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Functional models of the Lie algebra of a system of linear operators $\{A_1, A_2, A_3\}$

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Abstract. Functional models are constructed for a non-Abelian nilpotent Lie algebra of linear operators acting in the Hilbert space H. The algebra generators $\{A_1, A_2, A_3\}$ satisfy the relations $[A_1, A_3] = 0$, $[A_2, A_3] = 0$, $[A_1, A_2] = iA_3$, where $A_1x_1 + A_2x_2 + A_3x_3$ is not dissipative for all $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, and the space of non-Hermiticity $G = \operatorname{span}\{(A_k - A_k^*)h,\ k = 1, 2, 3,\ h \in H\}$ has dimension three.

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Introduction

Functional models of contracting (dissipative) operators first constructed by B. Sz.-Nagy and C. Foiaš [5] represent the operators of multiplication by an independent variable in the special spaces of functions. Construction of these models is associated with the Fourier transformation. For the non-dissipative operators, the construction of similar models is based on the study of the Branges transformation [1, p. 152] [8, p. 126].

The characteristic function is the main analytic object, in terms of which the functional models are constructed. L. L. Vaksman [7] showed that if the structure constants of the Lie algebras of linear nonself-adjoint operators are the same, and the corresponding characteristic functions coincide, then these algebras are unitarily equivalent. Thus, the model representations of a Lie algebra with assigned structure components built by the characteristic function are unitarily isomorphic.

For the Lie algebra of linear operators $\{A_1, A_2\}$ $[A_1, A_2] = iA_1$ [6, p. 10], the construction of functional models in the case where the operator A_1, for example, is dissipative is also based on the Fourier transformation.

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In [3, pp. 54–60], the functional models for an arbitrary commutative system of linear operators $\{A_1, A_2\}$ were constructed, and the functional models for an arbitrary Lie algebra of linear operators $\{A_1, A_2\}$ were constructed in [4, pp. 176–185] without the assumption about the dissipative property of the operators A_1, A_2. In this paper, we construct functional models for the Lie algebra of linear operators $\{A_1, A_2, A_3\}$ satisfying the relations $[A_1, A_3] = 0,\ [A_2, A_3] = 0,\ [A_1, A_2] = iA_3$ in the case where $\dim G = 3$ $[G = \operatorname{span}\{(A_k - A_k^*)h,\ k = 1, 2, 3,\ h \in H\}]$ without the assumption that the system contains dissipative operators.

1. Preliminary information

I. Consider a linear bounded operator A acting in a Hilbert space H. We recall that the family

$$\Delta = (A, H, \varphi, E, J) \tag{1.1}$$

is said to be *the local colligation* [2, p. 11], [8, p. 18] if the relation

$$A - A^* = i\varphi^* J\varphi \tag{1.2}$$

holds, where E is a Hilbert space, and φ, J are operators such that $\varphi : H \to E,\ J : E \to E$; moreover, $J = J^* = J^{-1}$.

The function

$$S(\lambda) = I - i\varphi(A - \lambda I)^{-1}\varphi^* J \tag{1.3}$$

is said to be *the characteristic function* [8, p. 24] of a colligation Δ (1.1).

Consider the case where $\dim E = 3$, and J is given by

$$J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}; \tag{1.4}$$

moreover, the spectrum of the operator A is real. Then it is well known [8, p. 66], [2, p. 71] that $S(\lambda)$ has the multiplicative representation

$$S(\lambda) = S_l(\lambda), \qquad S_x(\lambda) = \int_0^{\frown x} \exp\left\{\frac{iJdF_t}{\lambda - \alpha_t}\right\}, \tag{1.5}$$

where α_x is a real bounded function non-decreasing on $[0, l]$, $0 < l < \infty$, and F_t is a matrix-valued (3×3) non-decreasing function such that $trF_x = x$. Suppose that

$$dF_x = a_x\, dx, \tag{1.6}$$

where the matrix a_x is such that $a_x \geq 0,\ tr a_x = 1$,

$$a_x = \begin{pmatrix} a_{11}(x) & a_{12}(x) & a_{13}(x) \\ a_{21}(x) & a_{22}(x) & a_{23}(x) \\ a_{31}(x) & a_{32}(x) & a_{33}(x) \end{pmatrix}, \quad a_{ij} = \overline{a_{ji}}, \tag{1.7}$$

and $a_{ij}(x)$, $i, j = \overline{1,3}$, are functions on $[0, l]$.

Consider the following integral equation for the matrix-function $M_x(z)$:

$$M_x(z) + iz \int_0^x M_t(z)\, dF_t J = I, \tag{1.8}$$

where $x \in [0, l],\ z \in C$. It is easy to see that $M_x(z)$ can be represented by

$$M_x(z) = J S_x^*(\bar{z}^{-1}) J. \tag{1.9}$$

Define the row-vector $L_x(z) = \left[L_x^1(z), L_x^2(z), L_x^3(z)\right]$ as a solution of the integral equation

$$L_x(z) + iz \int_0^x L_t(z)\, dF_t J = (1, 1, 0) = L_x(0), \tag{1.10}$$

where $z \in \mathbb{C}$. It is obvious that

$$L_x(z) = (1, 1, 0) M_x(z) = (1, 1, 0) J S_x^*(\bar{z}^{-1}) J. \tag{1.11}$$

Consider the Hilbert space $L^2_{3,\ l}(F_t)$ [8, pp. 66–67]

$$L^2_{3,\ l}(F_x) = \left\{ f_x \in E^3;\ \int_0^l f_t\, dF_t\, f_t^* < \infty \right\} \tag{1.12}$$

assuming that the proper factorization by the metric kernel is already carried out.

