CHAOTIC MOTION OF DYNAMIC SYSTEMS WITH "ONE" DEGREE OF FREEDOM

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It is shown that the regimes with chaotic motion are also inherent in dynamic systems having only one degree of freedom. Such opportunity arises because such systems in the phase space have areas in which the conditions of the uniqueness theorem are broken, in particular, in the presence of singular solutions. Examples of such systems are given.

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1. INTRODUCTION

Now there is a common agreement that the regimes with chaotic motion in dynamic systems can exist only if they hold the number of degrees of freedom more or equal than 1.5. The main reason for existence of such opinion is the restriction that in the phase plane (system with one degree of freedom) the phase trajectories should not intersect. In other words, there is a requirement that the theorem of uniqueness be held (see, for example, [1]).

If we are limited to a class of ordinary differential equation (ODE) whose parameters satisfy this theorem, such statement, certainly, is correct. However, it is known that except the general solutions (for which the theorem of uniqueness is held), ODE have singular solutions. In points of the singular solutions the theorem of uniqueness is not held. Therefore, if we shall take into account dynamic systems in which there are singular solutions, the formulated restrictions on number of degrees of freedom (1.5) can be removed. In this case the modes with chaotic motion can be realized in the systems with one degree of freedom as well. In the present work we give examples of such systems.

2. SYSTEMS WITH SINGULAR SOLUTIONS

Below we shall show that in the systems with one degree of freedom at the presence of singular solutions the modes with chaos are possible. It is necessary to note that, as soon as the system is situated in the points of singular solutions, it, in general case, "does not know" its further trajectory. The choice of the further trajectory is determined by any external, even arbitrarily small perturbations. As a characteristic example we shall consider the dynamics of the system which is described by the following equations:

$$\dot{x}_0 = x_1; \quad \dot{x}_1 = \left(\frac{x_1^2}{2x_0}\right) - 0.5 \cdot x_0. \qquad (1)$$

The phase portrait of the system (1) is presented in Fig. 1. The integral curves in this case are circles (see below). Also, the centers of the circles are located on the axis x_1, and the radiuses of these circles are equal to distance of these centers from the zero point ($x_0 = 0$; $x_1 = 0$). This point is common for all circles. Besides, this point is the singular solution of the system (1) (see below). The system (1) was analyzed numerically. The results are presented in Figs. 1-6. In the second and third figures the characteristic time dependencies are presented. Comparing Fig. 1 with Fig. 2, it is possible to make the conclusion that the representing point, moving to one of circles after passage of a zero point, finds itself on another circle. Moreover, the transition from one circle to another occurs according to a random law. Really, the spectrum of this dynamics is wide (see Fig. 3), and the correlation function falls down quickly enough (see Fig. 4).

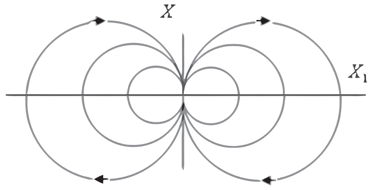

Fig. 1. *Phase portrait of the system (1)*

Let's note that the casual transitions from one circle to another at passage of the zero point depend on the accuracy of computing. The change, e.g., of the step of calculations changes the concrete character of these transitions. However, as a whole, statistically, the dynamics remains the same.

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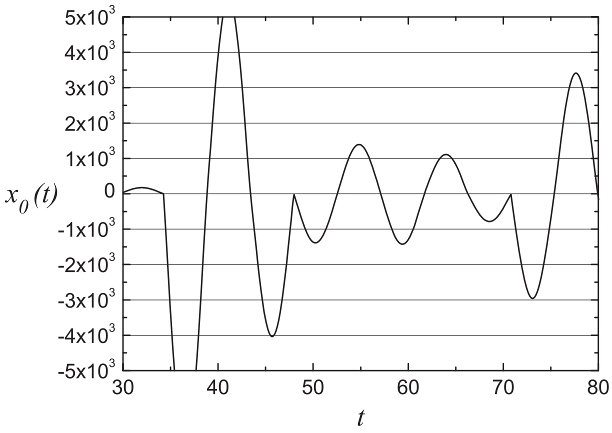

