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On convergence theory for Beltrami equations

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Abstract. This paper is devoted to convergence theorems which play an important role in our scheme for deriving theorems on the existence of solutions of the Beltrami equations.

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1. Introduction

Let D be a domain in the complex plane $\mathbb{C}$, i.e., a connected and open subset of $\mathbb{C}$, and let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. The *Beltrami equation* is an equation of the form

$$f_{\overline{z}} = \mu(z) \cdot f_z, \tag{1.1}$$

where $f_{\overline{z}} = \overline{\partial} f = (f_x + i f_y)/2$, $f_z = \partial f = (f_x - i f_y)/2$, $z = x + iy$, and f_x and f_y are partial derivatives of f with respect to x and y, respectively. The function μ is called the *complex coefficient* and

$$K_\mu(z) = \frac{1 + |\mu(z)|}{1 - |\mu(z)|} \tag{1.2}$$

the *maximal dilatation* or, in short, the *dilatation* of Eq. (1.1). The Beltrami equation (1.1) is said to be *degenerate* if $\operatorname{ess\,sup} K_\mu(z) = \infty$.

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Recall that a function $f : D \to \mathbb{C}$ is *absolutely continuous on lines*, abbr. $f \in \mathbf{ACL}$, if, for every closed rectangle R in D, whose sides are parallel to the coordinate axes, $f|R$ is absolutely continuous on almost all line segments in R which are parallel to the sides of R. In particular, f is ACL (possibly modified on a set of Lebesgue measure zero) if it belongs to the Sobolev class $W^{1,1}_{loc}$ of locally integrable functions with locally integrable first generalized derivatives and, conversely, if $f \in$ ACL has locally integrable first partial derivatives, then $f \in W^{1,1}_{loc}$, see, e.g., 1.2.4 in [9]. For a sense-preserving ACL homeomorphism $f : D \to \mathbb{C}$, the Jacobian $J_f(z) = |f_z|^2 - |f_{\overline{z}}|^2$ is nonnegative a.e. In this case, the *complex dilatation* μ_f of f is the ratio $\mu(z) = f_{\overline{z}}/f_z$, if $f_z \neq 0$ and $\mu(z) = 0$ otherwise, and the *dilatation* $K_f(z)$ of f is $K_\mu(z)$, see (1.2). Note that $|\mu(z)| \leq 1$ a.e. and $K_\mu(z) \geq 1$ a.e. Given a function $Q : D \to [1, \infty]$, a sense-preserving ACL homeomorphism $f : D \to \mathbb{C}$ is called a $Q(z)$-*quasiconformal mapping* if $K_f(z) \leq Q(z)$ a.e., see [11].

Recall also that, given a family of paths Γ in $\overline{\mathbb{C}}$, a Borel function $\rho : \overline{\mathbb{C}} \to [0, \infty]$ is called *admissible* for Γ, abbr. $\rho \in \operatorname{adm}\Gamma$, if

$$\int\limits_{\gamma} \rho(z)\,|dz| \geq 1 \tag{1.3}$$

for each $\gamma \in \Gamma$. The *modulus* of Γ is defined by

$$M(\Gamma) = \inf_{\rho \in \operatorname{adm}\Gamma} \int\limits_{\mathbb{C}} \rho^2(z)\,dx\,dy. \tag{1.4}$$

Motivated by the ring definition of quasiconformality in [6], we introduce the following notion that extends and localizes the notion of a quasiconformal mapping. Let D be a domain in $\mathbb{C}$, $z_0 \in \overline{D}$, and $Q : D \to [0, \infty]$ a measurable function. We say that a homeomorphism $f : D \to \overline{\mathbb{C}}$ is a *ring Q-homeomorphism* at the point z_0 if

$$M(\Delta(fC_0, fC_1, fD)) \leq \int\limits_{A \cap D} Q(z) \cdot \eta^2(|z - z_0|)\,dx\,dy \tag{1.5}$$

for every ring

$$A = A(z_0, r_1, r_2) = \{z \in \mathbb{C} : r_1 < |z - z_0| < r_2\}, \quad 0 < r_1 < r_2 < \infty,$$

and for every continua C_0 and C_1 in D which belong to the different components of the complement to the ring A in $\overline{\mathbb{C}}$, containing z_0 and ∞, respectively, and for every measurable function $\eta : (r_1, r_2) \to [0, \infty]$ such that

