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On the stability of solutions of systems of difference equations in a critical case

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Abstract. A system of nonlinear difference equations which admits the zero solution is considered. Its stability is studied by means of Lyapunov's direct method. Side by side with this system, a linearized system of difference equations is also considered. It is well known that if all roots of the characteristic equation of a linearized system lie within the unit circle on the complex plane, then the zero solution of the original full system is asymptotically stable. If at least one eigenvalue lies outside the unit disk, then the zero solution of the original system is unstable. In the case where the moduli of some eigenvalues are equal to unity, and the moduli of others are less than unity, the stability problem cannot be solved by considering only the linear terms. To solve this problem, it is necessary to use the terms of higher orders in expansions of the right-hand sides of the original system of difference equations in Maclaurin series. Such case is called a critical one. In this paper, we consider the critical case where one eigenvalue is equal to unity, and the stability problem can be solved by involving terms up to the third order in expansions of the right-hand sides of the initial equations in Maclaurin series.

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1. Introduction and basic definitions

The theory of discrete dynamical systems has grown tremendously in the last decade. Difference equations can arise in a number of ways. They may be the natural model of a discrete process (in combinatorics, for example), or they can be a discrete approximation of a continuous process. The development of the theory of difference systems has been strongly promoted by advanced technologies in the scientific computation and by a

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large number of applications to models in biology, engineering, and other sciences. For example, in papers [5,6,8,9,14], systems of difference equations were used as natural models of the dynamics of populations; in [10], the difference equations were applied to a simulation in genetics; in [25], the dynamics of an ecological system was also described by a system of difference equations. The method of construction of difference schemes for systems of differential equations is proposed in [2]. This method provides the consistency between differential and difference equations in the sense of the stability of the zero solution (we note that, as in the case of ordinary differential equations, the stability problem for any solution of a difference equation is reduced to that of the zero solution).

Many evolution processes are characterized by the fact that, at certain time moments, they experience abruptly a change of the state. These processes are subjected to short-term perturbations, whose duration is negligible in comparison with that of the process. Consequently, it is natural to assume that these perturbations act instantaneously, that is, in the form of impulses. Papers [31, 32] were the first articles in this direction. The early works on differential equations with impulse effect were summarized in monograph [33] in which the foundations of this theory were described. In recent years, the study of impulsive systems has attracted the increasing interest [16–18, 20–22, 28]. An impulsive system consists of a continuous system which is governed by ordinary differential equations and a discrete system which is governed by difference equations. So the dynamics of impulsive systems essentially depends on properties of the corresponding difference systems, and this confirms the importance of studying the qualitative properties of difference systems.

The stability of a discrete process is the ability of the process to resist the *a priori* unknown small influences. A process is said to be stable if such disturbances do not change it. This property turns out to be of utmost importance since an individual predictable process can be physically realized, in general, only if it is stable in the corresponding natural sense. One of the most powerful tools used in stability theory is Lyapunov's direct method. This method consists in the use of an auxiliary function (the Lyapunov function).

Consider the system of difference equations

$$x(n+1) = f(n, x(n)), \quad f(n, 0) = 0, \tag{1.1}$$

where $n = 0, 1, 2, \ldots$ is the discrete time, $x(n) = (x_1(n), \ldots, x_k(n))^T \in$

$\mathbb{R}^k$, and $f = (f_1, \dots, f_k)^T \in \mathbb{R}^k$. System (1.1) admits the trivial solution

$$x(n) = 0. \tag{1.2}$$

By $x(n, n_0, x^0)$, we denote the solution of system (1.1) coinciding with $x^0 = (x_1^0, x_2^0, \dots, x_k^0)^T$ for $n = n_0$. Let $\mathbb{Z}_+$ be the set of nonnegative real integers, and let $\mathbb{N}_{n_0}$ be the set of nonnegative real integers satisfying the inequality $n \geq n_0$, $B_r = \{x \in \mathbb{R}^k : \|x\| \leq r\}$.

By analogy with ordinary differential equations, we introduce the following definitions.

Definition 1.1. *The trivial solution of system* (1.1) *is said to be stable if, for any $\varepsilon > 0$, $n_0 \in \mathbb{Z}_+$, there exists $\delta = \delta(\varepsilon, n_0) > 0$ such that $\|x^0\| < \delta$ implies $\|x(n, n_0, x^0)\| < \varepsilon$ for $n > n_0$. Otherwise, the trivial solution of system* (1.1) *is called unstable. If δ in this definition can be chosen to be independent of n_0 (i.e. $\delta = \delta(\varepsilon)$), then the zero solution of system* (1.1) *is said to be uniformly stable.*

Definition 1.2. *Solution* (1.2) *of system* (1.1) *is said to be attracting if, for any $n_0 \in \mathbb{Z}_+$, there exists an $\eta = \eta(n_0) > 0$ such that, for any $\varepsilon > 0$ and $x^0 \in B_\eta$, there exists a $\sigma = \sigma(\varepsilon, n_0, x^0) > 0$ such that $\|x(n, n_0, x^0)\| < \varepsilon$ for all $n \geq n_0 + \sigma$.*

In other words, solution (1.2) of system (1.1) is called attracting if

$$\lim_{n \to \infty} \|x(n, n_0, x^0)\| = 0. \tag{1.3}$$

Definition 1.3. *The trivial solution of system* (1.1) *is said to be uniformly attracting if, for some $\eta > 0$ and for each $\varepsilon > 0$, there exists a $\sigma = \sigma(\varepsilon) \in \mathbb{N}$ such that $\|x(n, n_0, x^0)\| < \varepsilon$ for all $n_0 \in \mathbb{Z}_+,\ x^0 \in B_\eta$, and $n \in \mathbb{N}_{n_0+\sigma}$.*

In other words, solution (1.2) of system (1.1) is called uniformly attracting if (1.3) holds uniformly in $n_0 \in \mathbb{Z}_+,\ x^0 \in B_\eta$.