Define the kernel

$$K_x(z, w) = \frac{i}{\pi(z - \bar{w})} L_x(z) J L_x^*(\bar{w}). \tag{1.13}$$

It is obvious that

$$K_x(z, w) = \frac{i}{\pi(z - \bar{w})} \left(L_x^1(z) \overline{L_x^1(w)} - L_x^2(z) \overline{L_x^2(w)} - L_x^3(z) \overline{L_x^3(w)} \right). \tag{1.14}$$

The following theorem [8, pp. 118–119] takes place.

Theorem 1.1. *The row-vector* $L_x(z) = \left[L_x^1(z), L_x^2(z), L_x^3(z)\right]$*, which is a non-trivial solution* $(L_x(z) \neq (1,1,0))$ *of the integral equation* (1.10)*, is such that*

1) $L_x(z) \in L^2_{3,a}(F_t)$ *for all* $a \in [0,l]$ *and* $z \in \mathbb{C}$;

2) *for all* $z \in \mathbb{C}$ *and* $x \in [0,l]$

$$\left|L_x^1(z)\right| - \left|L_x^2(z)\right| - \left|L_3^3(z)\right| = \left\{ \begin{array}{ll} \geq 0, & \operatorname{Im} z > 0 \\ = 0, & \operatorname{Im} z = 0 \\ \leq 0, & \operatorname{Im} z < 0 \end{array} \right\} \tag{1.15}$$

is true.

II. Consider the following basis $\{e_k\}_1^3$ in E_3 :

$$\begin{aligned} e_1 &= (1,1,0); \\ e_2 &= (1,0,1); \\ e_3 &= (5,4,3). \end{aligned} \tag{1.16}$$

Similarly to (1.10), we define the vector-functions $N_x(z) = [N_x^1(z), N_x^2(z), N_x^3(z)]$ and $R_x(z) = \left[R_x^1(z), R_x^2(z), R_x^3(z)\right]$ as solutions of the integral equations

$$N_x(z) + iz \int_0^x N_t(z)\, dF_t J = (1,0,1) = N_x(0), \tag{1.17}$$

$$R_x(z) + iz \int_0^x R_t(z)\, dF_t J = (5,4,3) = R_x(0) \tag{1.18}$$

when $z \in \mathbb{C}$ and $x \in [0,l]$. For $N_x(z)$ and $R_x(z)$, the relations

$$N_x(z) = (1,0,1)M_x(z) = (1,0,1)JS_x^*\left(\bar{z}^{-1}\right)J, \tag{1.19}$$

$$R_x(z) = (5,4,3)M_x(z) = (5,4,3)JS_x^*\left(\bar{z}^{-1}\right)J \tag{1.20}$$

hold, as well as (1.11).

For the functions $N_x(z)$ and $R_x(z)$, the analog of Theorem 1.1 is true.

Definition 1.1. *Denote, by* $\mathbf{B}(L(z))$*, the linear space of the entire functions* $F(z)$*,* $z \in \mathbb{C}$*, such that*

A)
$$F(z) = \mathbf{B}_L f_t = \frac{1}{\pi} \int\limits_0^l f_t \, dF_t \, L_t^*(\bar{z}), \tag{1.21}$$

where $\mathbf{B}_L$ *is the Branges transform* [8, *p.* 125] *of the function* $f_t \in L^2_{3,l}(F_t)$*;*

B) *and let*
$$\|F(z)\|_{B(L(z))} = \|f_t\|_{L^2_{3,l}(F_t)}. \tag{1.22}$$

Theorem 1.2 ([1, p. 152], [8, pp. 126–127]). *Consider the family of Hilbert spaces* $\mathbf{B}(L_a(z))$*, where* $L_x(z)$ *is the vector-function which is a solution of the integral equation* (1.10) *on the interval* $[0, l]$ *for some matrix-valued measure* F_t*. Match every function* $h_t = \big(h^1(t), h^2(t), h^3(t)\big)$ *from* $L^2_{3,l}(F_t)$ *with the function given by*

$$F(z) = \frac{1}{\pi} \int\limits_0^a h_t \, dF_t \, L_t^*(\bar{z}), \tag{1.23}$$

where a is the inner point of the interval $[0, l]$*,* $0 < a < l$*. Then* $F(z) \in \mathbf{B}\left(L_a(z)\right).$

Definition 1.2. *The transform* $F(z)$ (1.21) *of the function* $h_t \in L^2_{3,l}(F_t)$ *is said to be the Branges transform of the function* h_t *by the measure* F_t*.*

Remark 1.1. Similarly, the Hilbert spaces $\mathbf{B}(N(z))$ and $\mathbf{B}(R(z))$ are defined. The Branges transformation of the function $h_t \in L^2_{3,l}(F_t)$ in the space $\mathbf{B}(N(z))$ is given by

$$\Phi_1(z) = \mathbf{B}_N h_t = \frac{1}{\pi} \int\limits_0^l h_t \, dF_t \, N_t^*(\overline{z}) \tag{1.24}$$

and the Branges transformation of the function $h_t \in L^2_{3,l}(F_t)$ in the space $\mathbf{B}(R(z))$, correspondingly, is

$$\Phi_2(z) = \mathbf{B}_R h_t = \frac{1}{\pi} \int\limits_0^l h_t \, dF_t \, R_t^*(\overline{z}), \tag{1.25}$$

where $z \in \mathbb{C}$.

