u\|_{\infty,Q^i}$. Then inequality (3.15) applied to a pair of cylinders $Q^i \subset Q^{i+1}$ takes the form

$$Y_i \leq \gamma\|u\|_{\theta,Q_0}^{\frac{\theta}{q+1}} Y_{i+1}^{\frac{(2-m-\lambda)(N+\lambda+1)}{(\lambda+1)(q+1)}} \sigma^{-i\frac{N+\lambda+1}{q+1}} t^{-\frac{N+\lambda+1}{(\lambda+1)(q+1)}} \times\left[\omega\left(\frac{Y_i^{-\theta}}{t\sigma^{\lambda+1}}\|u\|_{\theta,Q_0}^{\theta},\rho\right)\right]^{\frac{N}{q+1}}. \tag{3.16}$$

Applying the Young inequality with exponents $\frac{(\lambda+1)(q+1)}{(2-m-\lambda)(N+\lambda+1)}$, $\frac{(\lambda+1)(q+1)}{(\lambda+1)(q+1)-(2-m-\lambda)(N+\lambda+1)} = \frac{(\lambda+1)(q+1)}{K_\theta}$ to the right-hand side of (3.16), we get

$$Y_i \leq \delta Y_{i+1} + \gamma(\delta)\sigma^{-i\frac{(N+\lambda+1)(\lambda+1)}{K_\theta}} t^{-\frac{N+\lambda+1}{K_\theta}}\|u\|_{\theta,Q_0}^{\frac{\theta(\lambda+1)}{K_\theta}} \times\left[\omega\left(\frac{Y_0^{-\theta}}{t\sigma^{\lambda+1}}\|u\|_{\theta,Q_0}^{\theta},\rho\right)\right]^{\frac{N(\lambda+1)}{K_\theta}}. \tag{3.17}$$

Denoting $\gamma(\delta)\sigma^{-i\frac{(N+\lambda+1)(\lambda+1)}{K_\theta}} = \gamma b^i$, $b > 1$, we write (3.17) as

$$Y_i \le \delta Y_{i+1} + \gamma b^i t^{-\frac{N+\lambda+1}{K_\theta}} \|u\|_{\theta,Q_0}^{\frac{\theta(\lambda+1)}{K_\theta}} \left[\omega\left(\frac{Y_0^{-\theta}}{t}\|u\|_{\theta,Q_0}^{\theta}, \rho\right)\right]^{\frac{N(\lambda+1)}{K_\theta}}. \quad (3.18)$$

We now iterate (3.18) to get

$$Y_0 \le \delta^{i+1} Y_{i+1} + \sum_{k=0}^{i} (b\delta)^k \gamma t^{-\frac{N+\lambda+1}{K_\theta}} \|u\|_{\theta,Q_0}^{\frac{\theta(\lambda+1)}{K_\theta}} \left[\omega\left(\frac{Y_0^{-\theta}}{t}\|u\|_{\theta,Q_0}^{\theta}, \rho\right)\right]^{\frac{N(\lambda+1)}{K_\theta}},$$

$i = 0, 1, 2\ldots$ Choosing now $\delta = \frac{1}{2b}$ and letting $i \to \infty$, we get

$$Y_0 \le \gamma t^{-\frac{N+\lambda+1}{K_\theta}} \|u\|_{\theta,Q_0}^{\frac{\theta(\lambda+1)}{K_\theta}} \left[\omega\left(\frac{Y_0^{-\theta}}{t}\|u\|_{\theta,Q_0}^{\theta}, \rho\right)\right]^{\frac{N(\lambda+1)}{K_\theta}}.$$

Thus,

$$\|u\|_{\infty,Q_\infty} \le \gamma t^{-\frac{N+\lambda+1}{K_\theta}} \|u\|_{\theta,Q_0}^{\frac{\theta(\lambda+1)}{K_\theta}} \left[\omega\left(\frac{\|u\|_{\infty,Q_\infty}^{-\theta}}{t}\|u\|_{\theta,Q_0}^{\theta}, \rho\right)\right]^{\frac{N(\lambda+1)}{K_\theta}}. \quad (3.19)$$

Finally, the desired estimate (3.1) follows from (3.19) and the definition of ω. ☐

Remark 3.1. The proposition is still valid if we replace Ω_ρ by $A_\rho = \Omega_{2\rho} \setminus \Omega_\rho$ and $\Omega_{\rho(1+\sigma)}$ by $\tilde{A}_\rho = \Omega_{4\rho} \setminus \Omega_{\rho/2}$.