Definition 1.4. *The zero solution of system* (1.1) *is called*

- *asymptotically stable if it is both stable and attracting;*
- *uniformly asymptotically stable if it is both uniformly stable and uniformly attracting.*

Definition 1.5. *The trivial solution of system* (1.1) *is said to be exponentially stable if there exist* $M > 0$ *and* $\eta \in (0,1)$ *such that* $\|x(n, n_0, x^0)\| < M\|x^0\|\eta^{n-n_0}$ *for* $n \in \mathbb{N}_{n_0}$.

A great number of papers is devoted to the investigation of the stability of solution (1.2) of system (1.1). The general theory of difference systems and the foundations of the stability theory are presented in monographs [1, 7, 11, 13, 15, 19, 29, 30]. In paper [23], it is shown that if system (1.1) is autonomous (i.e. f does not depend on n explicitly) or periodic (i.e. there exists $\omega \in \mathbb{N}$ such that $f(n, x) \equiv f(n + \omega, x)$), then the stability of solution (1.2) implies its uniform stability, and the asymptotic stability implies its uniform asymptotic stability. The asymptotic stability of perturbed linear difference systems with periodic coefficients was studied in [3]. Papers [4, 12, 24, 26, 27] considered the stability of solutions of periodic and almost periodic systems.

Let us formulate the main theorems of Lyapunov's direct method about the stability of the zero solution of the system of autonomous difference equations

$$x(n+1) = f(x(n)). \tag{1.4}$$

These statements have been mentioned in [13, Theorems 4.20 and 4.27]. They are related to the existence of an auxiliary function $V(x)$; and the analog of its derivative is the variation of V relative to (1.4) which is defined as $\Delta V(x) = V(f(x)) - V(x)$.

Theorem A. *If there exists a positive definite continuous function* $V(x)$ *such that* $\Delta V(x)$ *relative to* (1.4) *is a negative semidefinite function or identically equals zero, then the trivial solution of system* (1.4) *is stable.*

Theorem B. *If there exists a positive definite continuous function* $V(x)$ *such that* $\Delta V(x)$ *relative to* (1.4) *is negative definite, then the trivial solution of system* (1.4) *is asymptotically stable.*

Theorem C. *If there exists a continuous function* $V(x)$ *such that* $\Delta V(x)$ *relative to* (1.4) *is negative definite, and the function* V *is not positive semidefinite, then the trivial solution of system* (1.4) *is unstable.*

Consider the autonomous system

$$x(n+1) = Ax(n) + X(x(n)), \tag{1.5}$$

where A is a $k \times k$ nonsingular matrix, and X is a function such that

$$\lim_{\|x\|\to 0} \frac{\|X(x)\|}{\|x\|} = 0. \tag{1.6}$$

According to [13, p. 175], we denote $\rho(A) = \max_{1\le i\le k} |\lambda_i|$, where $\lambda_i \quad (i = 1,\dots,k)$ are the roots of the characteristic equation

$$\det(A - \lambda I_k) = 0. \tag{1.7}$$

Here, I_k is the unit $k \times k$ matrix. In [1, Corollary 5.6.3 and Theorem 5.6.4], the following theorem was proved.

Theorem 1.1. *If $\rho(A) < 1$, then the zero solution of system* (1.5) *is asymptotically stable* (*moreover, the exponential stability holds in this case*)*. If $\rho(A) > 1$, then the zero solution of system* (1.5) *is unstable. If $\rho(A) \le 1$ and the moduli of some eigenvalues of A are equal to unity, then a function $X(x)$ in system* (1.5) *can be chosen to be such that the zero solution of system* (1.5) *is either stable or unstable.*

Thus, it follows from Theorem 1.1 that the problem of stability of the zero solution of system (1.5) can be solved by means of the system of the linear approximation

$$x(n+1) = Ax(n) \tag{1.8}$$

(when $\rho(A) < 1$ or $\rho(A) > 1$). In the case $\rho(A) = 1$, we have a critical case where the solution of the stability problem requires to use the terms of higher degrees.

For studying the stability of the zero solution of system (1.5), Elaydi [13] proposed to employ Lyapunov functions as a quadratic form

$$V(x) = \sum_{\substack{i_1+i_2+\cdots+i_k=2,\\ i_j\ge 0\ (j=1,\dots,k)}} b_{i_1,i_2,\dots,i_k} x_1^{i_1} x_2^{i_2} \dots x_k^{i_k}, \tag{1.9}$$

where $b_{i_1,i_2,\dots,i_k}$ are constants. He formulated the following statement [13, Corollary 4.31] without any proof.

Proposition 1.1. *If $\rho(A) > 1$, then there exist a quadratic form $V(x)$ which is not positive semidefinite and a negative definite quadratic form $W(x)$ such that*

$$W(x) = V(Ax) - V(x).$$

We will show here that Proposition 1.1 is not true and study the stability problem in the critical case where one eigenvalue of the matrix A is equal to unity. First, we show that Proposition 1.1 is false. To do this, let us consider the system

$$x(n+1) = Ax(n),$$

where $x = \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) \in \mathbb{R}^2$, $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right)$. Numbers 1 and 2 are the roots of its characteristic equation, $\rho(A) = 2 > 1$. But, for **any** quadratic form

$$V(x) = b_{2,0}x_1^2 + b_{1,1}x_1x_2 + b_{0,2}x_2^2,$$

we have

$$W(x) = V(Ax) - V(x) = b_{1,1}x_1x_2 + 3b_{0,2}x_2^2. \tag{1.10}$$

The quadratic form (1.10) can be neither positive definite nor negative definite. This example shows that Proposition 1.1 is not true.

