CLUSTER EXPANSION IN CITE PERCOLATION PROBLEM ON CUBIC LATTICE

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The cluster expansion of the Bernoulli random field percolation probability of the cubic lattice has been built. On its basis, it has been obtained the upper guaranteed estimate of the percolation threshold and corresponding accuracy estimates are proposed when some approximations are built.

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1. INTRODUCTION

We consider the set $\mathbf{Z}^3$ consisting of elements which are named vertexes. Let φ be the binary symmetric relation defined on this set. It is named the *adjacency* and it is such that for each vertex $\mathbf{x} = \langle n_1, n_2, n_3 \rangle$, all vertexes $\mathbf{y} \in \{\mathbf{x} \pm \mathbf{e}_k; k = 1, 2, 3\}$ where $\mathbf{e}_1 = \langle 1, 0, 0 \rangle$, $\mathbf{e}_2 = \langle 0, 1, 0 \rangle$, $\mathbf{e}_3 = \langle 0, 0, 1 \rangle$ are adjacent to $\mathbf{x}$. Such adjacent relation transforms $\mathbf{Z}^3$ to the infinite periodic graph [1] which is named the *cubic lattice* and, further, we denote it by the same symbol $\mathbf{Z}^3$.

Let $\{\tilde{c}(\mathbf{x}); \mathbf{x} \in \mathbf{Z}^3\}$ be the random uniform Bernoulli field with values $\{0, 1\}$ on the graph $\mathbf{Z}^3$. It is completely defined by the probability $c = \Pr\{\tilde{c}(\mathbf{x}) = 1\}$ named the *concentration*. For each random realization $\tilde{c}(\mathbf{x})$, $\mathbf{x} \in \mathbf{Z}^3$, there exists the set $\tilde{W} = \{\mathbf{x} : \tilde{c}(\mathbf{x}) = 1\}$ one-to-one compared to it. It is named the *configuration* $\tilde{W}$. The probability distribution of random configurations $\tilde{W}$ is completely defined by the random field $\{\tilde{c}(\mathbf{x}); \mathbf{x} \in \mathbf{Z}^3\}$.

The sequence $\gamma = \langle \mathbf{x}_0, \mathbf{x}_1, ..., \mathbf{x}_n \rangle$ consisting of vertexes belonging to the configuration $\tilde{W}$ and such that $\mathbf{x}_{i-1} \varphi \mathbf{x}_i$, $i = 1 \div n$ and $\mathbf{x}_j \neq \mathbf{x}_k$ at $j \neq k$ is named the *nonintersecting path* γ on the $\tilde{W}$ having the length n. Paths define the equivalence relation on $\tilde{W}$ which is named the *connection*.

If there is the infinite nonintersecting path $\gamma(\mathbf{x})$ on $\tilde{W}$ having infinite length and the initial vertex $\mathbf{x}$ then one may say that there exists the *percolation* on $\tilde{W}$ from the vertex $\mathbf{x}$. If the percolation probability $P(c, \mathbf{x}) = \Pr\{\exists(\gamma(\mathbf{x}) \subset \tilde{W})\} > 0$ then one may say that the Bernoulli random field has the percolation from the vertex $\mathbf{x}$ at the *concentration c*. For the cubic lattice, due to its uniformity, the percolation probability does not depend on the vertex $\mathbf{x}$, $P(c, \mathbf{x}) \equiv P(c)$. Therefore, there exists the value $c_* = \inf\{c : P(c) > 0\}$ named the *percolation threshold*.

2. CLUSTER EXPANSION

Since the connection relation is defined on configurations by means of paths settled in them, it decomposes each configuration on connected nonintersecting components as each equivalence relation is done. They are named *clusters*. If the cluster belonging to the configuration $\tilde{W}$ contains the vertex $\mathbf{x}$, it is denoted by $\tilde{W}(\mathbf{x})$. If the cluster $\tilde{W}(\mathbf{x})$ is infinite, and only in this case, there exists the percolation on the corresponding configuration. Let Γ be the class of all finite clusters containing the vertex $\mathbf{0}$. Consequently, the probability $\Pr\{\tilde{c}(\mathbf{0}) = 1\} = c$ of the event when the zero vertex is fulled up is summarized of the following probabilities. Firstly, it is all probabilities of events when the chosen vertex is contained in one of clusters from the class Γ, and, secondly, it is the probability of the event when this vertex is contained in the infinite cluster. Consequently, it is fulfilled the so-called *cluster expansion* [2]

$$c = P(c) + \sum_{W \in \Gamma} P(W), \qquad (1)$$

here $P(W) = \Pr\{W \subset \tilde{W}\}$ is the probability of the fact that the cluster W is contained in the configuration $\tilde{W}$. The cluster expansion is converged at each concentration c according to its building. Therefore, it may be the source of approximations of the percolation threshold in that case when one may find some upper estimates of remainders corresponding to initial terms of series (1). In such a case one may obtain the upper estimate of the percolation threshold c_*. Besides, generally speaking, more accurate upper estimates of the percolation threshold are obtained by more accurate estimates of series remainders.