4. Local estimate of the norm L_1 of a solution

Proposition 4.1. *Let $u_0 \ge 0$, $u_0 \in L_{1,loc}(\Omega)$. Then, for any $t > 0$, $\rho \ge 1$, the following estimate holds true:*

$$\sup_{0<\tau<t} \int_{\Omega_\rho} u(x,\tau)\,dx \le \gamma\left(\left[\frac{t}{\rho^{\lambda+1}}\right]^{\frac{1}{2-m-\lambda}} |\Omega_{2\rho}|_N + \int_{\Omega_{2\rho}} u_0\,dx\right). \quad (4.1)$$

Proof. Let $\zeta(x)$ be a nonnegative smooth cutoff function of $\Omega_{2\rho}$, which is equal to one in Ω_ρ and such that $|D\zeta| \le \gamma\rho^{-1}$. Denote $Q_\rho = \Omega_\rho \times (0,t)$.

Multiplying (1.1) by $\zeta(x)^{\lambda+1}$ and integrating over $Q_{2\rho}$, we obtain

$$\sup_{0<\tau<t} \int_{\Omega_\rho} u(x,\tau)\,dx \le \int_{\Omega_{2\rho}} u_0\,dx + \gamma\rho^{-1} \iint_{Q_{2\rho}} |Du|^\lambda\, u^{m-1}\, \zeta^\lambda\,dx\,d\tau. \quad (4.2)$$

Applying the Hölder estimate, we estimate the second integral on the right-hand side of (4.2):

$$\iint\limits_{Q_{2\rho}} |Du|^{\lambda}\ u^{m-1}\ \zeta^{\lambda}\, dx\, d\tau$$

$$\leq \Bigg(\iint\limits_{Q_{2\rho}} |Du|^{\lambda+1}\ u^{m-1}\ (t-\tau)^{\frac{1}{\lambda+1}}\ (u+\epsilon)^{-\theta}\ \zeta^{\lambda+1}\, dx\, d\tau \Bigg)^{\frac{\lambda}{\lambda+1}}$$

$$\times \Bigg(\iint\limits_{Q_{2\rho}} u^{m-1}\ (t-\tau)^{-\frac{\lambda}{\lambda+1}}\ (u+\epsilon)^{\theta\lambda}\, dx\, d\tau \Bigg)^{\frac{1}{\lambda+1}}. \quad (4.3)$$

To estimate the second integral on the right-hand side of (4.3), we multiply Eq. (1.1) by $\phi(x,\tau) = (t-\tau)^{\frac{1}{\lambda+1}}(u+\epsilon)^{1-\theta}\zeta^{\lambda+1}$, where $\zeta = \zeta(x)$ and θ: $1 < \theta < 2$ will be chosen later on, and integrate by parts over $Q_{2\rho}$. Then, performing simple calculations, we get

$$\iint\limits_{Q_{2\rho}} |Du|^{\lambda+1}\ u^{m-1}\ (t-\tau)^{\frac{1}{\lambda+1}}\ (u+\epsilon)^{-\theta}\ \zeta^{\lambda+1}\, dx\, d\tau$$

$$\leq \gamma\Big(1 + \frac{t}{\rho^{\lambda+1}\epsilon^{2-m-\lambda}}\Big) \iint\limits_{Q_{2\rho}} (u+\epsilon)^{2-\theta}\ (t-\tau)^{-\frac{\lambda}{\lambda+1}}\, dx\, d\tau. \quad (4.4)$$

Taking (4.4) into account, we have

$$\iint\limits_{Q_{2\rho}} |Du|^{\lambda}\ u^{m-1}\ \zeta^{\lambda}\, dx\, d\tau$$

$$\leq \gamma\Bigg[\Big(1 + \frac{t}{\rho^{\lambda+1}\epsilon^{2-m-\lambda}}\Big) \iint\limits_{Q_{2\rho}} (u+\epsilon)^{2-\theta}\ (t-\tau)^{-\frac{\lambda}{\lambda+1}}\, dx\, d\tau\Bigg]^{\frac{\lambda}{\lambda+1}}$$

$$\times \Bigg[\iint\limits_{Q_{2\rho}} (t-\tau)^{-\frac{\lambda}{\lambda+1}}\ (u+\epsilon)^{m-1+\theta\lambda}\, dx\, d\tau\Bigg]^{\frac{1}{\lambda+1}}. \quad (4.5)$$