$$\int_{r_1}^{r_2} \eta(r)\, dr = 1. \tag{1.6}$$

2. On convergence of Sobolev's functions

First of all, let us recall the necessary definitions and basic facts on the Sobolev spaces $W^{l,p}$ and $L^p, p \in [1, \infty]$. Given an open set U in $\mathbb{R}^n$ and a natural number l, $C_0^l(U)$ denotes a collection of all functions $\varphi : U \to \mathbb{R}$ with compact support having all partial continuous derivatives of order at most l in U. $\varphi \in C_0^\infty(U)$ if $\varphi \in C_0^l(U)$ for all $l = 1, 2, \ldots$. A vector $\alpha = (\alpha_1, \ldots, \alpha_n)$ with natural coordinates is called a *multiindex.* Every multiindex α is associated with the differential operator $D^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}$ where $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

Now, let u and $v : U \to \mathbb{R}$ be locally integrable functions. The function v is called the *generalized derivative* $D^\alpha u$ of u if

$$\int_\Omega u\, D^\alpha \varphi\, dx = (-1)^{|\alpha|} \int_\Omega v\, \varphi\, dx \qquad \forall\, \varphi \in C_0^\infty. \tag{2.1}$$

The concept of the generalized derivative was introduced by Sobolev in [13]. The *Sobolev class* $W^{l,p}(\Omega)$ consists of all functions $u : U \to \mathbb{R}$ in $L^p(U),\ p \geq 1$, with generalized derivatives of order l summable of order p. A function $u : U \to \mathbb{R}$ belongs to $W^{l,p}_{loc}(U)$ if $u \in W^{l,p}(U_*)$ for every open set U_* with compact closure $\overline{U}_* \subset U$. A similar notion introduced for vector-functions $f : U \to \mathbb{R}^m$ in the component-wise sense.

A function $\omega : \mathbb{R}^n \to \mathbb{R}$ with a compact support in $\mathbb{B}$ is called a *Sobolev averaging kernel* if ω is nonnegative, belongs to $C_0^\infty(\mathbb{R}^n)$, and

$$\int_{\mathbb{R}^n} \omega(x)\, dx = 1. \tag{2.2}$$

The well-known example of such a function is $\omega(x) = \gamma\varphi(|x|^2 - \frac{1}{4})$, where $\varphi(t) = e^{1/t}$ for $t < 0$ and $\varphi(t) \equiv 0$ for $t \geq 0$, and the constant γ is chosen so that (2.2) holds. Later on, we use only ω depending on $|x|$.

Let U be a nonempty bounded open subset of $\mathbb{R}^n$ and $f \in L^1(U)$. Extending f by zero outside of U, we set

$$f_h = \omega_h * f = \int\limits_{|y|\le 1} f(x+hy)\,\omega(y)\,dy = \frac{1}{h^n}\int\limits_{U} f(z)\,\omega\left(\frac{z-x}{h}\right)\,dz, \quad (2.3)$$

where $f_h = \omega_h * f$, $\omega_h(y) = \omega\left(y/h\right),\ h > 0$, is called the *Sobolev mean function* for f. It is known that $f_h \in C_0^\infty(\mathbb{R}^n)$, $\|f_h\|_p \le \|f\|_p$ for every $f \in L^p(U)$, $p \in [1,\infty]$, and $f_h \to f$ in $L^p(U)$ for every $f \in L^p(U)$, $p \in [1,\infty)$ (see, e.g., 1.2.1 in [9]). It is clear that if f has a compact support in U, then f_h also has a compact support in U for small enough h.

A sequence $\varphi_k \in L^1(U)$ is called *weakly fundamental* if

$$\lim_{k_1,k_2\to\infty}\int\limits_{U}\Phi(x)\,(\varphi_{k_1}(x)-\varphi_{k_2}(x))\,dx = 0 \qquad \forall\,\Phi \in L^\infty(U)$$

It is well known that the space $L^1(U)$ is *weakly complete*, i.e., every weakly fundamental sequence $\varphi_k \in L^1(U)$ converges weakly in $L^1(U)$ (see, e.g., Theorem IV.8.6 in [3]). Give also the following useful statement (see, e.g., Theorem 1.2.5 in [7]).