Fig. 2. *Evolution in time of the variable* x_0*. One can see the transitions from one circle to another*

The system (1) is not unique. Below we shall show how the sets of such systems can be constructed. But now let's consider the dynamics of one more system, probably, more interesting from the point of view of chaotic dynamics:

$$\begin{aligned} \frac{dx_0}{dt} &= x_0 \cdot x_1 + \gamma \cdot x_1 \equiv F_1, \\ \frac{dx_1}{dt} &= x_1^2 - x_0^4 - \gamma \cdot x_0 \equiv F_2. \end{aligned} \tag{2}$$

A phase portrait of the system (2) is presented in Fig. 5.

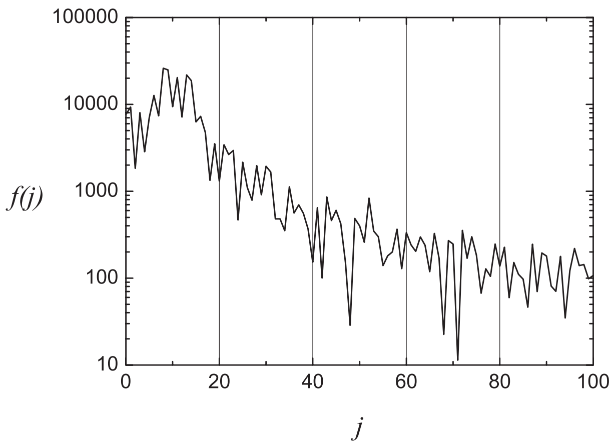

Fig. 3. *Spectrum of* x_0

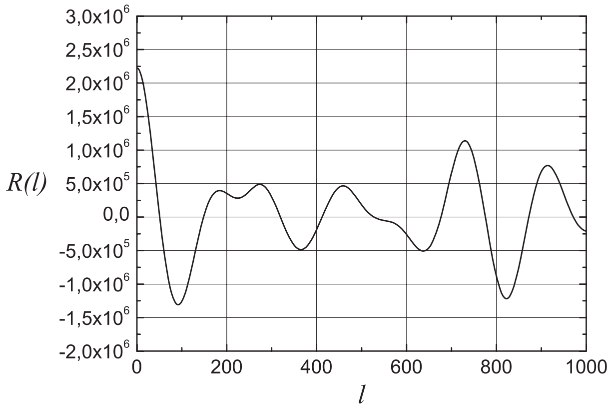

Fig. 4. *Correlation function for variable* x_0

Addition of linear terms ($\gamma \cdot x_i$) to the system (2) has allowed to change the character of a special zero-point. From the complex quasisaddle this point has turned into a "centre" point. If such updating is not done, the representing points will remain on the way to this special point during an unlimited time. To get rid of this feature this additional term has been introduced into the model. The characteristic dependence of dynamics variable on time is presented in Fig. 6.