Set $\theta = \frac{3-m}{\lambda+1}$. Then relation (4.5) yields

$$\iint\limits_{Q_{2\rho}} |Du|^{\lambda}\ u^{m-1}\ \zeta^{\lambda}\, dx\, d\tau$$

$$\leq \gamma\Big(1+\frac{t}{\rho^{\lambda+1}\epsilon^{2-m-\lambda}}\Big)^{\frac{\lambda}{\lambda+1}} \iint\limits_{Q_{2\rho}} (t-\tau)^{-\frac{\lambda}{\lambda+1}} \, (u+\epsilon)^{\frac{2\lambda+m-1}{\lambda+1}} \, dx\, d\tau. \quad (4.6)$$

Denote

$$\rho_n = \rho\sum_{i=0}^{n} 2^{-i}, \qquad Q_n = \Omega_{\rho_n}\times(0,t), \qquad M_n = \sup_{0<\tau<t}\int\limits_{\Omega_{\rho n}} u(x,\tau)\,dx.$$

Next, by (4.6) from (4.2), we get

$$M_n \leq \int\limits_{\Omega_{2\rho}} u_0\,dx + \gamma\rho^{-1}\Big(1+\frac{t}{\rho^{\lambda+1}\epsilon^{2-m-\lambda}}\Big)^{\frac{\lambda}{\lambda+1}} \times \iint\limits_{Q_{n+1}} (t-\tau)^{-\frac{\lambda}{\lambda+1}} \, (u+\epsilon)^{\frac{2\lambda+m-1}{\lambda+1}} \, dx\, d\tau. \quad (4.7)$$

Then (4.7) implies

$$M_n \leq \int\limits_{\Omega_{2\rho}} u_0\,dx + \gamma\rho^{-1}\Big(1+\frac{t}{\rho^{\lambda+1}\epsilon^{2-m-\lambda}}\Big)^{\frac{\lambda}{\lambda+1}} \times 2^n\Bigg(\int\limits_0^t\int\limits_{\Omega_{\rho_{n+1}}} (t-\tau)^{-\frac{\lambda}{\lambda+1}} \, \epsilon^{\frac{2\lambda+m-1}{\lambda+1}} \, dx\, d\tau + t^{\frac{1}{\lambda+1}} \sup_{0<\tau<t}\int\limits_{\Omega_{\rho_{n+1}}} u^{\frac{2\lambda+m-1}{\lambda+1}}\,dx\Bigg). \quad (4.8)$$

Thus, relation (4.8) yields

$$M_n \leq \int\limits_{\Omega_{2\rho}} u_0\,dx + \gamma\rho^{-1}\Big(1+\frac{t}{\rho^{\lambda+1}\epsilon^{2-m-\lambda}}\Big)^{\frac{\lambda}{\lambda+1}} 2^n\Bigg(\epsilon^{\frac{2\lambda+m-1}{\lambda+1}} t^{\frac{1}{\lambda+1}} |\Omega_{\rho_{n+1}}|_N + t^{\frac{1}{\lambda+1}} \sup_{0<\tau<t}\int\limits_{\Omega_{\rho_{n+1}}} u^{\frac{2\lambda+m-1}{\lambda+1}}\,dx\Bigg). \quad (4.9)$$

Applying the Hölder inequality with exponents $\frac{\lambda+1}{2\lambda+m-1}$ and $\frac{\lambda+1}{2-m-\lambda}$ to the last integral on the right-hand side of (4.9), we obtain

$$M_n \le \int\limits_{\Omega_{2\rho}} u_0\,dx + \gamma 2^n \rho^{-1}\Big(1+\frac{t}{\rho^{\lambda+1}\epsilon^{2-m-\lambda}}\Big)^{\frac{\lambda}{\lambda+1}} \times \Big(\epsilon^{\frac{2\lambda+m-1}{\lambda+1}} t^{\frac{1}{\lambda+1}} |\Omega_{\rho_{n+1}}|_N + t^{\frac{1}{\lambda+1}} M_{n+1}^{\frac{2\lambda+m-1}{\lambda+1}} |\Omega_{\rho_{n+1}}|_N^{\frac{2-m-\lambda}{\lambda+1}}\Big), \quad (4.10)$$

