DYNAMIC CHAOS GENERATED BY LINEAR SYSTEMS

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It is shown that regimes with chaotic behavior are also inherent in linear systems. A dynamic chaos which can arise in essentially quantum systems (not quasi-classical) is of special interest. In particular, it is shown that the diffusion of quantum systems in the energy space can be considerably more effective than transitions at multiphoton processes. It is shown that taking into account the singular solutions allows to realize regimes with chaotic behavior even in systems with "one" degree of freedom.

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1. INTRODUCTION

Practically till now it is considered that the regimes with dynamic chaos (DC) arise only in nonlinear systems. For this reason nobody carries out the search of regimes with DC at the study of linear systems. However, it is well known that by the transformation of the dependent variable the linear systems often can be transformed into nonlinear systems. In such systems regimes with DC are widely presented. Well-known examples are transitions from the equations of quantum mechanics to the equations of classical mechanics, and also transitions from the Maxwell equations to the equations of geometrical optics.

Thus, today there are known at least two examples at which in linear systems at certain values of their parameters the regimes with DC are possible. In works [1-3] it is shown that this situation is considerably more widespread and that the regimes with dynamic chaos are inherent to a large number of linear systems. In the present work the results of the analysis of this feature for a number of classical and quantum systems are given from unified positions. A possible mechanism of occurrence of unpredictability is discussed. For these purposes it is convenient to use the concept of a measure. The introduction of stochastic conjugated functions is also useful. These functions are irregular. However, some of their combinations behave regularly. The examples of such functions are given.

The significant role of singular solutions in the origin of chaotic dynamics is shown.

2. DYNAMIC CHAOS AT INTERACTION OF THREE LINEAR OSCILLATORS

For definiteness we shall consider the most simple and at the same time one of the most important linear physical system, in which the mode with DC is possible. This system is three connected linear oscillators:

$$\ddot{q}_0 + q_0 = -\mu_1 q_1 - \mu_2 q_2;\ \ddot{q}_1 + q_1 = -\mu_1 q_0; \quad \ddot{q}_2 + (1+\delta)\, q_2 = -\mu_2 q_0, \tag{1}$$

where $\dot{q} \equiv \frac{dq}{d\tau}$, $\mu_i \ll 1,\ \delta \ll 1$.

The r.h.s terms (factors of connection) of the system are small. That is why the solutions of the system (1) are convenient to be searched as:

$$q_i = A_i(\tau)\exp\left(i\omega_i t\right). \tag{2}$$

For finding complex amplitudes $A_i(\tau)$ it is possible to get the following system of equations:

$$2i\dot{A}_0 = -\mu_1 A_1 - \mu_2 A_2 \exp(i\delta\tau), \quad 2i\dot{A}_1 = -\mu_1 A_0,\ 2i\dot{A}_2 = -\mu_2 A_0 \exp(-i\delta\tau). \tag{3}$$

The system of equations (3) is linear. For further analysis of dynamics of $A_i(\tau)$ we shall present them as:

$$A_i(\tau) = a_i(\tau)\exp(i\varphi(\tau)), \tag{4}$$

here a_i, φ_i are real amplitudes and real phases. Substituting (4) in (3) for finding a_i and φ_i we shall get the following equations:

$$\begin{aligned}
&\dot{a}_0 = -\left(\mu_1/2\right) a_1 \cdot \sin\Phi - \left(\mu_2/2\right) \cdot a_2 \cdot \sin\Phi_1;\\
&\dot{a}_1 = \left(\mu_1/2\right) a_0 \cdot \sin\Phi;\ \dot{a}_2 = \left(\mu_2/2\right) a_0 \cdot \sin\Phi_1;\\
&\dot{\Phi} = \left(\frac{\mu_1}{2}\right)\left(\frac{a_0}{a_1} - \frac{a_1}{a_0}\right)\cos\Phi - \left(\frac{\mu_2}{2}\right)\left(\frac{a_2}{a_0}\right)\cos\Phi_1,\\
&\dot{\Phi}_1 = \left(\frac{\mu_2}{2}\right)\left(\frac{a_0}{a_2} - \frac{a_2}{a_0}\right)\cos\Phi_1 - \left(\frac{\mu_1}{2}\right)\left(\frac{a_1}{a_0}\right)\cos\Phi + \delta,
\end{aligned} \tag{5}$$