III. Consider the matrix T_1

$$T_1 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1.26}$$

Apply T_1 from the right to Eq. (1.10),

$$L_x T_1 + iz \int_0^x L_t(z)\, dF_t\, JT_1 = L_x(0)T_1.$$

Since $L_x(0)T_1 = N_x(0)$, this relation can be rewritten as

$$L_x(z)T_1 + iz \int_0^x L_t(z)T_1T_1^{-1}\, dF_t\, JT_1 = N_x(0). \tag{1.27}$$

Obviously, T_1^{-1} exists and is equal

$$T_1^{-1} = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

It is easy to see that

$$JT_1 = \tilde{T}_1 J, \tag{1.28}$$

where

$$\tilde{T}_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \tag{1.29}$$

therefore

$$L_x(z)T_1 + iz \int_0^x L_t(z)T_1T_1^{-1} a_t \tilde{T}_1 J\, dt = N_x(0). \tag{1.30}$$

Suppose

$$a_t \tilde{T}_1 = T_1 a_t. \tag{1.31}$$

Then relation (1.30) implies that $L_x(z)T_1$ satisfies Eq. (1.17), and this signifies in view of the uniqueness of the solution of (1.17) that

$$L_x(z)T_1 = N_x(z), \tag{1.32}$$

for all $x \in [0, l]$, $z \in \mathbb{C}$.

Consider $\Phi_1(z) = \mathbf{B}_N f_t$.

$$\mathbf{B}_N f_t = \frac{1}{\pi}\int_0^l f_t a_t\, dt\, N_t^*(\overline{z}) = \frac{1}{\pi}\int_0^l f_t a_t\, dt\, \tilde{T}_1^* L_1^*(z)$$
$$= \frac{1}{\pi}\int_0^l f_t \tilde{T}_1^* a_t\, dt\, L_t^*(\bar{z}) = \mathbf{B}_L(f_t\tilde{T}_1^*)$$

by virtue of (1.31).

Thus,

$$\mathbf{B}_N f_t = \mathbf{B}_L(f_t\tilde{T}_1^*). \tag{1.33}$$

Denote, by $\varphi_1(t)$, the function

$$\varphi_1(t) = f_t\tilde{T}_1^* = \left(f^1(t), f^2(t), f^3(t)\right)\tilde{T}_1^*. \tag{1.34}$$

It is obvious that $\varphi_1(t)$ belongs to the space $L^2_{3,l}(F_t)$, if $f_t \in L^2_{3,l}(F_t)$. So,

$$\Phi_1(z) = \mathbf{B}_N f_t = \mathbf{B}_L(f_t\tilde{T}_1^*) = \mathbf{B}_L\varphi_1(t). \tag{1.35}$$

Therefore, there exists the transformation $\psi_1 : \mathbf{B}(L(z)) \to \mathbf{B}(N(z))$, given by the formula

$$(\psi_1 G)(z) = G_1(z). \tag{1.36}$$

Here, $G(z) \in \mathbf{B}(L(z))$, and $G_1(z) \in \mathbf{B}(N(z))$, i.e. $G(z) = \mathbf{B}_L f_t$, where $f_t \in L^2_{3,l}(F_t)$ and $\psi_1 G(z) = \psi_1\mathbf{B}_L f_t = G_1(z)$. Since $G_1(z) \in \mathbf{B}(N(z))$, we have $G_1(z) = \mathbf{B}_N f_t$, where $f_t \in L^2_{3,l}(F_t)$, $\psi_1\mathbf{B}_L f_t = \mathbf{B}_N f_t$. Thus, by virtue of (1.33),

$$\psi_1\mathbf{B}_L f_t = \mathbf{B}_L\tilde{T}_1^* f_t, \tag{1.37}$$

i.e., $\psi_1\mathbf{B}_L = \mathbf{B}_L\tilde{T}_1^*$ and

$$\psi_1 = \mathbf{B}_L\tilde{T}_1^*\mathbf{B}_L^{-1}. \tag{1.38}$$

Definition 1.3. *The transformation $\mathbf{B}_L^{-1}$ is said to be inverse to the Branges transformation B_L for the function $f_t \in L^2_{3,l}(F_t)$.*