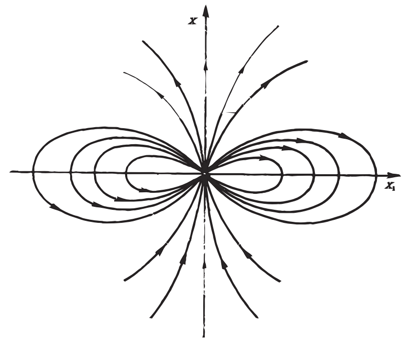

Fig. 5. *Phase portrait of the system* (2) *at* $\gamma = 0$

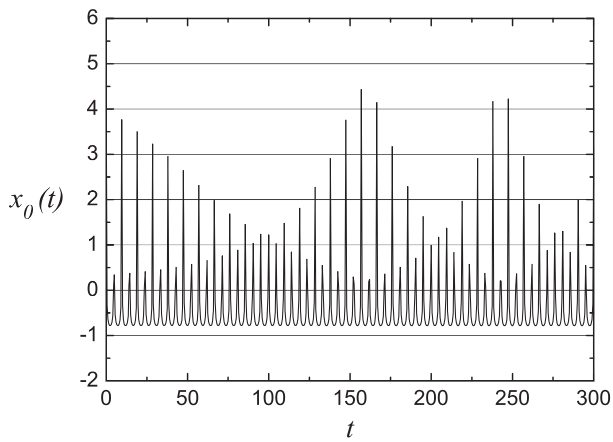

Fig. 6. *Characteristic view of realization of variable* x_0

From this figure it is already visible that the dynamics of transitions from one integrated trajectory to another at the passage in a vicinity of the zero point is complex. Moreover, these transitions occur casually. The randomness of them is visible when we are looking at the spectrum and the correlation function (Figs. 7 and 8). In these two figures one can see that the spectrum is wide enough and the correlation function falls down while oscillating. Now let us show how the set of systems with one degree of freedom similar to the systems (1) and (2) can be constructed. Let us take an integral curve which is given by the equation $\varphi(x_0, x_1) = 0$.

Then the system of equations whose integral will be this integral curve can be represented in the following form:

$$\begin{aligned} \frac{dx_0}{dt} &= F_1(\varphi, x_0, x_1) - \frac{\partial\varphi}{\partial x_1} M(x_0, x_1), \\ \frac{dx_1}{dt} &= F_2(\varphi, x_0, x_1) + \frac{\partial\varphi}{\partial x_0} M(x_0, x_1), \end{aligned} \tag{3}$$

where $F_s(\varphi, x_0, x_1)$ are arbitrary functions which have such property: $F_s(0, x_0, x_1) = 0$; $M(x_0, x_1)$ is an arbitrary function.