or

$$M_n \le \int\limits_{\Omega_{2\rho}} u_0\,dx + \gamma 2^n \rho^{-1}\Big(1+\frac{t}{\rho^{\lambda+1}\epsilon^{2-m-\lambda}}\Big)^{\frac{\lambda}{\lambda+1}} \epsilon^{\frac{2\lambda+m-1}{\lambda+1}} t^{\frac{1}{\lambda+1}} |\Omega_{\rho_{n+1}}|_N + \gamma 2^n \rho^{-1}\Big(1+\frac{t}{\rho^{\lambda+1}\epsilon^{2-m-\lambda}}\Big)^{\frac{\lambda}{\lambda+1}} t^{\frac{1}{\lambda+1}} M_{n+1}^{\frac{2\lambda+m-1}{\lambda+1}} |\Omega_{\rho_{n+1}}|_N^{\frac{2-m-\lambda}{\lambda+1}}. \quad (4.11)$$

Next, we apply the Young inequality with exponents $\frac{\lambda+1}{2\lambda+m-1}$ and $\frac{\lambda+1}{2-m-\lambda}$ to the last term in (4.11):

$$M_n \le \delta M_{n+1} + \int\limits_{\Omega_{2\rho}} u_0\,dx + \gamma 2^n \rho^{-1}\left(1+\frac{t}{\rho^{\lambda+1}\epsilon^{2-m-\lambda}}\right)^{\frac{\lambda}{\lambda+1}} \epsilon^{\frac{2\lambda+m-1}{\lambda+1}} t^{\frac{1}{\lambda+1}} |\Omega_{\rho_{n+1}}|_N + \gamma(\delta)\left(2^n\rho^{-1}\left(1+\frac{t}{\rho^{\lambda+1}\epsilon^{2-m-\lambda}}\right)^{\frac{\lambda}{\lambda+1}} t^{\frac{1}{\lambda+1}}\right)^{\frac{\lambda+1}{2-m-\lambda}} |\Omega_{\rho_{n+1}}|_N. \quad (4.12)$$

Iterating inequality (4.12) and proceeding as in Proposition 3.1, we obtain

$$M_0 = \sup_{0<\tau<t} \int\limits_{\Omega_\rho} u(x,\tau)\,dx \le \gamma\left(\left[\rho^{-1}\left(1+\frac{t}{\rho^{\lambda+1}\epsilon^{2-m-\lambda}}\right)^{\frac{\lambda}{\lambda+1}} \epsilon^{\frac{2\lambda+m-1}{\lambda+1}} t^{\frac{1}{\lambda+1}} + \rho^{-\frac{\lambda+1}{2-m-\lambda}}\left(1+\frac{t}{\rho^{\lambda+1}\epsilon^{2-m-\lambda}}\right)^{\frac{\lambda}{2-m-\lambda}} t^{\frac{1}{2-m-\lambda}}\right] |\Omega_{2\rho}|_N + \int\limits_{\Omega_{2\rho}} u_0\,dx\right). \quad (4.13)$$

Let us choose

$$\epsilon = \left[\frac{t}{\rho^{\lambda+1}}\right]^{\frac{1}{2-m-\lambda}}.$$

Then (4.13) leads to

$$M_0 \leq \gamma\Bigg(\Bigg[\rho^{-1}\left[\frac{t}{\rho^{\lambda+1}}\right]^{\frac{2\lambda+m-1}{(\lambda+1)(2-m-\lambda)}} t^{\frac{1}{\lambda+1}} + \left[\frac{t}{\rho^{\lambda+1}}\right]^{\frac{1}{2-m-\lambda}}\Bigg]|\Omega_{2\rho}|_N + \int\limits_{\Omega_{2\rho}} u_0\, dx\Bigg). \quad (4.14)$$

Therefore, from (4.14), we get

$$M_0 \leq \gamma\left(\left[\frac{t}{\rho^{\lambda+1}}\right]^{\frac{1}{2-m-\lambda}}|\Omega_{2\rho}|_N + \int\limits_{\Omega_{2\rho}} u_0\, dx\right).$$

□

5. Bound of the moment

Proposition 5.1. *The following estimate holds true:*

$$\sup_{0<\tau<t} \int\limits_{\Omega\setminus\Omega_\rho} u(x,t)\,\frac{|x|}{V(|x|)}\,dx \leq \gamma\mu(0) + \gamma\left(\frac{t}{\rho^{2\lambda+m-1}}\right)^{\frac{1}{2-m-\lambda}}. \quad (5.1)$$