Proposition 2.1. *Let f and $g \in L^1_{loc}(U)$. If*

$$\int f\,\varphi\,dx = \int g\,\varphi\,dx \qquad \forall\,\varphi \in C_0^\infty(U), \quad (2.4)$$

then $f = g$ a.e.

Later on, in comparison with [11], we apply the following lemma instead of Lemma III.3.5 in [10] which is not valid for $p = 1$.

Lemma 2.1. *Let U be a bounded open set in $\mathbb{R}^n$, and let $f_k : U \to \mathbb{R}$ be a sequence of functions of the class $W^{1,1}(U)$. Suppose that $f_k \to f$ as $k \to \infty$ weakly in $L^1(U)$, $\partial f_k/\partial x_j$, $k = 1,2,\ldots$, $j = 1,2,\ldots,n$ are uniformly bounded in $L^1(U)$ and their indefinite integrals are absolutely equicontinuous. Then $f \in W^{1,1}(U)$ and $\partial f_k/\partial x_j \to \partial f/\partial x_j$ as $k \to \infty$ weakly in $L^1(U)$.*

Remark 2.1. The weak convergence $f_k \to f$ in $L^1(U)$ implies that

$$\sup_k \|f_k\|_1 < \infty$$

(see, e.g., IV.8.7 in [3]). The latter together with

$$\sup_k \|\partial f_k/\partial x_j\|_1 < \infty,$$

$j = 1, 2, \ldots, n$, implies that $f_k \to f$ by the norm in L^q for every $1 < q < n/(n-1)$, the limit function f belongs to $BV(U)$, the class of functions of bounded variation, but, generally speaking, not to the class $W^{1,1}(U)$ (see, e.g., Remark in 4.6 and Theorem 5.2.1 in [4]). Thus, the additional condition of Lemma 2.1 on absolute equicontinuity of the indefinite integrals of $\partial f_k/\partial x_j$ is essential (cf. also Remark to Theorem I.2.4 in [10]).

Proof of Lemma 2.1. It is known that the space L^1 is weakly complete (see Theorem IV.8.6 in [3]). Thus, it suffices to prove that the sequences $\frac{\partial f_k}{\partial x_j}$ are weakly fundamental in L^1.

Indeed, by the definition of generalized derivatives, we have

$$\int\limits_U \varphi(x) \frac{\partial f_k}{\partial x_j}\, dx = -\int\limits_U f_k(x) \frac{\partial \varphi}{\partial x_j}\, dx \qquad \forall\, \varphi \in C_0^\infty(U). \tag{2.5}$$

Note that the integrals on the right-hand side in (2.5) are bounded linear functionals in $L^1(U)$, and the sequence f_k is weakly fundamental in $L^1(U)$ because $f_k \to f$ weakly in $L^1(U)$. Hence, in particular,

$$\int\limits_U \varphi(x) \left(\frac{\partial f_{k_1}}{\partial x_j} - \frac{\partial f_{k_2}}{\partial x_j} \right) dx \to 0 \qquad \forall\, \varphi \in C_0^\infty(U)$$

as k_1 and $k_2 \to \infty$.

Now, let $\Phi \in L^\infty(U)$. Then $\|\Phi_h\|_\infty \le \|\Phi\|_\infty$ and $\Phi_h \to \Phi$ in the norm of $L^1(U)$ for its Sobolev mean functions Φ_h, and, hence, $\Phi_h \to \Phi$ in measure as $h \to 0$. Set $\varphi_m = \Phi_{h_m}$, where $\Phi_{h_m} \to \Phi$ a.e. as $m \to \infty$. Considering the restrictions of Φ to compacta in U, we may assume that $\varphi_m \in C_0^\infty(U)$. By the Egoroff theorem, $\varphi_m \to \Phi$ uniformly on a set $S \subset U$ such that $|U \setminus S| < \delta$, where $\delta > 0$ can be arbitrary small (see, e.g., III.6.12 in [3]). Given $\varepsilon > 0$, we have