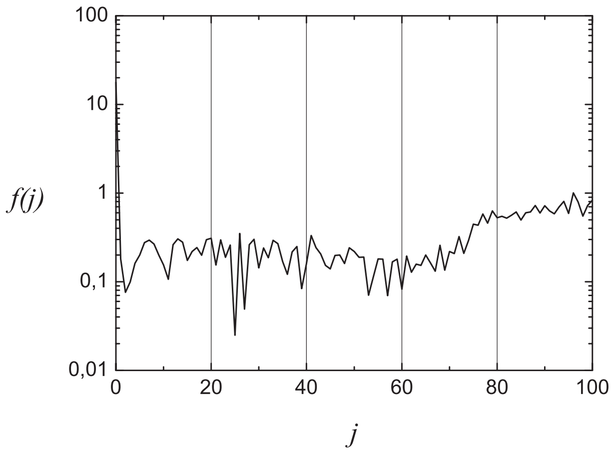

Fig. 7. *Spectrum of* x_0

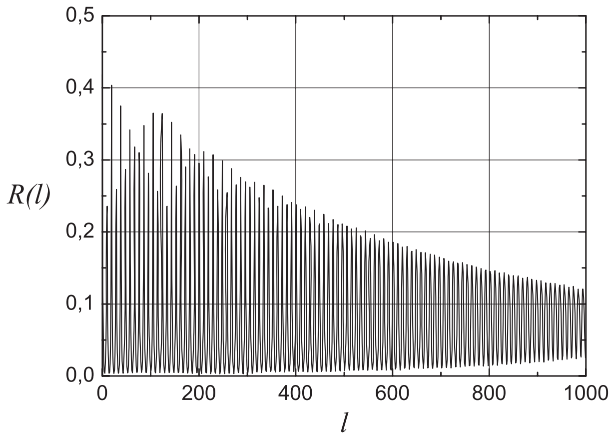

Fig. 8. *Correlation function for variable* x_0

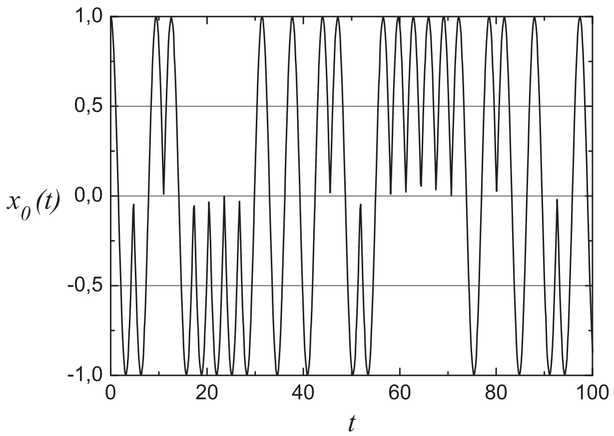

Fig. 9. *Dependence of amplitude on time*

Using the system (3), it is possible to construct a large variety of dynamic systems which will possess the necessary properties. As an example we shall consider a case that the integral curves are a family of circles with the radius R:

$$\varphi = (x_0 - R)^2 + x_1^2 - R^2 = 0. \tag{4}$$

The set of integral curves (4) is presented in Fig. 1. By choosing the functions F_s and M it is possible to achieve an elimination of the parameter R from the system (3). Really, we shall choose (for example) such functions as: $F_1 = 0$; $F_2 = \varphi \cdot f(x_0, x_1)$; $M = -x_0 \cdot f(x_0, x_1)$. Here $f(x_0, x_1)$ is an arbitrary function. Substituting these expressions in the system (3), we shall get a set of systems of equations in which the parameter R is already eliminated:

$$\begin{aligned} \frac{dx_0}{dt} &= 2x_0 \cdot x_1 \cdot f(x_0, x_1), \\ \frac{dx_0}{dt} &= \left(x_1^2 - x_0^2\right) \cdot f(x_0, x_1). \end{aligned} \tag{5}$$

Choosing function $f(x_0, x_1)$ as $f(x_0, x_1) = 1/2x_0$ we shall get the system of equations (1).

The dynamics of this system is chaotic. This is visible from Figs. 2-4. It is necessary to notice the following: the integral curves of the system (5) will be a family of circles with a common point $(0, 0)$. Despite this fact, the character of this point and the dynamics of the system will depend essentially on the form of the function $f(x_0, x_1)$.

Let's show now that the zero point is a singular solution of the system (1). Really, this point belongs to the family of circles (4). The same circles are integral curves of the system (1). These integral curves are convenient to be rewritten as: $x_1^2/x_0 + x_0 = R$. From these curves it follows that in the vicinity of zero point the Lipchitz conditions for the system (1) are broken. Really, the Lipchitz condition for the system (1) can be written down as

$$\left| \frac{\tilde{x}_1^2}{\tilde{x}_0} - \frac{x_1^2}{x_0} \right| \le L\left(|\tilde{x}_0 - x_0| + |\tilde{x}_1 - x_1|\right), \tag{6}$$

where L is a positive constant.

In the vicinity of zero the left part of the inequality (6) can be estimated as $\left|\tilde{R} - R\right|$, where $\tilde{R}$ and R are radiuses of two arbitrary circles. Generally, the differences of these radiuses can be of any size. Thus, in a zero point the Lipchitz conditions are not satisfied, i.e. the conditions of the theorem of uniqueness for the system (1) are broken. Besides taking the partial derivative of the function (4) with respect to R and equating it to zero, we find that the point $(x_0 = 0;\ x_1 = 0)$ is a singular solution of the system (1), and also envelope of the integral curves.

Let's consider now the system of equations (2). If $\gamma = 0$ then this system has the following family of integral curves:

$$\varphi = x_0^4 + x_1^2 - Cx_0^2 = 0. \tag{7}$$

Here C is a certain constant.