2. Critical case where one eigenvalue is equal to unity

In this section, we consider the critical case where one root of the characteristic equation (1.7) is equal to unity, i.e. we assume that Eq. (1.7) has one root $\lambda_1 = 1$, and other roots satisfy the conditions $|\lambda_i| < 1 \quad (i = 2, 3, \ldots, k)$. The function $X = (X_1, \ldots, X_k)^T$ is supposed to be holomorphic, and its expansion into a Maclaurin series begins with terms of the second order of smallness. So, system (1.5) has the form

$$\begin{aligned} x_j(n+1) = a_{j1}x_1(n) + a_{j2}x_2(n) + \cdots + a_{jk}x_k(n) \\ + X_j(x_1(n), \ldots, x_k(n)) \quad (j = 1, \ldots, k). \end{aligned} \tag{2.1}$$

Henceforth, we consider the critical case where the characteristic equation of the system in the first approximation

$$x_j(n+1) = a_{j1}x_1(n) + a_{j2}x_2(n) + \cdots + a_{jk}x_k(n) \quad (j = 1, \ldots, k) \tag{2.2}$$

has one root equal to unity, and other $k-1$ roots have moduli which are less than unity.

In system (2.2), we introduce the variable y instead of one of the variables x_j by means of the substitution

$$y = \beta_1x_1 + \beta_2x_2 + \cdots + \beta_kx_k, \tag{2.3}$$

where $\beta_j \quad (j = 1, \ldots, k)$ are some constants which we choose to be such that

$$y(n+1) = y(n). \tag{2.4}$$

Relations (2.3) and (2.4) yield

$$\begin{aligned} y(n+1) &= \beta_1 x_1(n+1) + \beta_2 x_2(n+1) + \cdots + \beta_k x_k(n+1) \\ &= \beta_1[a_{11}x_1(n) + a_{12}x_2(n) + \cdots + a_{1k}x_k(n)] \\ &+ \beta_2[a_{21}x_1(n) + a_{22}x_2(n) + \cdots + a_{2k}x_k(n)] \\ &+ \cdots + \beta_k[a_{k1}x_1(n) + a_{k2}x_2(n) + \cdots + a_{kk}x_k(n)] \\ &= \beta_1 x_1(n) + \beta_2 x_2(n) + \cdots + \beta_k x_k(n). \end{aligned}$$

Equating the coefficients corresponding to $x_j(n)$ $(j = 1, 2, \ldots, k)$, we obtain the system of linear homogeneous algebraic equations for β_j $(j = 1, \ldots, k)$,

$$a_{1j}\beta_1 + a_{2j}\beta_2 + \cdots + a_{kj}\beta_k = \beta_j, \tag{2.5}$$

or, in the matrix form,

$$(A^T - I_k)\beta = 0,$$

where $\beta = (\beta_1, \ldots, \beta_k)^T$. Since the equation $\det(A^T - \lambda I_k) = 0$ has the root $\lambda = 1$, the determinant of system (2.5) is equal to zero. Therefore, this system has a solution in which not all constants are equal to zero. To be definite, we assume that $\beta_k \neq 0$. Then we can use the variable y instead of the variable x_k. Other variables x_j $(j = 1, \ldots, k-1)$ are preserved without any change. Denoting

$$c_{ji} = a_{ji} - \frac{\beta_i}{\beta_k} a_{jk}, \qquad c_j = \frac{a_{jk}}{\beta_k} \quad (i, j = 1, 2, \ldots, k-1),$$

we transform Eqs. (2.2) to the form

$$x_j(n+1) = c_{j1}x_1(n) + c_{j2}x_2(n) + \cdots + c_{j,k-1}x_{k-1}(n) + c_j y(n) \quad (j = 1, \ldots, k-1), \tag{2.6}$$

$$y(n+1) = y(n), \tag{2.7}$$

where c_{ji} and c_j are constants.

The characteristic equation of system (2.6), (2.7) is reduced to two equations:

$$\lambda - 1 = 0$$

and

$$\det(\mathcal{C} - \lambda I_{k-1}) = 0, \tag{2.8}$$

where $\mathcal{C} = (c_{ij})_{i,j=1}^{k-1}$. Since the characteristic equation is invariant with respect to linear transformations and, in this case, has $k-1$ roots whose moduli are less than unity, Eq. (2.8) has $k-1$ roots, and their moduli are less than unity. We denote

$$x_j = y_j + l_j y \quad (j = 1, \ldots, k-1), \tag{2.9}$$

where $l_j \quad (j = 1, \ldots, k-1)$ are constants which we choose to be such that the right-hand sides of system (2.6) do not contain $y(n)$. In these designations with regard for (2.7), system (2.6) takes the form:

$$\begin{aligned} y_j(n+1) &= c_{j1}y_1(n) + c_{j2}y_2(n) + \cdots + c_{j,k-1}y_{k-1}(n) \\ &\quad + [c_{j1}l_1 + c_{j2}l_2 + \cdots + (c_{jj}-1)l_j + \cdots + c_{j,k-1}l_{k-1} + c_j]y(n) \\ &\qquad (j = 1, \ldots, k-1). \end{aligned}$$

We choose the constants l_j to be such that

$$c_{j1}l_1 + c_{j2}l_2 + \cdots + (c_{jj}-1)l_j + \cdots + c_{j,k-1}l_{k-1} = -c_j \quad (j = 1, \ldots, k-1). \tag{2.10}$$