Proof. Choose the cutoff function as

$$\zeta(x) = \begin{cases} 1, & \text{in } \Omega\setminus\Omega_{2\rho}, \\ 0, & \text{in } \Omega_\rho, \end{cases}$$

and $|D\zeta| \leq \gamma/\rho$. Let $\eta = \zeta^{\lambda+1}F(x)$, where F is as that in Lemma 2.2. For any fixed $\rho \geq 1$, we obtain

$$\sup_{0<\tau<t} \int\limits_{\Omega\setminus\Omega_\rho} \zeta^{\lambda+1}\,u(x,t)\,\frac{|x|}{V(|x|)}\,dx \leq \gamma\mu(0)$$
$$+ \gamma\int\limits_0^t \int\limits_{\Omega_{2\rho}\setminus\Omega_\rho} u^{m-1}\,|Du|^\lambda\,\zeta^\lambda\,|D\zeta|\,F(x)\,dx\,d\tau$$
$$+ \gamma\int\limits_0^t \int\limits_{\Omega\setminus\Omega_\rho} u^{m-1}\,|Du|^\lambda\,\zeta^{\lambda+1}\,|DF(x)|\,dx\,d\tau$$

$$\le \gamma\mu(0) + \gamma \int_0^t \int_{\Omega\setminus\Omega_\rho} u^{m-1}\, |Du|^\lambda\, \frac{1}{V(|x|)}\, \zeta^\lambda \, dx\, d\tau. \quad (5.2)$$

To estimate (5.2), we proceed as in Proposition 4.1:

$$\int_0^t \int_{\Omega\setminus\Omega_\rho} u^{m-1}\, |Du|^\lambda\, \frac{1}{V(|x|)}\, \zeta^\lambda \, dx\, d\tau$$

$$\le \left(\int_0^t \int_{\Omega\setminus\Omega_\rho} (t-\tau)^{\frac{1}{\lambda+1}}\, u^{m-1}\, |Du|^{\lambda+1}\, \zeta^{\lambda+1}\, u^{-\theta}\, dx\, d\tau \right)^{\frac{\lambda}{\lambda+1}}$$

$$\times \left(\int_0^t \int_{\Omega\setminus\Omega_\rho} u^{m-1} \left[\frac{1}{V(|x|)}\right]^{\lambda+1} u^{\theta\lambda}\, (t-\tau)^{-\frac{\lambda}{\lambda+1}}\, dx\, d\tau \right)^{\frac{1}{\lambda+1}}. \quad (5.3)$$

To estimate the first integral on the right-hand side of (5.3), we multiply (1.1) by $(t-\tau)^{\frac{1}{\lambda+1}} u^{1-\theta} \zeta^{\lambda+1}$ and integrate. We have

$$\int_0^t \int_{\Omega\setminus\Omega_\rho} (t-\tau)^{\frac{1}{\lambda+1}}\, u^{m-1}\, |Du|^{\lambda+1}\, \zeta^{\lambda+1}\, u^{-\theta}\, dx\, d\tau$$

$$\le \gamma \int_0^t \int_{\Omega\setminus\Omega_\rho} (t-\tau)^{-\frac{\lambda}{\lambda+1}}\, u^{2-\theta}\, dx\, d\tau$$

$$+ \gamma \int_0^t \int_{\Omega\setminus\Omega_\rho} \frac{t}{\rho^{\lambda+1} u^{2-m-\lambda}}\, (t-\tau)^{-\frac{\lambda}{\lambda+1}}\, u^{2-\theta}\, dx\, d\tau.$$

We may assume

$$\frac{t}{\rho^{\lambda+1} u^{2-m-\lambda}} < 1,$$

and therefore

$$\int_0^t \int_{\Omega\setminus\Omega_\rho} (t-\tau)^{\frac{1}{\lambda+1}}\, u^{m-1}\, |Du|^{\lambda+1}\, \zeta^{\lambda+1}\, u^{-\theta}\, dx\, d\tau$$

$$\le \gamma t^{\frac{1}{\lambda+1}} \sup_{0<\tau<t} \int_{\Omega\setminus\Omega_\rho} \zeta^{\lambda+1}\, u^{2-\theta}\, dx.$$