$$\left| \int\limits_S (\Phi(x) - \varphi_m(x)) \left(\frac{\partial f_{k_1}}{\partial x_j} - \frac{\partial f_{k_2}}{\partial x_j} \right) dx \right|$$
$$\le 2 \cdot \max_{x \in S} |\, \Phi(x) - \varphi_m(x)| \cdot \sup_{k=1,2,\ldots} \int\limits_U \left| \frac{\partial f_k}{\partial x_j} \right| dx \le \frac{\varepsilon}{3}$$

for all large enough m. Choosing one such m, we have

$$\left| \int\limits_U \varphi_m(x) \left(\frac{\partial f_{k_1}}{\partial x_j} - \frac{\partial f_{k_2}}{\partial x_j} \right) dx \right| \le \frac{\varepsilon}{3}$$

for k_1 and k_2 large enough. By the absolute equicontinuity of the indefinite integrals of $\partial f_k/\partial x_j$, there is $\delta > 0$ such that

$$\int\limits_E \left|\frac{\partial f_k}{\partial x_j}\right| dx \leq \frac{1}{12}\,\frac{\varepsilon}{\|\Phi\|_\infty}$$

for all $k = 1, 2, \ldots$ and every measurable set $E \subset U$ with $|E| < \delta$ (see IV.8.10 and IV.8.11 in [3]). Setting $E = U \setminus S$, we obtain

$$\left|\int\limits_U \Phi(x)\left(\frac{\partial f_{k_1}}{\partial x_j} - \frac{\partial f_{k_2}}{\partial x_j}\right) dx\right| \leq I_1 + I_2 + I_3,$$

where

$$I_1 = \left|\int\limits_E (\Phi(x) - \varphi_m(x))\left(\frac{\partial f_{k_1}}{\partial x_j} - \frac{\partial f_{k_2}}{\partial x_j}\right) dx\right|,$$

$$I_2 = \left|\int\limits_S (\Phi(x) - \varphi_m(x))\left(\frac{\partial f_{k_1}}{\partial x_j} - \frac{\partial f_{k_2}}{\partial x_j}\right) dx\right|,$$

$$I_3 = \left|\int\limits_U \varphi_m(x)\left(\frac{\partial f_{k_1}}{\partial x_j} - \frac{\partial f_{k_2}}{\partial x_j}\right) dx\right|,$$

and, hence by the above arguments,

$$\left|\int\limits_U \Phi(x)\left(\frac{\partial f_{k_1}}{\partial x_j} - \frac{\partial f_{k_2}}{\partial x_j}\right) dx\right| \leq \varepsilon$$

for large enough k_1 and k_2. Thus, $\frac{\partial f_k}{\partial x_j}$ is weakly fundamental in $L^1(U)$, and hence $\frac{\partial f_k}{\partial x_j}$ converges weakly in $L^1(U)$ just to $\frac{\partial f}{\partial x_j}$ by (2.5), see Proposition 2.1. □

3. On convergence of ACL homeomorphisms

Theorem 3.1. *Let D be a domain in $\mathbb{C}$, and let $f_n : D \to \mathbb{C}$ be a sequence of sense-preserving* ACL *homeomorphisms with complex dilatations μ_n such that*

$$\frac{1 + |\mu_n(z)|}{1 - |\mu_n(z)|} \leq Q(z) \in L^1_{\text{loc}} \qquad \forall\, n = 1, 2, \ldots \tag{3.1}$$

If $f_n \to f$ uniformly on each compact set in D, where f is a homeomorphism, then $f \in$ ACL and ∂f_n and $\overline{\partial} f_n$ converge weakly in L^1_{loc} to ∂f and $\overline{\partial} f$, respectively. Moreover, if, in addition, $\mu_n \to \mu$ a.e., then $\overline{\partial} f = \mu \partial f$ a.e.

Remark 3.1. In fact, it is easy to show that, under condition (3.1), f_n and f belong to $W^{1,1}_{\mathrm{loc}}$ (see, e.g., (3.2) below and II.3.27 in [3]). Moreover, if, in addition, $Q \in L^p_{\mathrm{loc}}$, then f_n and f belong to $W^{1,s}_{\mathrm{loc}}$, $\partial f_n \to \partial f$ and $\overline{\partial} f_n \to \overline{\partial} f$ weakly in L^s_{loc}, where $s = 2p/(1+p)$ (see, e.g., Lemma 2.2 in [1]). Finally, f is a $Q(z)$-quasiconformal mapping, see [11].