Unity is not a root of the characteristic equation (2.8); hence the determinant of system (2.10) is not equal to zero. Therefore, this system has the unique solution $(l_1, \ldots, l_{k-1})$. As a result of change (2.9), system (2.6), (2.7) transforms to the form

$$\begin{aligned} y_j(n+1) &= c_{j1}y_1(n) + c_{j2}y_2(n) + \cdots + c_{j,k-1}y_{k-1}(n) \\ &\qquad (j = 1, \ldots, k-1), \\ y(n+1) &= y(n), \end{aligned}$$

and nonlinear system (2.1) takes the form

$$\begin{aligned} y_j(n+1) &= c_{j1}y_1(n) + c_{j2}y_2(n) + \cdots + c_{j,k-1}y_{k-1}(n) \\ &\quad + Y_j(y_1(n), \ldots, y_{k-1}(n), y(n)) \quad (j = 1, \ldots, k-1), \\ y(n+1) &= y(n) + Y(y_1(n), \ldots, y_{k-1}(n), y(n)), \end{aligned} \tag{2.11}$$

where Y_j $(j = 1, \ldots, k-1)$ and Y are holomorphic functions of $y_1, \ldots, y_{k-1}, y$. Their expansions in power series lack constant and first-degree terms:

$$\begin{aligned} Y_j(y_1, y_2, \ldots, y_{k-1}, y) &= \sum_{i_1+i_2+\cdots+i_{k-1}+i_k=2}^{\infty} v^{(j)}_{i_1,i_2,\ldots,i_{k-1},i_k} y_1^{i_1} y_2^{i_2} \cdots y_{k-1}^{i_{k-1}} y^{i_k} \\ &\qquad (j = 1, \ldots, k-1), \\ Y(y_1, y_2, \ldots, y_{k-1}, y) &= \sum_{i_1+i_2+\cdots+i_{k-1}+i_k=2}^{\infty} v_{i_1,i_2,\ldots,i_{k-1},i_k} y_1^{i_1} y_2^{i_2} \cdots y_{k-1}^{i_{k-1}} y^{i_k}. \end{aligned}$$

By virtue of (2.9), it is clear that the stability problem for the trivial solution of system (2.1) is equivalent to that for the zero solution of system (2.11). Further, the form of (2.11) will be basic for studying the stability of the zero solution in the case where this problem can be solved with the use of terms of the first and second orders.

Theorem 2.1. *If the function* Y *is such that the coefficient* $v_{0,0,\ldots,0,2}$ *is not equal to zero, then the solution*

$$y_1 = 0, \quad y_2 = 0, \quad \ldots, \quad y_{k-1} = 0, \quad y = 0$$

of system (2.11) *is unstable.*

Proof. Let

$$V_1(y_1, \ldots, y_{k-1}) = \sum_{s_1+s_2+\cdots+s_{k-1}=2} B_{s_1,s_2,\ldots,s_{k-1}} y_1^{s_1} y_2^{s_2} \ldots y_{k-1}^{s_{k-1}}$$

be a quadratic form such that

$$V_1(c_{11}y_1 + \cdots + c_{1,k-1}y_{k-1}, \ldots, c_{k-1,1}y_1 + \cdots + c_{k-1,k-1}y_{k-1}) \\ - V_1(y_1, \ldots, y_{k-1}) = y_1^2 + y_2^2 + \cdots + y_{k-1}^2. \quad (2.12)$$

Since the moduli of all eigenvalues of the matrix $\mathcal{C} = (c_{ij})_{I,j=1}^{k-1}$ are less than unity, such quadratic form is unique according to [13, Theorem 4.30] and negative definite. Consider the Lyapunov function

$$V(y_1, \ldots, y_{k-1}, y) = V_1(y_1, \ldots, y_{k-1}) + \alpha y, \quad (2.13)$$

where $\alpha = \text{const}$. Let us find ΔV:

$$\Delta V\Big|_{(2.11)} \\ = \sum_{s_1+\cdots+s_{k-1}=2} B_{s_1,\ldots,s_{k-1}} \{[c_{11}y_1 + \cdots + c_{1,k-1}y_{k-1} + Y_1(y_1, \ldots, y_{k-1}, y)]^{s_1} \\ \cdots \times [c_{k-1,1}y_1 + \cdots + c_{k-1,k-1}y_{k-1} + Y_{k-1}(y_1, \ldots, y_{k-1}, y)]^{s_{k-1}} \\ - y_1^{s_1} \ldots y_{k-1}^{s_{k-1}}\} + \alpha Y(y_1, \ldots, y_{k-1}, y).$$

Taking (2.12) into account, ΔV can be written in the form

$$\Delta V\Big|_{(2.11)} = W(y_1, \ldots, y_{k-1}, y) + W_*(y_1, \ldots, y_{k-1}, y),$$

where

$$\begin{aligned} W = (y_1^2 + y_2^2 + \cdots + y_{k-1}^2) + \alpha v_{0,0,\ldots,0,2} y^2 \\ + \alpha(v_{2,0,\ldots,0} y_1^2 + v_{1,1,\ldots,0} y_1 y_2 + \cdots + v_{1,0,\ldots,1,0} y_1 y_{k-1} \\ + v_{1,0,\ldots,0,1} y_1 y + v_{0,2,\ldots,0} y_2^2 + \cdots + v_{0,0,\ldots,1,1} y_{k-1} y), \end{aligned}$$

and W_* is a holomorphic function whose Maclaurin series begins with terms of the third power in $y_1, \ldots, y_{k-1}$, y. We choose the sign of α such that $\alpha v_{0,\ldots,0,2} > 0$. We now show that α can be chosen so small that the quadratic form W is positive definite. To do this, we show that α can be chosen so that the principal minors of the matrix