where $\Phi \equiv \varphi_1 - \varphi_0,\ \Phi_1 \equiv \varphi_2 - \varphi_0 + \delta\tau$.

The system (5) is already nonlinear. Basically, the dynamics of such system can be chaotic. The analysis of system (5) gives the following estimation for the condition of occurrence of regimes with DC: $(\mu_1 + \mu_2) > \delta$. It is possible to find more detailed information about the system (5), and also about the results of numerical investigation in [1-5]. Let us note

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only that at fulfilling the conditions for onset of DC the dynamics of amplitudes (a_i) and phases (φ_i) had been found as chaotic. The spectra are wide, correlation functions fall down quickly, the main Lyapunov indexes are positive.

3. WAVES SCATTERING IN THE MEDIUM WITH WEAK PERIODIC INHOMOGENEITY

Properties similar to the considered case of regular or chaotic dynamics in the system of three linear coupled oscillators take place in a large number of other linear systems. Here we consider a model of an electromagnetic wave scattering on a non-uniform dielectric medium filling the half-space ($z > 0$ for unambiguity). The wave comes from the upper ($z < 0$) homogeneous half-space. The wave of electrical field satisfies the well known wave equation:

$$\Delta E + k^2 \varepsilon E = 0,$$

where $k = \omega/c$.
We consider the case that the permittivity is described by the formula:

$$\varepsilon = 1 + \sum_{i=1}^{2} \mu_i \cdot \cos\left(\vec{\kappa}_i \cdot \vec{r}\right).$$

It is assumed that the heterogeneity is small, $\mu_i \ll 1$. In this case the electromagnetic wave scattering gives diffracted waves of the minus first order of diffraction. The complete field can be represented in the form:

$$E = \sum_{i=0}^{2} E_i = \sum_{i=0}^{2} A_i(z) \cdot \exp\left(i\vec{k}_i\vec{r}\right).$$

Here the first term corresponds to the incident wave, the second and third ones correspond to the minus first order of diffraction by the first and the second heterogeneities accordingly. Let the relation between the wave vectors satisfy the following expressions:

$$\vec{k}_1 = \vec{k}_0 - \vec{\kappa}_1, \vec{k}_2 = \vec{k}_0 - \vec{\kappa}_2 + \vec{\delta}, \vec{\delta} = (0, 0, \delta).$$

In this model we consider the case that all the wave amplitudes depend only on the coordinate z directed into the lower half- space. Substituting equation for the field into the wave equation and applying averaging to find the slow varying amplitudes of the interactive waves we obtain the following system of equations:

$$\begin{aligned} 2iA_0' &= \mu_1 A_1 + \mu_2 A_2 \exp(i\delta z), \\ 2iA_1' &= \mu_1 A_0 \cdot (k_{0z}/k_{1z}), \\ 2iA_2' &= \mu_2 A_0 \cdot (k_{0z}/k_{2z}) \exp(-i\delta z), \end{aligned} \quad (6)$$

where $A' \equiv dA/dz$ and the following dimensionless parameters and independent variables are introduced: $\mu_1 \equiv \mu_1 \cdot k/k_{0z}$, $\mu_2 \equiv \mu_2 \cdot k/k_{0z}$, $\delta \equiv \delta/k$, $z \equiv k \cdot z$.