Proof of Theorem 3.1. To prove the first part of the theorem, it suffices by Lemma 2.1 to show that ∂f_n and $\overline{\partial} f_n$ are uniformly bounded in L^1_{loc} and have locally absolute equicontinuous indefinite integrals. So, let C be a compact set in D, and let V be an open set with their compact closure $\overline{V}$ in D such that $C \subset V$, say $V = \{z \in D : \mathrm{dist}(z, C) < r\}$, where $r < \mathrm{dist}(C, \partial D)$. Note that

$$|\overline{\partial} f_n| \le |\partial f_n| \le |\partial f_n| + |\overline{\partial} f_n| \le Q^{1/2}(z) \cdot J_n^{1/2}(z) \qquad \text{a.e.},$$

where J_n is the Jacobian of f_n. Consequently, by the Hölder inequality and Lemma III.3.3 in [8],

$$\int_E |\partial f_n| \; dx\, dy \le \left| \int\limits_E Q(z)\, dx\, dy \right|^{1/2} |f_n(C)|^{1/2}$$

for every measurable set $E \subseteq C$. Hence, by the uniform convergence of f_n to f on C,

$$\int\limits_E |\partial f_n| \; dx\, dy \le \left| \int\limits_E Q(z)\, dx\, dy \right|^{1/2} |f(V)|^{1/2} \tag{3.2}$$

for large enough n and, thus, the first part of the proof is completed.

We now assume that $\mu_n(z) \to \mu(z)$ a.e. Set $\zeta(z) = \overline{\partial} f(z) - \mu(z)\, \partial f(z)$ and show that $\zeta(z) = 0$ a.e. Indeed, for every disk B with $\overline{B} \subset D$, by the triangle inequality

$$\left| \int\limits_B \zeta(z)\, dx\, dy \right| \le I_1(n) + I_2(n),$$

where

$$I_1(n) = \left| \int\limits_B \left(\overline{\partial} f(z) - \overline{\partial} f_n(z) \right) \, dx\, dy \right|$$

and

$$I_2(n) = \left| \int\limits_B (\mu(z)\,\partial f(z) - \mu_n(z)\,\partial f_n(z))\; dx\, dy \right|$$

Note that $I_1(n) \to 0$ because $\overline{\partial} f_n \to \overline{\partial} f$ weakly in $L^1_{\rm loc}$ by the first part of the proof. Next, $I_2(n) = I_2'(n) + I_2''(n)$, where

$$I_2'(n) = \left| \int\limits_B \mu(z)(\partial f(z) - \partial f_n(z))\, dx\, dy \right|$$

and

$$I_2''(n) = \left| \int\limits_B (\mu(z) - \mu_n(z))\partial f_n(z)\, dx\, dy \right|.$$

Again, by the weak convergence $\partial f_n \to \partial f$ in $L^1_{\rm loc}$, we have that $I_2'(n) \to 0$ because $\mu \in L^\infty$. Moreover, given $\varepsilon > 0$, by (3.2)

$$\int\limits_E |\partial f_n(z)|\, dx\, dy < \varepsilon, \quad n = 1, 2, \dots, \tag{3.3}$$

whenever E is every measurable set in B with $|E| < \delta$ for small enough $\delta > 0$.

Further, by the Egoroff theorem (see, e.g., III.6.12 in [3]), $\mu_n(z) \to \mu(z)$ uniformly on some set $S \subset B$ such that $|E| < \delta$, where $E = B \setminus S$. Hence, $|\mu_n(z) - \mu(z)| < \varepsilon$ on S and, by (3.3),

$$\begin{aligned} I_2''(n) &\le \varepsilon \int\limits_S |\partial f_n(z)|\, dx\, dy + 2 \int\limits_E |\partial f_n(z)|\, dx\, dy \\ &\le \varepsilon \left\{ \left(\int\limits_B Q(z)\, dx\, dy \right)^{1/2} \cdot |f(\lambda B)|^{1/2} + 2 \right\} \end{aligned}$$

for some $\lambda > 1$ and for all large enough n, i.e. $I_2''(n) \to 0$, because $\varepsilon > 0$ is arbitrary. Thus, $\int_B \zeta(z)\, dx\, dy = 0$ for all disks B with $\overline{B} \subset D$. Finally, by the Lebesgue theorem on the differentiability of indefinite integrals (see, e.g., IV(6.3) in [12]), $\zeta(z) = 0$ a.e. in D. □