$$\begin{pmatrix} 1 + \alpha v_{2,0,\ldots,0} & \frac{1}{2}\alpha v_{1,1,\ldots,0} & \cdots & \frac{1}{2}\alpha v_{1,0,\ldots,1,0} & \frac{1}{2}\alpha v_{1,0,\ldots,0,1} \\ \frac{1}{2}\alpha v_{1,1,\ldots,0} & 1 + \alpha v_{0,2,\ldots,0} & \cdots & \frac{1}{2}\alpha v_{0,1,\ldots,1,0} & \frac{1}{2}\alpha v_{0,1,\ldots,0,1} \\ \frac{1}{2}\alpha v_{1,0,1,\ldots,0} & \frac{1}{2}\alpha v_{0,1,1,\ldots,0} & \cdots & \frac{1}{2}\alpha v_{0,0,1,\ldots,1,0} & \frac{1}{2}\alpha v_{0,0,1,\ldots,0,1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{2}\alpha v_{1,0,\ldots,1,0} & \frac{1}{2}\alpha v_{0,1,\ldots,1,0} & \cdots & 1 + \alpha v_{0,\ldots,0,2,0} & \frac{1}{2}\alpha v_{0,\ldots,0,1,1} \\ \frac{1}{2}\alpha v_{1,0,\ldots,0,1} & \frac{1}{2}\alpha v_{0,1,\ldots,0,1} & \cdots & \frac{1}{2}\alpha v_{0,0,\ldots,1,1} & \frac{1}{2}\alpha v_{0,0,\ldots,0,2} \end{pmatrix}$$

are positive. In fact, any principal minor Δ_s of this matrix is a continuous function of α: $\Delta_s = \Delta_s(\alpha)$. Note that $\Delta_s(0) = 1$ for $s = 1, 2, \ldots, k-1$. Thus, there exists $\alpha_* > 0$ such that, for $|\alpha| < \alpha_*$, we have $\Delta_s(\alpha) \geq \frac{1}{2}$ $(s = 1, 2, \ldots, k-1)$. We now prove that the inequality $\Delta_k > 0$ holds for sufficiently small $|\alpha|$. To do this, we expand Δ_k in terms of the elements of the last row. We obtain $\Delta_k = \frac{1}{2}\alpha v_{0,0,\ldots,0,2}\Delta_{k-1} + \alpha^2\Delta_*$, where Δ_* is a polynomial in α and $v_{i_1,i_2,\ldots,i_k}$ $(i_1 + i_2 + \cdots + i_k = 2,\ i_j \geq 0)$. Hence, we have $\Delta_k > 0$ for sufficiently small $|\alpha|$. That is, the quadratic form W is positive definite for α, whose absolute value is small enough and whose sign coincides with the sign of $v_{0,0,\ldots,2}$. Therefore, the sum $W + W_*$ is also positive definite in a sufficiently small neighborhood of the origin. At the same time, the function V of form (2.13) is alternating. Hence, the zero solution of system (2.11) is unstable. The proof is completed. □

Thus, in the case where $v_{0,0,\ldots,2} \neq 0$, the stability problem has been solved independently of the terms, whose degrees are higher than two. Consider now the case where $v_{0,0,\ldots,2} = 0$. We transform system (2.11) to the form where $v^{(j)}_{0,0,\ldots,2} = 0$ $(j = 1, 2, \ldots, k-1)$. We denote

$$y_j = \xi_j + m_j y^2 \qquad (j = 1, 2, \ldots, k-1), \tag{2.14}$$

where m_j are constants. In these designations, system (2.11) takes the form

$$\begin{aligned}\xi_j(n+1) &= c_{j1}\xi_1(n) + c_{j2}\xi_2(n) + \cdots + c_{j,k-1}\xi_{k-1}(n)\\ &\quad + y^2(n)(c_{j1}m_1 + c_{j2}m_2 + \cdots + c_{j,k-1}m_{k-1})\\ &\quad + Y_j(\xi_1(n) + m_1y^2(n), \dots, \xi_{k-1}(n) + m_{k-1}y^2(n), y(n))\\ &\quad - m_j[y^2(n) + 2y(n)Y(\xi_1(n) + m_1y^2(n), \dots, \xi_{k-1}(n) + m_{k-1}y^2(n), y(n))\\ &\quad + Y^2(\xi_1(n) + m_1y^2(n), \dots, \xi_{k-1}(n) + m_{k-1}y^2(n), y(n))], \quad (2.15)\end{aligned}$$

$$y(n+1) = y(n) + Y(\xi_1(n) + m_1y^2(n), \dots, \xi_{k-1}(n) + m_{k-1}y^2(n), y(n)). \quad (2.16)$$

We choose constants $m_1, \dots, m_{k-1}$ to be such that the coefficients corresponding to $y^2(n)$ on the right-hand sides of system (2.15) are equal to zero. Equating the corresponding coefficients to zero, we obtain the system of linear algebraic equations for $m_1, \dots, m_{k-1}$:

$$c_{j1}m_1 + c_{j2}m_2 + \cdots + c_{j,k-1}m_{k-1} = m_j - v^{(j)}_{0,0,\dots,2} \qquad (j = 1, 2, \dots, k-1).$$

This system has a unique solution, because unity is not an eigenvalue of the matrix $\mathcal{C}$. Substituting the obtained values $m_1, \dots, m_{k-1}$ to (2.15) and (2.16), we get the system

$$\begin{aligned}\xi_j(n+1) &= c_{j1}\xi_1(n) + c_{j2}\xi_2 + \cdots + c_{j,k-1}\xi_{k-1}(n)\\ &\quad + \Xi_j(\xi_1(n), \dots, \xi_{k-1}(n), y(n))\\ &\quad (j = 1, \dots, k-1), \quad (2.17)\end{aligned}$$

$$y(n+1) = y(n) + Y_*(\xi_1(n), \dots, \xi_{k-1}(n), y(n)), \qquad (2.18)$$

where

$$\begin{aligned}\Xi_j(\xi_1, \dots, \xi_{k-1}, y) &= Y_j(\xi_1 + m_1y^2, \dots, \xi_{k-1} + m_{k-1}y^2, y)\\ &\quad - 2m_jyY(\xi_1 + m_1y^2, \dots, \xi_{k-1} + m_{k-1}y^2, y)\\ &\quad - m_jY^2(\xi_1 + m_1y^2, \dots, \xi_{k-1} + m_{k-1}y^2, y) - v^{(j)}_{0,0,\dots,2}y^2,\end{aligned}$$

$$Y_*(\xi_1, \dots, \xi_{k-1}, y) = Y(\xi_1 + m_1y^2, \dots, \xi_{k-1} + m_{k-1}y^2, y).$$