Consider $\psi_1^{-1} : \mathbf{B}(N(z)) \to \mathbf{B}(L(z))$ and $\psi_1^{-1} = \mathbf{B}_L\tilde{T}_1^{*-1}B_L^{-1}$, i.e.,

$$\left(\psi_1^{-1}\Phi_1\right)(z) = \psi_1^{-1}\mathbf{B}_N f_t = \psi_1^{-1}\mathbf{B}_L\tilde{T}_1^* f_t$$
$$= \mathbf{B}_L\tilde{T}_1^{*-1}B_L^{-1}\mathbf{B}_L\tilde{T}_1^* f_t = B_L f_t = \hat{F}_1(z)$$

takes place for all functions $\Phi_1(z) \in \mathbf{B}(N(z))$, where $\hat{F}_1(z) \in \mathbf{B}(L(z))$. Thus, there exists $\hat{h}_t$ from $L^2_{3,l}(F_t)$ such that

$$\hat{F}_1(z) = \frac{1}{\pi}\int_0^l \hat{h}_t\, dF_t\, L_t^*(\overline{z}), \tag{1.39}$$

$$\hat{F}_1(z) = \mathbf{B}_L \hat{h}_t. \tag{1.40}$$

IV. Similar considerations can be carried out for the space $\mathbf{B}(R(z))$. Namely, there exists the matrix T_2 given by

$$T_2 = \begin{pmatrix} 2 & 3 & 0 \\ 3 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1.41}$$

By applying T_2 to Eq. (1.10) from the right, we get

$$L_x(z)T_2 + iz\int_0^x L_t(z)T_2T_2^{-1}dF_tJT_2 = R_x(0). \tag{1.42}$$

Obviously, the matrix T_2^{-1} exists. It is easy to see that

$$JT_2 = \tilde{T}_2 J, \tag{1.43}$$

where

$$\tilde{T}_2 = \begin{pmatrix} 2 & -3 & 0 \\ -3 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1.44}$$

Therefore, supposing that

$$a_t\tilde{T}_2 = T_2 a_t, \tag{1.45}$$

we obtain that $L_x(z)T_2$ satisfies Eq. (1.18). This signifies in view of the uniqueness of the solution of (1.18) that

$$L_x(z)T_2 = P_x(z), \tag{1.46}$$

for all $x \in [0, l]$, $z \in \mathbb{C}$.

In exactly the same way, let us consider the function $\varphi_2(t)$ given by

$$\varphi_2(t) = f_t\tilde{T}_2. \tag{1.47}$$

Similarly,

$$\Phi_2(z) = \mathbf{B}_R f_t = \frac{1}{\pi}\int_0^l f_t a_t\, dt\, R_t^*(\overline{z})$$

$$= \frac{1}{\pi}\int_0^l f_t a_t\, dt\, \tilde{T}_2^* L_t^*(\overline{z}) = \frac{1}{\pi}\int_0^l f_t \tilde{T}_2^* a_t\, dt\, T_1 L_t^*(\overline{z});$$

$$\Phi_2(z) = B_L(f_t\tilde{T}_2^*), \tag{1.48}$$

by virtue of (1.45).

That is, $\mathbf{B}_R f_t = \mathbf{B}_L f_t \tilde{T}_2$ or $\mathbf{B}_R f_t = \mathbf{B}_L(f_t\tilde{T}_2^*) = \mathbf{B}_L\varphi_2(t)$. Carrying out similar considerations, we obtain that there exists the map $\psi_2 : \mathbf{B}(R(z)) \to \mathbf{B}(L(z))$ given by the formula

$$(\psi_2 G)(z) = G_2(z), \tag{1.49}$$

where $G_2(z) \in \mathbf{B}(R(z)),\ G_2(z) = \mathbf{B}_R f_t,\ f_t \in L^2_{3,l}(F_t)$ and

$$\Psi_2\mathbf{B}_L f_t = \mathbf{B}_L\tilde{T}_2^* f_t, \tag{1.50}$$

i.e., $\psi_2\mathbf{B}_L = \mathbf{B}_L\tilde{T}_2^*$ и

$$\psi_2 = \mathbf{B}_L\tilde{T}_2^*\mathbf{B}_L^{-1}. \tag{1.51}$$

Consider $\psi_2^{-1} : \mathbf{B}(R(z)) \to \mathbf{B}(L(z))$ and $\psi_2^{-1} = \mathbf{B}_L\tilde{T}_2^{*-1}\mathbf{B}_L^{-1}$, i.e.,

$$\begin{aligned}\left(\psi_2^{-1}\Phi_2\right)(z) &= \psi_2^{-1}\mathbf{B}_R f_t = \psi_2^{-1}\mathbf{B}_L\tilde{T}_2^* f_t \\ &= \mathbf{B}_L\tilde{T}_2^{*-1}(t)B_L^{-1}\mathbf{B}_L\tilde{T}_2^* f_t = B_L f_t = \hat{F}_2(z),\end{aligned}$$

where $\hat{F}_2(z) \in \mathbf{B}(L(z))$ takes place for every function $\Phi_2(z) \in \mathbf{B}(R(z))$. Thus,

$$\hat{F}_2(z) = \frac{1}{\pi}\int_0^l \hat{h}_t\, dF_t\, L_t^*(\overline{z}), \tag{1.52}$$

$$\hat{F}_1(z) = \mathbf{B}_L\hat{h}_t, \tag{1.53}$$

where $\hat{h}_t \in L^2_{3,l}(F_1)$.