The system (6) is similar to the system of equations (3) if the derivative with respect to time is substituted by the derivative with respect to the coordinate z. Therefore, the dynamics of the systems (3) and (6) has the same qualitative description. They both have the areas of parameters in which their behaviour is chaotic. The chaos criterion, for example for the system (6), in the symmetric case ($\mu_1 \approx \mu_2 = \mu$), is given by the inequality:

$$\delta < (k^2 \cdot \mu)/4\sqrt{k_{1z} \cdot k_{2z}}.$$

4. CHAOS IN QUANTUM SYSTEMS

Anyone who uses words "quantum" and "chaos" in the same sentence should be hung by his thumbs on a tree in the park behind the Niels Bohr Institute

Joseph Ford

This epigraph (from [6]) reflects the attitude of many scientists to researches on quantum chaos. This attitude, basically, is caused by the fact that the equations of quantum mechanics are linear equations. Below we shall show that at the appropriate transformation of dependent variable the equations of quantum mechanics become nonlinear, in which the regimes with DC exist.

Let us consider a quantum system which is under perturbation. Its Hamiltonian is $\hat{H} = \hat{H}_0 + \hat{H}_1(t)$. We will use the conventional theory of perturbation. In this case the solution can be represented in such a form:

$$\psi(t) = \sum_n A_n(t) \cdot \varphi_n \cdot \exp(i\omega_n t).$$

Unknown amplitudes $A_n(t)$ must be found from the following system of equations:

$$i\hbar \cdot \dot{A}_n = \sum_m U_{n\,m}(t) \cdot A_m, \quad (7)$$

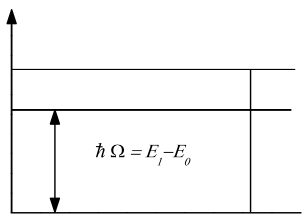
where

$$U_{n\,m} = \int \varphi_m^* \, \hat{H}_1(t) \, \varphi_n \, \exp[i\,t\,(E_n - E_m)/\hbar] \, dq.$$

Let us consider a more simple case when in our system a harmonic perturbation $\hat{H}_1(t) = \hat{U} \cdot \exp(i\Omega t)$ is present. In this case the matrix elements of interaction get such forms:

$$\begin{aligned} U_{n\,m} &= V_{n\,m} \exp\{i \cdot t \cdot [(E_n - E_m)/\hbar + \Omega]\}, \\ V_{n\,m} &= \int \varphi_n^* \cdot \hat{U} \cdot \varphi_m dq. \end{aligned} \quad (8)$$

Let us consider the dynamics of three-level system ($|0\rangle$, $|1\rangle$, $|2\rangle$) (see the figure).



Scheme of energy levels

We will suppose that the frequency perturbation and eigenvalue of energies of these levels satisfy such relations:

$$m = 1,\ \ n = 0,\ \ \hbar\Omega = E_1 - E_0,\ \ m = 2,\ \ n = 0,$$
$$\hbar(\Omega + \delta) = E_2 - E_0,\ \ |\delta| \ll \Omega. \tag{9}$$

Using these relations one can limit oneself only to three equations in the system (7):

$$\begin{gathered} i\hbar\dot{A}_0 = V_{01}A_1 + V_{02}A_2\,\exp(i\delta t), \\ i\hbar\dot{A}_1 = V_{1\,0}A_0, \quad i\hbar\dot{A}_2 = V_{2\,0}A_0\exp(-i\delta t). \end{gathered} \tag{10}$$

The system (10) at $V_{i\,0} = V_{0\,i}$, ($i = 1,\,2$) is equivalent to the system (3).

Let us make such replacement of the dependent variable $A_i(\tau) = a_i(\tau)\exp(i\varphi(\tau))$. Then for a definition of variables a_i and φ_i we shall get a system of nonlinear equations which are equivalent to the system (5). A condition of the development of DC in it will be the inequality $(\mu_1 + \mu_2) > \delta$ (here $\mu_i = 2\cdot V_{0\,i}/\hbar\cdot\Omega$). This inequality has a simple physical sense. It means that Rabi frequency $(V_{0,\,1}/\hbar)$ of transitions between zero and first levels should be more than the distance between the first and the second levels.