Choose θ: $2-\theta = m-1+\theta\lambda$, i.e., $\theta = \frac{3-m}{\lambda+1}$. Then, taking the last estimate from (5.3) into account, we get

$$\int_0^t \int_{\Omega\setminus\Omega_\rho} u^{m-1}\,|Du|^\lambda\,\frac{1}{V(|x|)}\,\zeta^\lambda\,dx\,d\tau$$
$$\le \gamma t^{\frac{1}{\lambda+1}} \left(\sup_{0<\tau<t}\int_{\Omega\setminus\Omega_\rho} u^{\frac{2\lambda+m-1}{\lambda+1}}\,dx\right)^{\frac{\lambda}{\lambda+1}}$$
$$\times\left(\sup_{0<\tau<t}\int_{\Omega\setminus\Omega_\rho} u^{\frac{2\lambda+m-1}{\lambda+1}}\left[\frac{1}{V(|x|)}\right]^{\lambda+1} dx\,d\tau\right)^{\frac{1}{\lambda+1}}$$
$$\equiv \gamma t^{\frac{1}{\lambda+1}} \sup_{0<\tau<t}\left(J_1^{\frac{\lambda}{\lambda+1}} J_2^{\frac{1}{\lambda+1}}\right). \quad (5.4)$$

We now estimate J_i, $i=1,2$.

$$J_1 = \int_{\Omega\setminus\Omega_\rho}\left[\frac{|x|}{V(|x|)}\cdot u\right]^{\frac{2\lambda+m-1}{\lambda+1}}\left[\frac{|x|}{V(|x|)}\right]^{-\frac{2\lambda+m-1}{\lambda+1}} dx$$
$$\le \left(\int_{\Omega\setminus\Omega_\rho}\frac{|x|}{V(|x|)}\cdot u\,dx\right)^{\frac{2\lambda+m-1}{\lambda+1}}$$
$$\times\left(\int_{\Omega\setminus\Omega_\rho}\left[\frac{|x|}{V(|x|)}\right]^{-\frac{2\lambda+m-1}{2-m-\lambda}} dx\right)^{\frac{2-m-\lambda}{\lambda+1}}. \quad (5.5)$$

We have

$$\int_{\Omega\setminus\Omega_\rho}\left[\frac{|x|}{V(|x|)}\right]^{-\frac{2\lambda+m-1}{2-m-\lambda}} dx$$
$$= \int_\rho^\infty \left[\frac{\tau}{V(\tau)}\right]^{-\frac{2\lambda+m-1}{2-m-\lambda}} \frac{d}{d\tau}V(\tau)\,d\tau$$
$$\le \gamma\int_\rho^\infty F^{-\frac{2\lambda+m-1}{2-m-\lambda}}\,\frac{d}{d\tau}V(\tau)\,d\tau, \quad (5.6)$$

where F is the function defined in Lemma 2.2. Integrating by parts and applying (1.6) from (5.6), we get

$$\int\limits_{\Omega\setminus\Omega_\rho} \left[\frac{|x|}{V(|x|)}\right]^{-\frac{2\lambda+m-1}{2-m-\lambda}} dx \le \gamma\left[\frac{\tau}{V(\tau)}\right]^{-\frac{2\lambda+m-1}{2-m-\lambda}} V(\tau)\Big|_\rho^\infty$$

$$+\gamma\int\limits_\rho^\infty \left[\frac{\tau}{V(\tau)}\right]^{-\frac{2\lambda+m-1}{2-m-\lambda}-1} \frac{1}{V(\tau)}\, V(\tau)\, d\tau$$

$$\le \gamma\int\limits_\rho^\infty \left[\frac{\tau}{V(\tau)}\right]^{-\frac{2\lambda+m-1}{2-m-\lambda}-1} d\tau$$

$$\le \gamma\int\limits_\rho^\infty \left[\frac{V(\tau)}{\tau^{1-\beta}}\right]^{\frac{\lambda+1}{2-m-\lambda}} \tau^{-\beta\frac{\lambda+1}{2-m-\lambda}}\, d\tau$$

$$\le \gamma\left[\frac{V(\rho)}{\rho^{1-\beta}}\right]^{\frac{\lambda+1}{2-m-\lambda}} \int\limits_\rho^\infty \tau^{-\beta\frac{\lambda+1}{2-m-\lambda}}\, d\tau. \quad (5.7)$$

Therefore,

$$\int\limits_{\Omega\setminus\Omega_\rho} \left[\frac{|x|}{V(|x|)}\right]^{-\frac{2\lambda+m-1}{2-m-\lambda}} dx \le \gamma\rho\left[\frac{\rho}{V(\rho)}\right]^{-\frac{\lambda+1}{2-m-\lambda}}. \quad (5.8)$$