Definition 1.4. *The function* $\hat{h}_t = \left(\hat{h}^1(t), \hat{h}^2(t), \hat{h}^3(t)\right) \in L^2_{2,l}(F_t)$ *constructed by this rule is said to be the dual function to the function* $h_t = \left(h^1(t), h^2(t), h^3(t)\right) \in L^2_{3,l}(F_t)$.

Remark 1.2.

$$\Phi_1(z) = (\psi_1\hat{F}_1)(z), \tag{1.54}$$

$$\Phi_2(z) = (\psi_2\hat{F}_2)(z), \tag{1.55}$$

takes place.

2. Triangular models of an operator system

V. Consider the commutative system of linear bounded operators $\{A_1, A_2\}$ acting in a Hilbert space H, i.e., the relation

$$[A_1, A_2] = A_1A_2 - A_2A_1 = 0. \tag{2.1}$$

holds.

As is well known [2, pp. 11–15], the family

$$\Delta = (A_1, A_2, H, \varphi, E, \sigma_1, \sigma_2, \gamma, \widetilde{\gamma}), \tag{2.2}$$

where E is some Hilbert space, $\varphi, \sigma_1, \sigma_2, \gamma, \widetilde{\gamma}$ are operators such that $\varphi : H \to E$, $\sigma_1 : E \to E$, $\sigma_2 : E \to E, \gamma : E \to E$, $\widetilde{\gamma} : E \to E$, and $\sigma_k = \sigma_k^*$, $k = 1, 2$, $\gamma = \gamma^*$, $\widetilde{\gamma} = \widetilde{\gamma}^*$, is said to be *the commutative colligation* if the relations

$$\begin{aligned} &1. \quad A_k - A_k^* = i\varphi^*\sigma\varphi, \quad k = 1, 2\\ &2. \quad \gamma\varphi = \sigma_1\varphi A_2^* - \sigma_2\varphi A_1^* \,(\widetilde{\gamma}\varphi = \sigma_1\varphi A_2 - \sigma_2\varphi A_1)\\ &3. \quad \gamma - \widetilde{\gamma} = i(\sigma_1\varphi\varphi^*\sigma_2 - \sigma_2\varphi\varphi^*\sigma_1)\end{aligned} \tag{2.3}$$

hold.

Definition 2.1. *The matrix-function $S(\lambda_1)$ given by*

$$S(\lambda_1) = I - i\varphi\,(A_1 - \lambda_1 I)^{-1}\,\varphi^*\sigma_1, \tag{2.4}$$

is said to be the characteristic function of colligation (2.2) *corresponding to the operator A_1. If dim $E = 3$, and the spectrum of the operator A_1 is real, then, for $S(\lambda_1)$* [2, *p.* 71], *the multiplicative representation* (1.5) *takes place.*

Let $\sigma_1 = J$, where J (1.4) and $\sigma_2 = \sigma$. Then the intertwining condition [9, p. 117]

$$(\sigma\lambda_1 + \gamma)JS(\lambda_1) = S(\lambda_1)(\sigma\lambda_1 + \widetilde{\gamma})J \tag{2.5}$$

takes place for function (2.4).

Suppose that $dF_1 = a_t dt$, where the matrix a_t is given by (1.7) and is such that $a_t \geq 0$ and $tra_t = 1$. Then the following theorem takes place [9, p. 118].

Theorem 2.1. *In order that the intertwining condition*

$$(\sigma\lambda + \gamma_x)JS_x(\lambda) = S_x(\lambda)(\sigma\lambda + \widetilde{\gamma})J, \tag{2.6}$$

for the matrix-function $S_x(\lambda)$ hold, it is necessary and sufficient that

$$1) \quad \frac{d}{dx}\gamma_x J = i[Ja_x, \sigma J]\gamma_0 = \widetilde{\gamma}, \tag{2.7}$$

$$2) \quad [Ja_x, (\sigma\alpha_x + \gamma_x)J] = 0. \tag{2.8}$$

VI. Consider now the system of linear bounded operators $\{A_1, A_2, A_3\}$ in H such that

$$
\begin{aligned}
[A_1, A_3] &= 0,\\
[A_2, A_3] &= 0, \qquad (2.9)\\
[A_1, A_2] &= 0.
\end{aligned}
$$

The triangular model realization of the Lie algebra (2.9) in the space $L^2_{3,l}(F_t)$ (1.12) is given by

$$
\begin{aligned}
\hat{A}_1 f_x &= f_x J(\gamma_{x,1} + \alpha_x \sigma_1) + i \int_x^l f_t a_t \, dt \, \sigma_1,\\
\hat{A}_2 f_x &= f_x J(\gamma_{x,2} + \alpha_x \sigma_2) + i \int_x^l f_t a_t \, dt \, \sigma_2, \qquad (2.10)\\
\hat{A}_3 f_x &= \alpha_x f_x + i \int_x^l f_t a_t \, dt \, \sigma_3.
\end{aligned}
$$