Expansions of Ξ_j and Y_* in power series begin with terms of the second degree, and the coefficients corresponding to y^2 in expansions of Ξ_j and Y_* are equal to zero. System (2.17) and (2.18) will be basic in our further investigation of the stability of the zero solution

$$\xi_1 = 0, \quad \xi_2 = 0, \quad \dots, \quad \xi_{k-1} = 0, \quad y = 0. \tag{2.19}$$

By $\Xi_j^{(0)}(y)$ $(j = 1, \dots, k-1)$ and $Y_*^{(0)}(y)$, we denote, respectively, the sum of all terms in the functions Ξ_j and Y_* which do not include $\xi_1, \dots, \xi_{k-1}$, so that

$$\Xi_j^{(0)}(y) = \Xi_j(0, \dots, 0, y) = h_j y^3 + \sum_{s=4}^{\infty} h_j^{(s)} y^s,$$

$$Y_*^{(0)}(y) = Y_*(0, \dots, 0, y) = h y^3 + \sum_{s=4}^{\infty} h^{(s)} y^s,$$

where $h, h_j, h^{(s)}, h_j^{(s)}$ $(j = 1, \dots, k-1;\ s = 4, 5, \dots)$ are constants.

Theorem 2.2. *Solution* (2.19) *of system* (2.17), (2.18) *is asymptotically stable for* $h < 0$ *and unstable for* $h > 0$.

Proof. We now show that there exists a Lyapunov function V such that it depends on $\xi_1, \dots, \xi_{k-1}, y$, and ΔV is positive definite. Consider the system of linear equations

$$\xi_j(n+1) = c_{j1}\xi(n) + c_{j2}\xi_2(n) + \dots + c_{j,k-1}\xi_{k-1}(n) \quad (j = 1, \dots, k-1). \tag{2.20}$$

Let $W = \sum_{i_1+\dots+i_{k-1}=2} w_{i_1,\dots,i_{k-1}} \xi_1^{i_1} \dots \xi_{k-1}^{i_{k-1}}$ be a quadratic form of the variables $\xi_1, \dots,$ ξ_{k-1} and such that

$$\Delta W \Big|_{(2.20)} = \xi_1^2 + \dots + \xi_{k-1}^2. \tag{2.21}$$

Since all eigenvalues of the matrix $\mathcal{C}$ are inside of the unit disk, the form W satisfying (2.21) exists and is unique and negative definite [13, Theorem 4.30].

If the functions Ξ_j $(j = 1, \dots, k-1)$ do not depend on y, then a variation ΔW of the function W along system (2.17), i.e. the expression

$$\sum_{i_1+\dots+i_{k-1}=2} w_{i_1,\cdots,i_{k-1}} \{[c_{11}\xi_1 + c_{12}\xi_2 + \dots + c_{1,k-1}\xi_{k-1} + \Xi_1]^{i_1}$$
$$\times \cdots [c_{k-1,1}\xi_1 + \dots + c_{k-1,k-1}\xi_{k-1} + \Xi_{k-1}]^{i_{k-1}} - \xi_1^{i_1} \cdots \xi_{k-1}^{i_{k-1}}\}, \tag{2.22}$$

is a positive definite function of the variables $\xi_1, \dots, \xi_{k-1}$ for sufficiently small $\xi_1, \dots, \xi_{k-1}$.

On the other hand, if the function Y_* does not depend on $\xi_1, \dots, \xi_{k-1}$ (i.e. if $Y_* = Y_*^{(0)}$), then the variation of $\frac{1}{2}hy^2$ is equal to

$$\Delta\Big(\frac{1}{2}hy^2\Big) = \frac{1}{2}h\Big[2yY_*^{(0)} + Y_*^{(0)2}\Big] = h^2y^4 + hh^{(4)}y^5 + o(y^5), \qquad (2.23)$$

and this variation is a positive definite function of y for sufficiently small $|y|$. Therefore, under these conditions, the variation of the function $V_1 = \frac{1}{2}hy^2 + W(\xi_1, \dots, \xi_{k-1})$ along the total system (2.17), (2.18) is a positive definite function of all variables $\xi_1, \dots, \xi_{k-1}, y$ in some neighborhood of the origin. In view of (2.21) and (2.23), this variation can be represented in the form

$$(h^2 + g_1)y^4 + \xi_1^2 + \dots + \xi_{k-1}^2 + \sum_{i,j=1}^{k-1} g_{ij}^{(1)} \xi_i \xi_j, \qquad (2.24)$$

where g_1 is a holomorphic function of the variable y, vanishing for $y = 0$, and $g_{ij}^{(1)}$ are holomorphic functions of the variables $\xi_1, \dots, \xi_{k-1}$, vanishing for $\xi_1 = \dots = \xi_{k-1} = 0$. But since the functions Ξ_j $(j = 1, \dots, k-1)$ include y, and the function Y_* includes $\xi_1, \dots, \xi_{k-1}$, the variation of the function V_1 along system (2.17), (2.18) is not, in general, positive definite. In this variation, there appear the terms breaking the positive definiteness.