Inequalities (5.5) and (5.8) imply

$$J_1 \le \gamma\Bigg(\int\limits_{\Omega\setminus\Omega_\rho} \frac{|x|}{V(|x|)}\cdot u\, dx\Bigg)^{\frac{2\lambda+m-1}{\lambda+1}} \rho^{\frac{2-m-\lambda}{\lambda+1}}\left[\frac{\rho}{V(\rho)}\right]^{-1}. \quad (5.9)$$

Analogously,

$$J_2 \le \gamma\Bigg(\int\limits_{\Omega\setminus\Omega_\rho} \frac{|x|}{V(|x|)}\cdot u\, dx\Bigg)^{\frac{2\lambda+m-1}{\lambda+1}} \rho^{\lambda+1-\frac{2-m-\lambda}{\lambda+1}}\left[\frac{\rho}{V(\rho)}\right]^{\lambda}. \quad (5.10)$$

We now apply estimates (5.9) and (5.10) to (5.4) and obtain

$$\int\limits_0^t \int\limits_{\Omega\setminus\Omega_\rho} u^{m-1}\,|Du|^\lambda\,\frac{1}{V(|x|)}\,\zeta^\lambda\,dx\,d\tau$$

$$\le \gamma t^{\frac{1}{\lambda+1}} \sup_{0<\tau<t}\Bigg(\int\limits_{\Omega\setminus\Omega_\rho} \frac{|x|}{V(|x|)}\cdot u\, dx\Bigg)^{\frac{2\lambda+m-1}{\lambda+1}} \left(\rho^{\frac{2-m-\lambda}{\lambda+1}}\left[\frac{\rho}{V(\rho)}\right]^{-1}\right)^{\frac{\lambda}{\lambda+1}}$$

$$\times \left(\rho^{\lambda+1-\frac{2-m-\lambda}{\lambda+1}} \left[\frac{\rho}{V(\rho)}\right]^{\lambda}\right)^{\frac{1}{\lambda+1}}$$

$$= \gamma t^{\frac{1}{\lambda+1}} \sup_{0<\tau<t} \left(\int\limits_{\Omega\setminus\Omega_\rho} \frac{|x|}{V(|x|)} \cdot u\,dx \right)^{\frac{2\lambda+m-1}{\lambda+1}} \rho^{-\frac{2\lambda+m-1}{\lambda+1}}. \quad (5.11)$$

By Young's inequality, (5.2), and (5.11), we get

$$\sup_{0<\tau<t} \int\limits_{\Omega\setminus\Omega_\rho} \zeta^{\lambda+1} \frac{|x|}{V(|x|)} u(x,t)\,dx \le \gamma\mu(0) + \gamma\left(\frac{t}{\rho^{2\lambda+m-1}}\right)^{\frac{1}{2-m-\lambda}}.$$

□

6. Proof of Theorem 1.1

We approximate our problem by the sequence of problems

$$u_{nt} - \operatorname{div}(u_n^{m-1}|Du_n|^{\lambda-1}Du_n) = 0, \qquad \text{in } \Omega_n \times (0,\infty),$$

$$u_n(x,t) = 0, \qquad \text{on } (\partial\Omega_n \cap \Omega) \times (0,\infty),$$

$$u_n^{m-1}|Du_n|^{\lambda-1}\frac{\partial u_n}{\partial \overrightarrow{n}} = 0, \qquad \text{on } (\partial\Omega_n \cap \partial\Omega) \times (0,\infty),$$

$$u_n(x,0) = u_{0n} \qquad x \in \Omega_n.$$

Here, $u_n \ge 0,\ n \ge 1$; $u_{0n} \in C_\infty(\overline{\Omega}_n)$, and u_{0n} converges to u_0 in $L_1(\Omega)$. We may assume that

$$\|u_{0n}\|_{1,\Omega} \le \gamma\|u_0\|_{1,\Omega}, \qquad \int\limits_\Omega \frac{|x|}{V(|x|)} u_{0n}(x)\,dx \le \gamma\mu(0).$$

We define u_n on Ω, setting $u_n \equiv 0$ outside of Ω_n.

The classical compactness arguments of [16] (see also [9] and [14]) yield the global solvability of the above-described problem.