The appearance of these integral curves is presented in Fig. 5. Using this formula the right part of the second equation of the system (2) at $\gamma = 0$ can be rewritten in the following form: $F_2 = -2 \cdot x_0^4 + Cx_0^2$. Using this expression, it is easy to see that the left part of the Lipchitz inequality in a vicinity of small value of x_0 will be proportional to $x_0^2\left|(C - \tilde{C})\right|$. Taking into account that the constants C and $\tilde{C}$ also can be essentially different, in a vicinity of zero point (but not namely in this point) the Lipchitz conditions will not be satisfied. However such "cutting down" of the system of equations (2) will behave quite regularly.

Moreover, very quickly all the initial points will find themselves in the vicinity of the zero point and will remain in its vicinity arbitrarily long time. To remove such feature of dynamics in the system (2), the linear terms were added. They do not permit the representing points to stop in the zero point.

Let's point out another opportunity of construction of the systems of differential equations with one degree of freedom whose dynamics can show chaotic behavior. Let us have two families of integral curves $\varphi_1(x_0, x_1) = 0$ and $\varphi_2(x_0, x_1) = 0$. Using the system of equations (3), it is easy to find a system of differential equations whose solutions are such integral curves:

$$\begin{aligned} \frac{dx_0}{dt} &= \frac{\partial\varphi_2}{\partial x_1} F_1(\varphi_1,\, x_0,\, x_1,\, t) - \\ &\quad - \frac{\partial\varphi_1}{\partial x_1} F_2(\varphi_2,\, x_0,\, x_1,\, t), \\ \frac{dx_1}{dt} &= -\frac{\partial\varphi_2}{\partial x_0} F_1(\varphi_1,\, x_0,\, x_1,\, t) + \\ &\quad + \frac{\partial\varphi_1}{\partial x_0} F_2(\varphi_2,\, x_0,\, x_1,\, t). \end{aligned} \tag{8}$$

Here $F_s(\varphi_s,\, x_0,\, x_1,\, t)$ are arbitrary functions possessing the property $F_s(0,\, x_0,\, x_1,\, t) = 0$. From the system (8) it is possible to obtain the following equation for φ_s:

$$\frac{d\varphi_s}{dt} = \left[\frac{\partial\varphi_2}{\partial x_1}\frac{\partial\varphi_1}{\partial x_0} - \frac{\partial\varphi_2}{\partial x_0}\frac{\partial\varphi_1}{\partial x_1}\right] F_s \equiv \Delta \cdot F_s. \tag{9}$$

From (9) it follows that if we want these integral curves to be stable, the fulfillment of the following inequalities is necessary:

$$\left[\Delta \cdot \frac{\partial F_s}{\partial \varphi_s}\right]_{\varphi_s = 0} < 0. \tag{10}$$

For our purposes (the occurrence of chaotic dynamics) is of interest the case that the integral curves $\varphi_1(x_0, x_1) = 0$ and $\varphi_2(x_0, x_1) = 0$ have a common point. However it is obvious, that if integral curves are stable, and the inequality (10) is fulfilled, the common points will be stable too. In this case, it is obvious that the dynamics will be regular. Therefore it is necessary to require that in common points (in points of crossing of integral curves) the stability must be broken. It can be achieved if one requires

$$\left[\Delta \cdot \frac{\partial F_s}{\partial \varphi_s}\right]_{\varphi_s = 0} \begin{cases} < 0 \\ \geq 0. \end{cases} \tag{11}$$

The upper inequality in (11) should be fulfilled in all points, where $\varphi_1 \neq \varphi_2$. The lower inequality should be fulfilled in the crossing points $\varphi_1 = \varphi_2 = 0$.