In this case, we suppose that

$$
\sigma_3 = J,
$$

$$
\sigma_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad (2.11)
$$

$$
\sigma_1 = \begin{pmatrix} 0 & b & 0 \\ b & 0 & b \\ 0 & b & 0 \end{pmatrix},
$$

$$
\gamma_{x,1} = \begin{pmatrix} \beta_{11}(x) & \beta_{12}(x) & \beta_{13}(x) \\ \bar{\beta}_{12}(x) & \beta_{22}(x) & \beta_{23}(x) \\ \bar{\beta}_{13}(x) & \bar{\beta}_{23}(x) & \beta_{33}(x) \end{pmatrix}, \qquad (2.12)
$$

where $b \in \mathbb{R}$, $\beta_{ij}(x)$ are some functions, and $\gamma_{0,1} = \gamma_1$. In addition,

$$
\gamma_{x,2} = \begin{pmatrix} d_{11}(x) & d_{12}(x) & d_{13}(x) \\ d_{12}(x) & d_{22}(x) & d_{23}(x) \\ d_{13}(x) & d_{23}(x) & d_{33}(x) \end{pmatrix}, \qquad (2.13)
$$

where $d_{ij}(x)$ are some functions, and $\gamma_{0,2} = \gamma_2$. Moreover, the relation

$$
\gamma_2 - \gamma_2^* = i\sigma_3 \qquad (2.14)
$$

holds for γ_2.

In order that the conditions of Theorem 2.1 hold for the commutative operators $\{A_1, A_3\}$ and $\{A_2, A_3\}$, namely, in order that (2.7) and (2.8) take place and condition (2.9) hold, the matrix a_x must be given by

$$a_x = \begin{pmatrix} 1-a_2(x) & ia_1(x) & a_2(x) \\ -ia_1(x) & 1-2a_2(x) & -ia_1(x) \\ a_2(x) & ia_1(x) & 3a_2(x)-1 \end{pmatrix}, \tag{2.15}$$

and $\gamma_{x,1}$ and $\gamma_{x,2}$ must satisfy the relation

$$\gamma_{x,1} = b\gamma_{x,2} + c, \tag{2.16}$$

where c is a constant matrix given by

$$c = \begin{pmatrix} -\beta - ib/2 & -i/2 & 0 \\ i/2 & \beta + ib/2 & -i/2 \\ 0 & i/2 & \beta + ib/2 \end{pmatrix}. \tag{2.17}$$

In this case, γ_1, γ_2 are

$$\begin{gathered} \gamma_1 = \begin{pmatrix} -\beta & b\alpha - i/2 & 0 \\ b\bar{\alpha} + i/2 & \beta & b\alpha - i/2 \\ 0 & b\bar{\alpha} + i/2 & \beta \end{pmatrix}, \\ \gamma_2 = \begin{pmatrix} i/2 & \alpha & 0 \\ \bar{\alpha} & -i/2 & \alpha \\ 0 & \bar{\alpha} & -i/2 \end{pmatrix}, \end{gathered} \tag{2.18}$$

where $\beta \in R$, $\alpha = ik$, $k \in R$. The matrix $\gamma_{x,2}$ is such that

$$\frac{d}{dx}\,\gamma_{x,2} = \begin{pmatrix} 2ia_1(x) & 2(1-a_2(x)) & 0 \\ -2(1-a_2(x)) & 4ia_1(x) & 2(3a_2(x)-1) \\ 0 & -2(3a_2(x)-1) & -2ia_1(x) \end{pmatrix}. \tag{2.19}$$

3. Functional models of the Lie algebra of operators $\{\hat{A}_1, \hat{A}_2, \hat{A}_3\}$

VII. Consider the operator system $\{\hat{A}_1, \hat{A}_2, \hat{A}_3\}$ (2.10) acting in $L^2_{3,l}(F_t)$ (1.12); moreover, $\{\sigma_1, \sigma_2, \sigma_3\}$ (2.11), γ_1, γ_2 (2.18) respectively, $\gamma_{x,1}, \gamma_{x,2}$ satisfy relation (2.16); moreover, relation (2.19) holds for $\gamma_{x,2}$.

Let us study how the action of each of the operators $\{\hat{A}_1, \hat{A}_2, \hat{A}_3\}$ changes after the Branges transformation (1.21)

$$\pi\hat{A}_3F(z)=\int_0^l\left(\int_t^l f_s\,dF_s\,J\right)dF_tL_t^*(\bar z)$$

$$=\int_0^l f_t\,dF_t\,\frac{L_t^*(\bar z)-L_t^*(0)}{z}=\pi\,\frac{F(z)-F(0)}{z}.$$

That is,

$$\widetilde{A}_3F(z)=\frac{F(z)-F(0)}{z}, \tag{3.1}$$

$\widetilde{A}_3F(z)\in\mathbf{B}(L_l(z))$.