Next, we need the following estimate of u_n:

$$\|u(t)\|_{1,\Omega} \le \|u_0\|_{1,\Omega}, \qquad t > 0. \quad (6.1)$$

By Proposition 3.1 with $\theta = 1$ and (6.1), we get

$$\|u(t)\|_{\infty,\Omega_\rho} \le \gamma \max\Bigg(t^{-\frac{N}{K}} \|u_0\|_{1,\Omega}^{\frac{\lambda+1}{K}},$$

$$t^{-\frac{1}{2\lambda+m-1}} \|u_0\|_{1,\Omega}^{\frac{\lambda+1}{2\lambda+m-1}} \left[\frac{\rho}{V(\rho)}\right]^{\frac{\lambda+1}{2\lambda+m-1}}, \left[\frac{t}{\rho^{\lambda+1}}\right]^{\frac{1}{2-m-\lambda}} \Bigg). \quad (6.2)$$

Next, we use Remark 3.1. For $\rho \geq \gamma P(\tau)$, we have

$$\|u(t)\|_{\infty,A_\rho} \leq \gamma \max\Bigg(t^{-\frac{N}{k}} \|u_0\|_{1,\Omega}^{\frac{\lambda+1}{k}}, \quad t^{-\frac{1}{2\lambda+m-1}} \Bigg(\frac{\rho}{V(\rho)} \sup_{0<\tau<t} \int\limits_{\tilde{A}_\rho} u(x,t)\,dx \Bigg)^{\frac{\lambda+1}{2\lambda+m-1}} \Bigg). \tag{6.3}$$

We have

$$\frac{\rho}{V(\rho)} \sup_{0<\tau<t} \int\limits_{\tilde{A}_\rho} u(x,t)\,dx \leq \gamma \sup_{0<\tau<t} \int\limits_{\tilde{A}_\rho} \frac{|x|}{V(|x|)} u(x,t)\,dx. \tag{6.4}$$

From estimates (6.3), (6.4), and (6.2) with $\rho = P(\tau)$, we arrive at

$$\|u(t)\|_{\infty,\Omega} \leq \gamma \max\Bigg(t^{-\frac{N}{K}} \|u_0\|_{1,\Omega}^{\frac{\lambda+1}{K}}, \quad t^{-\frac{1}{2\lambda+m-1}} \Bigg[\sup_{0<\tau<t} \int\limits_{\Omega\setminus\Omega_\rho} \frac{|x|}{V(|x|)} u(x,t)\,dx \Bigg]^{\frac{\lambda+1}{2\lambda+m-1}}, \quad t^{-\frac{1}{2\lambda+m-1}} \|u_0\|_{1,\Omega}^{\frac{\lambda+1}{2\lambda+m-1}} \Bigg[\frac{P(\tau)}{V(P(\tau))} \Bigg]^{\frac{\lambda+1}{2\lambda+m-1}} \Bigg).$$

According to (5.1) with $\rho = P(\tau)$, we get (1.9). The right-hand side estimate of (1.11) follows from Remark 1.1. It remains to prove the lower bound in (1.11). By the mass conservation law for any $t > 0$, we have

$$\int\limits_{\Omega} u_0\,dx = \int\limits_{\Omega_R} u(x,t)\,dx + \int\limits_{\Omega\setminus\Omega_R} u(x,t)\,dx. \tag{6.5}$$

In view of (5.1), relation (6.5) yields

$$\int\limits_{\Omega} u_0\,dx \leq \|u(t)\|_{\infty,\Omega} V(R) + \frac{V(R)}{R} \int\limits_{\Omega\setminus\Omega_R} \frac{|x|}{V(|x|)}\, u\,dx$$
$$\leq \|u(t)\|_{\infty,\Omega} V(R) + \gamma \Bigg[\mu(0) + \Bigg(\frac{t}{R^{2\lambda+m-1}} \Bigg)^{\frac{1}{2-m-\lambda}} \Bigg] \frac{V(R)}{R}. \tag{6.6}$$

We put $R = CP(\tau) = CP(t\|u_0\|_{1,\Omega}^{m+\lambda-2})$ in (6.6). Then, by definition of $P(\tau)$ (see (1.10)), it follows that

$$\Bigg(\frac{t}{P(\tau)^{2\lambda+m-1}} \Bigg)^{\frac{1}{2-m-\lambda}} \frac{V(P(\tau))}{P(\tau)} = \|u_0\|_{1,\Omega}.$$

Finally, the lower bound in (1.11) for t large enough follows from (6.6). Theorem 1.1 is proved.

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