Let us now assume that the conditions of realization of DC are fulfilled. The investigated systems in this case will casually wander on power levels. It is interesting to give an estimation of time for transition of the system from one level to another and to compare this value to the regular transitions. The dimensionless time in the regular case is equal to the ratio of frequency of external perturbation to Rabi frequency: $\tau_r \sim 1/\mu = \hbar\Omega/U_{01}$.

We can get such estimation for probability to find our system on first energetic level $\left\langle a_1^2\right\rangle \sim \mu^2\cdot\left\langle a_0^2\right\rangle\cdot\tau$ in stochastic regime. The squares of amplitudes vary from 0 to 1. From this one can estimate the average transition time between the levels in a stochastic regime by such expression: $\tau_{ch} \sim (\tau_r)^2 \sim (\hbar\cdot\Omega/U_{01})^2$. Thus, the time of diffusion in energy space at distance ΔE can be estimated by formula: $\tau_D \sim (\Delta E/\hbar\Omega)\,(\hbar\Omega/U_{01})^2$. It is interesting to compare this time to the time of multiphoton transitions. The time of multiphoton transitions at distance ΔE can be estimated as: $\tau_{MPh} \sim (\hbar\Omega/U_{01})^{(\Delta E/\hbar\Omega)}$. If the inequality $(\Delta E/\hbar\Omega) \gg 1$ holds, the efficiency of transitions at the stochastic motion will be much higher than the one due to the multiphoton transitions: $(\tau_{D,CH}/\tau_{MPh}) \ll 1$.

5. CHANGE OF PROBABILITY AT TRANSFORMATIONS OF DEPENDENT VARIABLES

Such question arises: which replacement of dependent variables will lead to a system of nonlinear equations which have regimes with DC? Now there is no exhausting answer to this question. However, there are considerations that the measure and density of probability can be useful in this case. Really, let's introduce the measure $\Delta\mu = p\left(\vec{x}_i\right)\cdot\Delta\vec{x}$. Here $\Delta\vec{x}$ is the volume of the phase space, $p\left(\vec{x}_i\right)$ is the probability to find our system in the point $\vec{x}_i$ of the phase space. Let's make the replacement $\vec{z}_k = f(\vec{x}_i)$. Thus the new density of probability is connected to the old density of probability by the following formula (see, for example, [7]):

$$g(\vec{z}) = \sum_i p\left(\vec{x}_i\right)\frac{\Delta\vec{x}_i}{\Delta\vec{z}} = \sum_i \frac{p\left(\vec{x}_i\right)}{|J|}, \tag{11}$$

here J is the Jacobian of transformation from the old variables to the new ones.

If initially we have a linear system, then for its density of probability it is possible to choose function: $p(\vec{x}_i) \sim \delta(\vec{x}_i - \vec{x}_i(t))$. From the formula (11) it follows that we can lose determinacy if Jacobian tends to infinity. Let us consider which Jacobian has our replacement (5). In our case we have $A_k = A_k' + iA_k'' = a_k\exp\left(i\varphi_k\right)$; $A_k' = a_k\cos\varphi_k$, $A_k'' = a_k\sin\varphi_k$. The Jacobian of this transformation is $|J| = 1/\left|a_k\right|$. Thus, if as a result of dynamics of investigated system the amplitude tends to zero ($a_k \to 0$), the Jacobian will infinitely grow. This area of the new phase space will be a source of uncertainty.

6. CONCLUSIONS

It is unconditional that the most important result of the carried out researches is the fact that the transformation of the dependent variable in linear systems can transform them in nonlinear systems which can have regimes with DC. Let us note that such regimes appeared in the systems which were not exposed to coarsening.