Note that expression (2.24) remains positive definite if the function g_1 contains not only the variable y, but also the variables $\xi_1, \dots, \xi_{k-1}$, and the functions $g_{ij}^{(1)}$ contain not only variables $\xi_1, \dots, \xi_{k-1}$, but also the variable y. It is only important that the functions g_1 and $g_{ij}^{(1)}$ vanish for $\xi_1 = \dots = \xi_{k-1} = y = 0$. Taking into account this fact, we write the second variation of the function V_1 along system (2.17), (2.18) in the form

$$\begin{aligned}\Delta V_1 = \Delta\Big(\frac{1}{2}hy^2\Big) + \Delta W = hyY_* + \frac{1}{2}hY_*^2 \\ + \sum_{i_1+\dots+i_{k-1}=2} w_{i_1,\dots,i_{k-1}} \{[c_{11}\xi_1 + c_{12}\xi_2 + \dots + c_{1,k-1}\xi_{k-1} + \Xi_1]^{i_1} \\ \times \cdots [c_{k-1,1}\xi_1 + \dots + c_{k-1,k-1}\xi_{k-1} + \Xi_{k-1}]^{i_{k-1}} - \xi_1^{i_1} \cdots \xi_{k-1}^{i_{k-1}}\} \\ = [h^2 + g_1(\xi_1, \dots, \xi_{k-1}, y)]y^4 + \xi_1^2 + \dots + \xi_{k-1}^2 \\ + \sum_{i,j=1}^{k-1} g_{ij}^{(1)}(\xi_1, \dots, \xi_{k-1}, y)\xi_i\xi_j + Q(\xi_1, \dots, \xi_{k-1}, y), \quad (2.25)\end{aligned}$$

where the functions g_1 and $g_{ij}^{(1)}$ $(i,j=1,\dots,k-1)$ vanish for $\xi_1=\dots=\xi_{k-1}=y=0$, and Q is the sum of all terms which can be included neither to the expression

$$g_1(\xi_1,\dots,\xi_{k-1},y)y^4 \tag{2.26}$$

nor to the expression

$$\sum_{i,j=1}^{k-1} g_{ij}^{(1)}(\xi_1,\dots,\xi_{k-1},y)\xi_i\xi_j. \tag{2.27}$$

All terms which are included into Q can be divided into four following groups: the terms free of $\xi_1,\dots,\xi_{k-1}$, the terms linear with respect to $\xi_1,\dots,\xi_{k-1}$, the terms quadratic with respect to $\xi_1,\dots,\xi_{k-1}$, and the terms having degree higher than two with respect to $\xi_1,\dots,\xi_{k-1}$. It is evident that all terms of the last group can be included into expression (2.27); therefore, we consider only three first groups of terms.

All terms free of $\xi_1,\dots,\xi_{k-1}$ are obviously included in expressions (2.23) (where they have been written explicitly) and in

$$\sum_{i_1+\dots+i_{k-1}=2} w_{i_1,\dots,i_{k-1}} \Xi_1^{(0)^{i_1}} \cdots \Xi_{k-1}^{(0)\ \ i_{k-1}}$$

(where there are summands of the sixth and higher degrees with respect to y). All these summands can be included into expression (2.26). Hence, the function Q does not include the terms free of $\xi_1,\dots,\xi_{k-1}$.

Terms linear with respect to $\xi_1,\dots,\xi_{k-1}$ are included into expression (2.25) both by means of summands from $hyY_* + \frac{1}{2}hY_*^2$ and from (2.22). If these terms have order not less than the fourth one with respect to y, then it is clear that they can be included into expression (2.26). Thus, the function Q has only those terms linear with respect to $\xi_1,\dots,\xi_{k-1}$ which have degrees two and three with respect to y.

Finally, consider the terms quadratic with respect to $\xi_1,\dots,\xi_{k-1}$. If these terms have the total degree higher than two, then they can be included into expression (2.27); and, therefore, they are not included in the function Q. All quadratic terms with respect to $\xi_1,\dots,\xi_{k-1}$ having the second degree (i.e. the terms with constant coefficients) are included into the expression

$$\sum_{i_1+\dots+i_{k-1}=2} w_{i_1,\cdots,i_{k-1}} \{[c_{11}\xi_1 + c_{12}\xi_2 + \dots + c_{1,k-1}\xi_{k-1}]^{i_1}$$

$$\times \cdots [c_{k-1,1}\xi_1 + \cdots + c_{k-1,k-1}\xi_{k-1}]^{i_{k-1}} - \xi_1^{i_1} \dots \xi_{k-1}^{i_{k-1}}\} \\ = \xi_1^2 + \cdots + \xi_{k-1}^2$$

and, hence, are not included into the function Q.

Thus, the function Q has the form

$$Q = y^2 Q_2(\xi_1, \dots, \xi_{k-1}) + y^3 Q_3(\xi_1, \dots, \xi_{k-1}), \tag{2.28}$$

where Q_2 and Q_3 are linear forms with respect to $\xi_1, \dots, \xi_{k-1}$:

$$Q_2 = q_1^{(2)}\xi_1 + q_2^{(2)}\xi_2 + \cdots + q_{k-1}^{(2)}\xi_{k-1},$$

$$Q_3 = q_1^{(3)}\xi_1 + q_2^{(3)}\xi_2 + \cdots + q_{k-1}^{(3)}\xi_{k-1}.$$

The presence of summand (2.28) in (2.25) breaks the positive definiteness of ΔV_1. To get rid of the summand $y^2 Q_2(\xi_1, \dots, \xi_{k-1})$, we add the summand $y^2 P_2(\xi_1, \dots, \xi_{k-1}) = y^2(p_1^{(2)}\xi_1 + p_2^{(2)}\xi_2 + \cdots + p_{k-1}^{(2)}\xi_{k-1})$ to the function V_1. Here, $p_j^{(2)}$ $(j = 1, \dots, k-1)$ are constants. In other words, consider the function

$$V_2 = \frac{1}{2}hy^2 + W(\xi_1, \dots, \xi_{k-1}) + y^2 P_2(\xi_1, \dots, \xi_{k-1}) \tag{2.29}$$