Proposition 3.1. *Let D be a domain in $\overline{\mathbb{C}}$ and $f_n : D \to \overline{\mathbb{C}}$, n*

Proof. Indeed, suppose that $f(z_1) = f(z_2)$ for some $z_1 \neq z_2$ in D. For small $t > 0$, let D_t be a disk of the spherical radius t centered at z_1 such that $\overline{D_t} \subset D$ and $z_2 \notin \overline{D_t}$. Then, for all n, $f_n(\partial D_t)$ separates $f_n(z_1)$ from $f_n(z_2)$ and, hence, $s(f_n(z_1), f_n(\partial D_t)) < s(f_n(z_1), f_n(z_2))$. Thus, for every such t, there is $\zeta_n(t) \in \partial D_t$ such that $s(f_n(z_1), f_n(\zeta_n(t)) < s(f_n(z_1), f_n(z_2))$. Moreover, there is a subsequence $\zeta_{n_k}(t) \to \zeta_0(t) \in \partial D_t$, because the circle ∂D_t is a compact set. However, the locally uniform convergence $f_{n_k} \to f$ implies that $f_{n_k}(\zeta_{n_k}(t)) \to f(\zeta_0(t))$ (see, e.g., [2, p. 268]). Consequently, $s(f(z_1), f(\zeta_0(t)) \leq s(f(z_1), f(z_2))$. Then, since $f(z_1) = f(z_2)$, there is a point $z_t = \zeta_0(t)$ on ∂D_t such that $f(z_1) = f(z_t)$ for every small t contradicting the discreteness of f. □

Corollary 3.1. *Let D be a domain in $\mathbb{C}$ and $f_n : D \to \overline{\mathbb{C}}$, $n = 1, 2, \ldots,$ a sequence of quasiconformal mappings which satisfy* (3.1)*. If $f_n \to f$ locally uniformly, then either f is constant or f is an ACL homeomorphism, and ∂f_n and $\bar\partial f_n$ converge weakly in $L^1_{loc}(D \setminus \{f^{-1}(\infty)\})$ to ∂f and $\bar\partial f$, respectively. If, in addition, $\mu_n \to \mu$ a.e., then $\bar\partial f = \mu \partial f$ a.e.*

Proof. Consider the case where f is not constant in D. Let us show that then no point in D has a neighborhood of the constancy for f. Indeed, assume that there is at least one point $z_0 \in D$ such that $f(z) \equiv c$ for some $c \in \overline{\mathbb{C}}$ in a neighborhood of z_0. Note that the set Ω_0 of such points z_0 is open. The set $E_c = \{z \in D : s(f(z), c) > 0\}$, where s is the spherical (chordal) distance in $\overline{\mathbb{C}}$, is also open in view of continuity of f and not empty in the considered case. Thus, there is a point $z_0 \in \partial\Omega_0 \cap D$ because D is connected. By continuity of f, we have $f(z_0) = c$. However, by construction, there is a point $z_1 \in E_c = D \setminus \overline{\Omega_0}$ such that $|z_0 - z_1| < r_0 = \text{dist}\,(z_0, \partial D)$ and, thus, by the lower estimate of the distance $s(f(z_0), f(z))$ in Lemma 3.12 from [11], we obtain a contradiction for $z \in \Omega_0$. Then, again by Lemma 3.12 in [11], we obtain that f is discrete, and f is a homeomorphism by Proposition 3.1. All other assertions follow from Theorem 3.1. □

4. On convergence of ring Q-homeomorphisms

Theorem 4.1. *Let $f_n : D \to \overline{\mathbb{C}}$, $n = 1, 2, \ldots,$ be a sequence of ring Q-homeomorphisms at a point $z_0 \in \overline{D}$. If f_n converges locally uniformly to a homeomorphism $f : D \to \overline{\mathbb{C}}$, then f is also a ring Q-homeomorphism at z_0.*