We now calculate $\pi\hat{A}_1f_t$ ($\pi\hat{A}_2f_t$ can be obtained similarly)

$$\pi\hat{A}_1F(z)=\int_0^l(A_1f_t)\,dF_t\,L_t^*(\bar z)=\int_0^l f_t\,dF_t\,(A_1^*L_t(z))^*$$

$$=\int_0^l f_t\,dF_t\left(\alpha_tL_t(z)J\sigma_1+L_t(z)J\gamma_{t,1}-i\int_0^x L_s(z)\,dF_s\,\sigma_1\right)^*.$$

By virtue of the integral equation (1.10), we obtain

$$\pi\hat{A}_1F(z)=\int_0^l f_t\,dF_t\left(\frac{L_t(z)-L_t(0)}{z}\;J\sigma_1+L_t(z)J\gamma_{t,1}\right)^*$$

$$=\frac{1}{z}\int_0^l f_t\,dF_t\,(L_t(z)J(\sigma_1+\gamma_{t,1}z)-L_t(0)J\sigma_1)^*.$$

Remark 3.1. It is easy to see that

$$L_t(z)J(\sigma_1+\gamma_{t,1}z)|_{z=0}=L_t(0)J\sigma_1. \tag{3.2}$$

Remark 3.2. It is shown earlier that, for the pair of the operators $\{A_1,A_3\}$ forming the commutative operator system, the conditions of Theorem 2.1 are true, and, thus, the following intertwining property takes place, namely:

$$(\sigma_1\lambda+\gamma_{x,1})JS_x(\lambda)=S_x(\lambda)(\sigma_1\lambda+\gamma_1)J, \tag{3.3}$$

and setting $\lambda=\frac{1}{z}$ in this relation we obtain

$$(\sigma_1+\gamma_{x,1}z)JS_x\left(z^{-1}\right)=S_x\left(z^{-1}\right)(\sigma_1+\gamma_1z)J.$$

In view of relations (1.11), (1.17), and (1.18), we obtain

$$
\begin{aligned}
&L_x(z)J(\sigma_1+\gamma_{x,1}z)\\
&\qquad=(1,1,0)M(z)J(\sigma_1+\gamma_{x,1}z)\\
&\qquad=(1,1,0)JS_x^*\left(\bar z^{-1}\right)JJ(\sigma_1+\gamma_{x,1}z)J\\
&\qquad=(1,1,0)J(\sigma_1+\gamma_1 z)JS_x^*\left(\bar z^{-1}\right)J\\
&\qquad=(1,1,0)J(\sigma_1+\gamma_1 z)M_x(z).
\end{aligned}
$$

This relation can be represented in the form

$$
(1,1,0)J(\sigma_1+\gamma_1 z)M_x(z)=\sum_{j=1}^{3}\zeta_j(z)e_jM_x(z), \tag{3.4}
$$

where e_j $(j=1,2,3)$ are given by (1.16), and $\zeta_j(z)$, $(j=1,2,3)$ are some functions from z, $z\in\mathbb{C}$. Taking relations (1.11), (1.19), and (1.20) into account, we obtain

$$
\sum_{j=1}^{3}\zeta_j(z)e_jM_x(z)=\zeta_1(z)L_x(z)+\zeta_2(z)N_x(z)+\zeta_3(z)R_x(z),
$$

i.e.,

$$
L_x(z)J(\sigma_1+\gamma_{x,1}z)=\zeta_1(z)L_x(z)+\zeta_2(z)N_x(z)+\zeta_3(z)R_x(z). \tag{3.5}
$$

In the case where σ_1 and γ_1 are given by formulas (2.11) and (2.18), $\zeta_j(z)$, $(j=1,2,3)$ are given by

$$
\zeta_1(z)=pz-b;\quad \zeta_2(z)=pz+b;\quad \zeta_3(z)=-idz-b; \tag{3.6}
$$

where $p=-\beta+id$, $d=(2bk-1)/2$, $k:\alpha=ik$, $k\in R$. In addition, $\zeta_j(z)$ $(j=1,2,3)$ at the point $z=0$ are equal to

$$
\zeta_1(0)=-b;\quad \zeta_2(0)=b;\quad \zeta_3(0)=-b. \tag{3.7}
$$

Thus,

$$
\begin{aligned}
\pi\hat A_1F(z)=\frac{1}{z}\int_0^l f_t\,dF_t\,(&\zeta_1(z)L_t(z)-\zeta_1(0)L_t(0)+\zeta_2(z)N(z)\\
&-\zeta_2(0)N(0)+\zeta_3(z)R(z)-\zeta_3(0)R(0))^*\\
=\frac{1}{z}\{&\bar\zeta_1(z)F(z)-\bar\zeta_1(0)F(0)+\bar\zeta_2(z)\Phi_1(z)
\end{aligned}
$$

$$- \bar{\zeta}_2(0)\Phi_1(0) + \bar{\zeta}_3(z)\Phi_2(z) - \bar{\zeta}_3(0)\Phi_2(0)\}.$$

Taking relations (1.54) and (1.55) into consideration, we obtain

$$\begin{aligned}\widetilde{A}_1 F(z) = b\,\frac{F(0)-F(z)}{z} + \bar{p}F(z) \\ + b\,\frac{\big(\Psi_1\hat{F}_1\big)(z) - \big(\Psi_1\hat{F}_1\big)(0)}{z} + \bar{p}\,\big(\Psi_1\hat{F}_1\big)(z) \\ + b\,\frac{\big(\Psi_2\hat{F}_2\big)(0) - \big(\Psi_2\hat{F}_2\big)(z)}{z} + id\big(\Psi_2\hat{F}_2\big)(z).\end{aligned} \tag{3.8}$$

By carrying on similar considerations for the operator $\hat{A}_2$, we get

$$\begin{aligned}\pi\hat{A}_2 F(z) &= \int_0^l (A_2 f_t)\, dF_t\, L_t^*(\bar{z}) = \int_0^l f_t\, dF_t\, (A_1^* L_t(z))^* \\ &= \int_0^l f_t\, dF_t \left(\alpha_t L_t(z) J\sigma_2 + L_t(z) J\gamma_{t,2} - i\int_0^x L_s(z)\, dF_s\, \sigma_2\right).\end{aligned}$$

In this case, the corresponding analogs of Remarks 3.1 and 3.2 are also valid.