instead of the function V_1. The term $y^2 P_2(\xi_1, \dots, \xi_{k-1})$ brings the following summands to ΔV_2:

$$\Delta(y^2 P_2(\xi_1, \dots, \xi_{k-1})) = [y^2 + 2yY_*(\xi_1, \dots, \xi_{k-1}, y) + Y_*^2(\xi_1, \dots, \xi_{k-1}, y)]$$

$$\times \sum_{j=1}^{k-1} p_j^{(2)} [c_{j,1}\xi_1 + c_{j,2}\xi_2 + \cdots + c_{j,k-1}\xi_{k-1} + \Xi_j(\xi_1, \dots, \xi_{k-1}, y)]$$

$$-y^2[p_1^{(2)}\xi_1 + p_2^{(2)}\xi_2 + \cdots + p_{k-1}^{(2)}\xi_{k-1}]$$

$$= y^2 \left[\sum_{j=1}^{k-1} p_j^{(2)} (c_{j1}\xi_1 + c_{j2}\xi_2 + \cdots + c_{j,k-1}\xi_{k-1} - \xi_j) \right] + G(\xi_1, \dots, \xi_{k-1}, y),$$

where the function G is the sum of summands every of which can be included either to expression (2.26) or to (2.27). Let us choose constants $p_1^{(2)}, \dots, p_{k-1}^{(2)}$ such that the equality

$$\sum_{j=1}^{k-1} p_j^{(2)} (c_{j1}\xi_1 + c_{j2}\xi_2 + \cdots + c_{j,k-1}\xi_{k-1} - \xi_j) = -\sum_{j=1}^{k-1} q_j^{(2)}\xi_j \tag{2.30}$$

holds. To do this, let us equate the coefficients corresponding to ξ_j ($j = 1, \dots, k-1$) on the right- and left-hand sides of equality (2.30). We obtain the system of linear equations for $p_j^{(2)}$ ($j = 1, \dots, k-1$):

$$c_{1j}p_1^{(2)} + c_{2j}p_2^{(2)} + \cdots + (c_{jj} - 1)p_j^{(2)} + \cdots + c_{k-1,j}p_{k-1}^{(2)} = -q_j^{(2)} \quad (j = 1, \dots, k-1). \tag{2.31}$$

The determinant of this system is not equal to zero, because all eigenvalues of $\mathcal{C}$ are inside the unit disk. Therefore, system (2.31) has the unique solution. Substituting the obtained values $p_1^{(2)}, \dots, p_{k-1}^{(2)}$ in the expression $P_2(\xi_1, \dots, \xi_{k-1})$, we get

$$\Delta V_2 = [h^2 + g_2(\xi_1, \dots, \xi_{k-1}, y)]y^4 + (\xi_1^2 + \cdots + \xi_{k-1}^2) + \sum_{i,j=1}^{k-1} g_{ij}^{(2)}(\xi_1, \dots, \xi_{k-1}, y)\xi_i\xi_j + y^3 Q_3(\xi_1, \dots, \xi_{k-1}), \tag{2.32}$$

where g_2 and $g_{ij}^{(2)}$ are functions vanishing for $\xi_1 = \xi_2 = \cdots = \xi_{k-1} = y = 0$.

Similarly, we can show that it is possible to get rid of the summand $y^3 Q_3(\xi_1, \dots, \xi_{k-1})$ in expression (2.32). To do this, all we need is to add the summand

$$y^3 P_3(\xi_1, \dots, \xi_{k-1}) = y^3(p_1^{(3)}\xi_1 + p_2^{(3)}\xi_2 + \cdots + p_{k-1}^{(3)}\xi_{k-1})$$

to the function V_2, where $p_j^{(3)}$ ($j = 1, \dots, k-1$) are constants. In other words, consider the function

$$V = \frac{1}{2}hy^2 + W(\xi_1, \dots, \xi_{k-1}) + y^2 P_2(\xi_1, \dots, \xi_{k-1}) + y^3 P_3(\xi_1, \dots, \xi_{k-1}) \tag{2.33}$$

instead of the function V_2. Its variation along system (2.17), (2.18) is equal to

$$\Delta V = [h^2 + g(\xi_1, \dots, \xi_{k-1}, y)]y^4 + (\xi_1^2 + \cdots + \xi_{k-1}^2) + \sum_{i,j=1}^{k-1} g_{ij}(\xi_1, \dots, \xi_{k-1}, y)\xi_i\xi_j, \tag{2.34}$$

where g and g_{ij} are functions vanishing for $\xi_1 = \xi_2 = \cdots = \xi_{k-1} = y = 0$.

It follows from (2.34) that ΔV is positive definite in a sufficiently small neighborhood of the origin, and the function V of form (2.33) is negative definite for $h < 0$ and changes its sign for $h > 0$. Hence, according to Theorems B and C, we can conclude that solution (2.19) of system (2.17), (2.18) is asymptotically stable for $h < 0$ and unstable for $h > 0$. This completes the proof. □

Remark 2.1. Obviously, substitutions (2.3), (2.9), and (2.14) are such that the investigation of the stability of solution (2.19) of system (2.17), (2.18) is equivalent to the investigation of the stability of the zero solution of system (2.1).

Remark 2.2. In Theorems 2.1 and 2.2, there are the conditions under which the problem of the stability of the zero solution of system (2.1) can be solved in the critical case where one eigenvalue of the linearized system is equal to unity. The obtained criteria do not depend on nonlinear terms with degrees of smallness more than three. If we obtain $h = 0$, then the stability problem cannot be solved by terms of the first, second, and third degrees of smallness in the expansions of the right-hand sides of the system of difference equations. To solve this problem in this case, it is necessary to consider also the terms of higher degrees.

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