3. SYSTEM WITH BROKEN CONDITIONS OF UNIQUENESS

Above we have considered the systems which have singular solutions. In the points of singular solutions the uniqueness theorem is broken. This fact also has resulted in qualitative change of the dynamics of investigated system. Therefore it is clear that generally there is no necessity in availability of singular solutions. It is sufficient that in the phase space of the investigated system there was an area in which the uniqueness theorem is broken. Besides it is important that the trajectories of the investigated system find themselves often enough in this area of phase space. As the elementary example, in which such scenario of the development of chaotic dynamics can be realized, we shall consider the following simple system:

$$\dot{x} = y, \quad \dot{y} = -x. \tag{12}$$

Let functions $x(t)$, $y(t)$ be complex. Then two variants of substitutions are possible. The first one is $x = x_R + i \cdot x_I$, $y = y_R + i \cdot y_I$. In this case the dynamics of real and imaginary parts are independent. In this case new dynamics does not arise.

New dynamics arises at introduction of other dependent variables, namely

$$x = x_0 \cdot \exp(i \cdot x_2) \quad y = x_1 \cdot \exp(i \cdot x_3). \tag{13}$$

Here $x_k(t)$ $k = \{0, 1, 2, 3\}$ are real functions.

Substituting (13) in (12) we shall get the following system of ODE for finding $x_k(t)$:

$$\begin{aligned} &\dot{x}_0 = x_1 \cos(x_3 - x_2), \;\; \dot{x}_1 = -x_0 \cos(x_3 - x_2), \\ &\dot{x}_2 = \frac{x_1}{x_0} \sin(x_3 - x_2), \;\; \dot{x}_3 = \frac{x_0}{x_1} \sin(x_3 - x_2). \end{aligned} \tag{14}$$

These four equations can be rewritten as three ones:

$$\begin{aligned} \dot{x}_0 = x_1 \cos\Phi, \quad \dot{x}_1 = -x_0 \cos\Phi, \\ \dot{\Phi} = \left(\frac{x_0}{x_1} - \frac{x_1}{x_0}\right) \cdot \sin\Phi, \end{aligned} \tag{15}$$

where $\Phi = x_3 - x_2$.

From the first two equations of the system (14) or (15) it is visible that these equations have the following integral:

$$x_0^2 + x_1^2 = \text{const.} \tag{16}$$

Thus, the system of equations (15) has only one degree of freedom. The system (14) was investigated by numerical methods. The characteristics of their dynamics are presented in Figs. 9-11. From these figures it is visible that as soon as the value of amplitude appears in the vicinity of zero ($x_0 = 0$), the value of the phase (x_2) can undergo a jump. Occurrence of these jumps is called by the fact that in the point $x_0 = 0$ the theorem of uniqueness for the system (5) is not held. In this point arbitrarily small fluctuations can define the value of a phase. As a matter of fact, in the given elementary example the choice is not so rich: the phase can either remain continuous or undergo a jump to a value of π. The occurrence of these jumps, as it is visible from Fig. 10, is a stochastic function. Similar dynamics is observed for amplitude x_1 and for a phase x_3. As a result of these random jumps the dynamics of all system becomes chaotic. The degree of randomness can be characterized by correlation function (see

Fig. 11). For our case the correlation function falls down quickly enough. It is possible to show that also other statistical characteristics for the system (14) (spectra, main Lyapunov index) are the same as for the systems with chaotic dynamics.