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Upper estimates of the series (1) are built on the basis of the concept of external border $\partial_+ W$ [1] of each finite cluster W. The border ∂W of the finite cluster W is the set $\{\mathbf{x} \notin W : \exists(\mathbf{y}\varphi\mathbf{x} : \mathbf{y} \in W)\}$. The external border $\partial_+ W$ of the finite cluster W is the set $\{\mathbf{x} \notin W : \exists(\gamma(\mathbf{x}) \subset \mathbf{Z}^3 : \gamma(\mathbf{x}) \cap (\partial W \cup W \setminus \{\mathbf{x}\}) = \emptyset)\}$.

It is fulfilled the elementary estimate

$$P(W) < (1 - c)^{|\partial_+ W|}. \qquad (2)$$

Obtaining of upper estimates of series (1) remainders is based on the using of this inequality. However, in the connection with the inequality (2), it is plausible to do them by the following way: to introduce the class Δ of all sets such that each of them may be external border of anything finite cluster containing the vertex $\mathbf{0}$. In this case, the upper estimate of the sum of probabilities $P(W)$ on all clusters W corresponding to the same external border $S \in \Delta$ takes place. This estimate is analogous to (2),

$$P[S] \equiv \sum_{W \in \Delta \, : \, \partial_+ W = S} P(W) < (1 - c)^{|S|}. \qquad (3)$$

Then the upper estimate takes place

$$\sum_{S \in \Delta \, : \, |S| \geq n} P[S] < \sum_{l=n}^{\infty} (1 - c)^l N_l, \qquad (4)$$

where N_l is the number of sets belonging to the class Δ and having the "area" $|S|$ being less than l.

Estimation of the series convergence interval and, therefore, obtaining of the upper estimate of percolation threshold are reduced to calculation of remainder estimates of the last series in (4).

3. UPPER ESTIMATE OF NUMBER N_l

Thus, the important source of upper estimates of series remainders are produced by possibly more accurate upper estimates of main term of the asymptotic $\ln N_l$ when $l \to \infty$. In problems of discrete percolation theory on plane lattices, the upper estimates obtaining of the number N_l is based on the Kesten theorem which consists of the assertion: for each plane lattice, the class Δ consists of simple closed contours on the *conjugate* lattice which surround the vertex $\mathbf{0}$. Therefore, we obtain the reduction of the problem under consideration to the estimation of the number of all pointed out contours having the length l. We have proved geometrical theorem analogous to Kesten's one when the lattice is cubic.

Theorem 1. *For cubic lattice, the class Δ consists of all sets on it such that they are connected on the conjugate lattice and may be imbedded by homeomorphic way without edge intersection on closed oriented surface in $\mathbf{R}^3$. Each set of the class Δ decomposes the surface on foursided faces when this imbedding is done.*

The proved theorem permits to construct the enumeration algorithm of the class Δ and, therefore, to find upper estimates of the number N_l. Ii is proved the following assertion.

Theorem 2. *For cubic lattice $\mathbf{Z}^3$, it takes place the inequality*

$$N_l < \frac{l - 2}{4} 5^{l-1}. \qquad (5)$$

4. UPPER ESTIMATE OF PERCOLATION THRESHOLD

Upper estimates of percolation threshold are obtained on the basis of lower estimates of those concentrations $\Pr\{\tilde{c}(\mathbf{0}) = 1\} = c$ for which the series $\sum_l (1-c)^l N_l$ [3] is converged. In a result, as a consequence from the inequality (5), we have gone to the following assertion.

Theorem 3. *The percolation threshold of uniform Bernoulli field on cubic lattice satisfies the inequality $c_* < 0,8$.*

At last, the estimate (5) gives the possibility to find the accuracy of approximation of the percolation probability. In particular, in the case of restriction by one series term $P(W) = c(1 - c)^6$ corresponding to the cluster $\{\mathbf{0}\}$, we obtain that the accuracy of such a probability approximation is given by the estimate

$$c(1 - (1 - c)^6) - P(c) \leq \sum_{l=10, \, l-\text{even}}^{\infty} (1 - c)^l N_l \leq$$

$$\leq \frac{1}{20} \sum_{l=10, \, l-\text{even}}^{\infty} (l - 2)[5(1 - c)]^l,$$

when $c > 0,8$.

5. CONCLUSIONS

We have found the guaranteed estimates of percolation threshold in the cite percolation problem on three-dimensional lattice. Up to now rigorous mathematical results in discrete percolation theory connected with concrete evaluation or accurate estimation percolation characteristics are known only for two-dimensional lattices.

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КЛАСТЕРНОЕ РАЗЛОЖЕНИЕ В ПЕРКОЛЯЦИОННОЙ ЗАДАЧЕ УЗЛОВ НА КУБИЧЕСКОЙ РЕШЁТКЕ

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Построено кластерное разложение для вероятности перколяции бернуллиевского случайного поля на кубической решётке. На его основе получена верхняя гарантированная оценка для порога перколяции и даны оценки точности получаемых при использовании приближений.

КЛАСТЕРНИЙ РОЗКЛАД У ПЕРКОЛЯЦІЙНІЙ ПРОБЛЕМІ ВУЗЛІВ НА КУБІЧНІЙ РЕШІТЦІ

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Побудовано кластерний розклад для імовірності перколяції бернулієвського випадкового поля на кубічній решітці. На його основі одержана верхня гарантована оцінка для порогу перколяції і подані оцінки точності наближень, які застосовуються.