Above we saw that the dynamics of new dependent variable can be chaotic. At the same time the initial dependent variable obeys the linear equations. Therefore their movement should be regular. Really, the numerical investigation shows that despite of the fact that functions a_i and φ_i behave irregularly being separate, their combination such as $A_k' = a_k\cos\varphi_k$, $A_k'' = a_k\sin\varphi_k$ behaves quite regularly. Such combination of stochastic functions can be named as stochastic conjugated functions. But it is turned out that sometimes the combination of functions $A_k' = a_k\cos\varphi_k$, $A_k'' = a_k\sin\varphi_k$ behaves irregularly. There is a question, to what is it connected? The analysis shows that it occurs only in the case when the amplitudes a_i pass through zero. We saw that the information about initial dynamics in this case is lost. However, another factor can be included in the game: the one of violation of uniqueness of the solution. Really, at passage of functions through zero, as it is easy to see, the conditions of the theorem of uniqueness are broken.

Looking at all these investigations a question can arise: why is it necessary to transform a simple linear system into a complex nonlinear system? (We have accustomed to the return transformations, when nonlinear equations are transformed to linear ones. A classical example now is the method LA pair). The answer to this question is such: first of all at these transformations new sides of dynamics of the investigated systems are revealed, and secondly, as soon as we have defined the area of parameters of investigated system, in which the regimes with DC are realized, for the analysis of behavior of investigated system in new variable the methods of statistical physics can be used. In many cases it allows us to understand essentially better the behavior of considered system. In particular, above at the consideration of the dynamics of a quantum system we have seen that as soon as the Rabi frequency became higher than the frequency between two close levels, the DC develops. Moreover, the efficiency of the diffusion process in the energy space can be much greater than at multiphoton processes. Already this result justifies the use of such transformations.

Above we have considered only one type of transformations of dependent variable. Such transformation is most widely used in all physical researches. However it is obvious that other transformations can play a similar role too. As example of such new transformation one can point on the one which realizes a transformation of an ordinary second-order linear differential equation into a Ricatti equation.

It is also necessary to pay attention to the fact that now there is a common agreement that chaotic dynamics is possible only in the systems with the number of degrees of freedom equal or more than 1.5. This statement is caused by the fact that there were considered the systems for which the conditions of the uniqueness theorem were fulfilled. The systems having singular solutions were dropped out from the investigation. The systems in which the phase space had areas with violation of the uniqueness theorem were dropped out too. If we take into account such systems then we find out that the chaotic behavior will be inherent even in systems with one degree of freedom. Such example we saw above. More examples and more details of this situation are included in the next work.

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ДИНАМИЧЕСКИЙ ХАОС, ГЕНЕРИРУЕМЫЙ ЛИНЕЙНЫМИ СИСТЕМАМИ

В.А. Буц

Показано, что режимы с хаотическим поведением присущи также линейным динамическим системам. Особый интерес представляет динамический хаос, который может возникать в существенно квантовых системах (не квазиклассических). В частности, показано, что диффузия квантовых систем в пространстве энергии может быть значительно более эффективной, чем переходы при многофотонных процессах. Показано, что учет особых решений позволяет реализовать режимы с хаотическим поведением даже в системах с «одной» степенью свободы.

ДИНАМІЧНИЙ ХАОС, ЯКИЙ ГЕНЕРУЄТЬСЯ ЛІНІЙНИМИ СИСТЕМАМИ

В.А. Буц

Показано, що режими з хаотичним поводженням властиві також лінійним динамічним системам. Особливий інтерес представляє динамічний хаос, що може виникати в істотно квантових системах (не квазікласичних). Зокрема, показано, що дифузія квантових систем у просторі енергії може бути значно більш ефективною, ніж переходи при багатофотонних процесах. Показано, що облік особливих рішень дозволяє реалізувати режими з хаотичним поводженням навіть у системах з «одним» ступенем свободи.