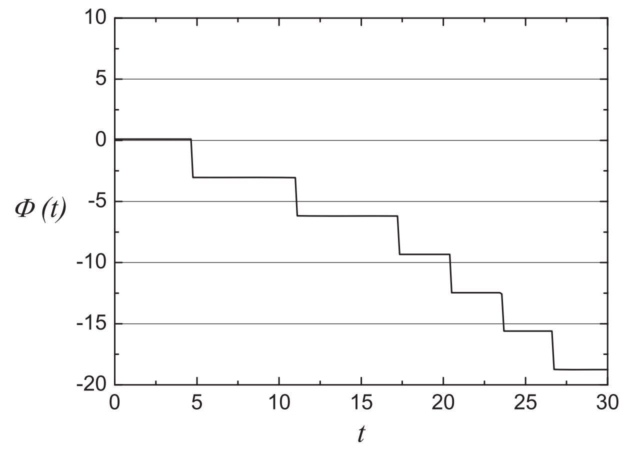

Fig. 10. *Evolution of phase in time*

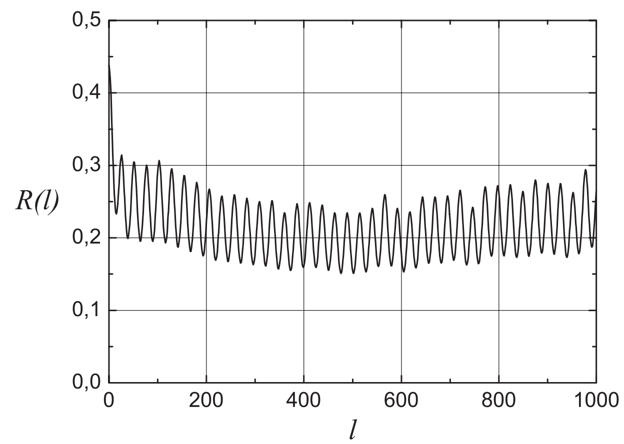

Fig. 11. *Correlation function for variable* x_0

4. CONCLUSIONS

Thus, if we abandon the conditions of uniqueness of the solutions, then in such systems the modes with chaotic motion can appear. The simplest systems in this case are the systems with one degree of freedom which we have considered above. However, it is clear that in the systems with a large number of degrees of freedom such mechanism of occurrence of chaotic regimes will appear too. It is necessary to say that in many systems which, for example, are described by the systems of equations (1) or (12) the randomness arises as a result of the presence of small fluctuations at numerical calculation. And, the more precise calculation is used, the smaller area in a vicinity of zero will define the chaotic motion. Thus, chaotic dynamics of these systems is practically caused by the presence (even arbitrarily small) of casual forces. Just for this reason the word "one" was taken in inverted commas in the title of this paper. However, these cases, apparently, do not exhaust all opportunities. If the dynamics of the system is such as the area of non-uniqueness in the phase space is not limited to one point, the chaotic dynamics of the system may not depend on numerical fluctuation. However, these questions require an additional study. It is necessary to notice that some aspects of the dynamics considered above are similar to the dynamics of singular point mapping which were studied, for example, in [2, 3].

References

1. A.J. Liechtenberg, M.A. Lieberman. *Regular and Stochastic Motion.* New York: "Springer-Verlag New York inc.", 1983, 499 p.
2. S.V. Slipushenko, A.V. Tur, V.V. Yanovsky. Mechanism of intermittency arising in singular conservative systems // *Izvestia vuzov "AND".* 2010, v. 18, N. 4, p. 91-110.
3. S.V. Slipushenko, A.V. Tur, V.V. Yanovsky. Intermittency without chaotic phases // *Functional materials.* 2006, v. 13, N. 4, p. 551-557.

ХАОТИЧЕСКОЕ ДВИЖЕНИЕ СИСТЕМ С "ОДНОЙ" СТЕПЕНЬЮ СВОБОДЫ

В.А. Буц

Показано, что режимы с хаотическим движением присущи также динамическим системам, имеющим всего одну степень свободы. Такая возможность возникает благодаря тому, что такие системы в своем фазовом пространстве имеют области, в которых нарушаются условия теоремы единственности, в частности, при наличии особых решений. Приведены примеры таких систем.

ХАОТИЧНИЙ РУХ СИСТЕМ З "ОДНИМ" СТУПЕНЕМ СВОБОДИ

В.А. Буц

Показано, що режими з хаотичним рухом властиві також динамічним системам, що мають усього один ступінь свободи. Така можливість виникає завдяки тому, що такі системи у своєму фазовому просторі мають області, у яких порушуються умови теореми єдності, зокрема, при наявності особливих рішень. Наведено приклади таких систем.