Proof. Note first that every point $w_0 \in D' = fD$ belongs to $D'_n = f_n D$ for all $n \geq N$ together with $\overline{D(w_0, \varepsilon)}$, where $D(w_0, \varepsilon) = \{w \in \overline{\mathbb{C}} : s(w, w_0) < \varepsilon\}$ for some $\varepsilon > 0$. Indeed, set $\delta = \frac{1}{2}\, s(z_0, \partial D)$, where

$z_0 = f^{-1}(w_0)$ and $\varepsilon_n = s(w_0, \partial f_n D(z_0, \delta))$. Note that the sets $f_n D(z_0, \delta)$ are open, and $\varepsilon_n > 0$ is the radius of the maximal closed disk centered at w_0 which is inside of $\overline{f_n D(z_0, \delta)}$. Assume that $\varepsilon_n \to 0$ as $n \to \infty$. Since $\partial D(z_0, \delta)$ and $\partial f_n D(z_0, \delta) = f_n \partial D(z_0, \delta)$ are compact, there exist $z_n \in \partial D(z_0, \delta)$, $s(z_n, z_0) = \delta$, such that $\varepsilon_n = s(w_0, f_n(z_n))$, and we may assume that $z_n \to z_* \in \partial D(z_0, \delta)$ as $n \to \infty$ and then $f_n(z_n) \to f(z_*)$ as $n \to \infty$ (see, e.g., [2, p. 268]). However, by construction, $s(w_0, f_n(z_n)) = \varepsilon_n \to 0$ as $n \to \infty$, and, hence, $f(z_*) = f(z_0)$, i.e., $z = z_*$. This contradiction disproves the above assumption. Thus, we obtain also that every compact set $C \subset D'$ belongs to D'_n for all $n \geq N$ for some N.

Now remark that $D' = \bigcup_{m=1}^{\infty} C_m$, where $C_m = \overline{D^*_m}$, D^*_m is a connected component of the open set $\Omega_m = \{w \in D' : s(w, \partial D') > 1/m\}$, $m = 1, 2, \dots$, including a fixed point $w_0 \in D'$. Indeed, every point $w \in D'$ can be joined with w_0 by a path γ in D'. Because $|\gamma|$ is compact, we have $s(|\gamma|, \partial D') > 0$ and, consequently, $w \in D^*_m$ for large enough $m = 1, 2, \dots$.

Next, take an arbitrary pair of continua E and F in D which belong to the different connected components of the complement of a ring $A = A(z_0, r_1, r_2) = \{z \in \mathbb{C} : r_1 < |z - z_0| < r_2\}$, $z_0 \in \overline{D}$, $0 < r_1 < r_2 < r_0 \leq \sup_{z \in D} |z - z_0|$. For $m \geq m_0$, the continua fE and fF belong to D^*_m. Fix one of such m. Then the continua $f_n E$ and $f_n F$ also belong to D^*_m for large enough n. As well known,

$$M(\Delta(f_n E, f_n F; D^*_m)) \to M(\Delta(fE, fF; D^*_m))$$

as $n \to \infty$, see [14, Theorem 1]. However, $D^*_m \subset f_n D$ for large enough n, and hence

$$M(\Delta(f_n E, f_n F; D^*_m)) \leq M(\Delta(f_n E, f_n F; f_n D))$$

and, thus, by (1.5),

$$M(\Delta(fE, fF; D^*_m)) \leq \int\limits_{A \cap D} Q(z) \cdot \eta^2(|z - z_0|)\, dx\, dy$$

for every measurable function $\eta : (r_1, r_2) \to [0, \infty]$ such that

$$\int\limits_{r_1}^{r_2} \eta(r)\, dr = 1.$$

Finally, since $\Gamma = \bigcup_{m=m_0}^{\infty} \Gamma_m$ where $\Gamma = \Delta(fE, fF; fD)$ and $\Gamma_m :\, = \Delta(fE, fF; D_m^*)$ is increasing in $m = 1, 2, \ldots$, we obtain that $M(\Gamma) = \lim_{m\to\infty} M(\Gamma_m)$ (see, e.g., [5, Theorem 7]), and, thus,

$$M(\Delta(fE, fF; fD)) \leq \int\limits_{A\cap D} Q(z) \cdot \eta^2(|z - z_0|)\, dx\, dy,$$

i.e., f is a ring Q-homeomorphism at z_0. □

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