Consider $L_t(z)J(\sigma_2 + \gamma_{t,2}z)$:

$$\begin{aligned}L_x(z)J(\sigma_2+\gamma_{x,2}z) &= (1,1,0)M_x(z)J(\sigma_2+\gamma_{x,2}z) \\ &= (1,1,0)JS_x^*(\bar{z}^{-1})JJ(\sigma_2+\gamma_{x,2}z)^*J \\ &= (1,1,0)J(\sigma_2+\gamma_2\bar{z}-i\sigma_3\bar{z})JS_x^*(\bar{z}^{-1})J.\end{aligned}$$

Remark 3.3. $(\sigma_2+\gamma_{x,2}z)^* = (\sigma_2+\gamma_{x,2}^*\bar{z})$ and, consequently, there is γ_2^* in the relations. Since γ_2 and γ_2^* satisfy relation (2.14), we have

$$(\sigma_2+\gamma_2^*\bar{z}) = (\sigma_2+\gamma_2\bar{z}-i\sigma_3\bar{z}). \tag{3.9}$$

Taking σ_3 from (2.11), we get $(\sigma_2+\gamma_2^* z) = (\sigma_2+\gamma_2\bar{z}) - iJ\bar{z}$,

$$L_x(z)J(\sigma_2+\gamma_{x,2}z) = (1,1,0)J(\sigma_2+\gamma_2 z - iJz)M_x(z). \tag{3.10}$$

As before, we have

$$L_x(z)J(\sigma_2+\gamma_{x,2}z) = \sum_{j=1}^{3}\eta_j(z)e_jM_x(z), \tag{3.11}$$

where e_j $(j = 1,2,3)$ are given by (1.16), and $\eta_j(z)$, $(j = 1,2,3)$ are some functions from z, $z \in \mathbb{C}$

$$L_x(z)J(\sigma_2 + \gamma_{x,2}z) = \eta_1(z)L_x(z) + \eta_2(z)N_x(z) + \eta_3(z)R_x(z). \quad (3.12)$$

In the case where σ_2 and σ_3 are defined by (2.11), $\eta_j(z)$ $(j = 1,2,3)$ are given by

$$\eta_1(z) = -1-iz\,(k+1/2)\,; \quad \eta_2(z) = 1+iz\,(k-1/2)\,; \quad \eta_3(z) = -1-ikz, \quad (3.13)$$

where $k : \alpha = ik,\ k \in R$. Note that, when $z = 0$,

$$\eta_1 = -1; \quad \eta_2 = 1; \quad \eta_3 = -1. \quad (3.14)$$

Thus, similarly to the aforesaid for the operator $\hat{A}_1$, we obtain

$$\widetilde{A}_2F(z) = \frac{F(0)-F(z)}{z} + \frac{i}{2}\,(1+2k)F(z) + \frac{\big(\Psi_1\hat{F}_1\big)(z) - \big(\Psi_1\hat{F}_1\big)(0)}{z} + \frac{i}{2}\,(1-2k)\big(\Psi_1\hat{F}_1\big)(z) + \frac{\big(\Psi_2\hat{F}_2\big)(0) - \big(\Psi_2\hat{F}_2\big)(z)}{z} + ik\,\big(\Psi_2\hat{F}_2\big)(z). \quad (3.15)$$

So, we obtain the following result.

Theorem 3.1. *Let* $\{\hat{A}_1, \hat{A}_2, \hat{A}_3\}$ *be a system of the model operators* (2.10) *acting in the space* $L^2_{3,l}(F_t)$ (1.12) ($dF_t = a_t dt$, (1.31), (1.45) *take place for* a_t) *satisfying the commutative relations* (2.9)*; in addition, let* $\{\sigma_1, \sigma_2, \sigma_3\}$ *be given by* (2.11) *and* γ_1, γ_2, *correspondingly, by* (2.18)*;* $\gamma_{x,1}$ *and* $\gamma_{x,2}$ *satisfy relation* (2.16)*, and let relation* (2.19) *be true for* $\gamma_{x,2}$.

If $F(z) \in \mathbf{B}(L(z))$ *is the Branges transform of the function* h_t *from* $L^2_{3,l}(F_t)$*, and if* $\hat{F}_1(z)$ *and* $\hat{F}_2(z)$ *are the Branges transforms* [*by* (1.36) *and* (1.49)*, correspondingly*] *for the dual function* $\hat{h}_t$ (*by Definition* 1.4)*, then the Branges transform* (1.21) *establishes the unitary equivalence between the triangular models* $\{\hat{A}_1, \hat{A}_2, \hat{A}_3\}$ (2.10) *and the functional models* $\{\tilde{A}_1, \tilde{A}_2, \hat{A}_3\}$ (3.14)*,* (3.15)*,* (3.1)*.*

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