# CRITICAL PHENOMENA AND CRITICAL DIMENSIONS IN ANISOTROPIC NONLINEAR SYSTEMS

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(Received October 30, 2011)

The model that allows one to generalize the notions of the multicritical and Lifshitz points is considered. The model under consideration includes the higher powers and derivatives of order parameters. Critical phenomena in such systems were studied. We assess the lower and upper critical dimensions of these systems. These calculation enable us to find the fluctuation region where the mean field theory description does not work.

PACS: 64.60.Kw, 64.60-i

## 1. INTRODUCTION

Second order phase transitions (PT) are ones of the most intensively investigated phenomena in theoretical physics. A lot of different kinds of PT have been observed and investigated in various physical systems [1–3]. Initially the basics objects of PT theory application were various condensed matter systems. However such fundamentals of PT theory as the spontaneous symmetry breaking, the group-theoretical approach allow one to use predictions of the theory of PT in the fields of physics which from the first view have nothing in common with above mentioned condensed matter systems.

One of the most important properties of the systems in vicinity of critical point is a strong increase of fluctuations influence. The influence of the fluctuations strongly depends on spatial dimension. In modern theory of critical phenomena the space dimensionality "*d*" is usually considered as continuous value [4,5]. It appears in thermodynamic relations as one of the parameters of the system. As mentioned above a critical behavior of the system strongly depends on it.

One of the effects of this dependence is an existence of the 2 critical (or borderline) dimensions. Lower critical dimension determines the range of the existence ordering states: there are no PT at nonzero temperature if the space dimensionality is less than lower critical dimension, or in other words at lower critical dimension Goldstone bosons start to interact strongly [6]. Upper critical dimension determines a range of the mean field based theories applicability. So if we plot an axis (Fig. 1) of dimensionality then the critical dimensions divide it into 3 regions ($d_l$ and $d_u$ are the upper and lower critical dimensions).

The critical dimensions are important not only as "borders". They are necessary elements for calculations of critical indexes in the fluctuation region by various methods based on the renormalization group. In condensed matter theory a consideration of systems with $d > 3$ is just a trick that allows one to calculate the critical indexes by the renormgroup methods. But in such fields as particle physics, general relativity and cosmology it is necessary to use models with higher space dimensionalities [7,8].

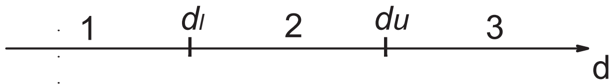


***Fig. 1.*** *1. No ordering. 2. Fluctuation region. PT are possible, but the mean field approximation is invalid. 3. The fluctuations are damped. The mean field approximation is valid*

## 2. MULTICRITICAL AND LIFSHITZ POINTS

In the simplest model of PT an expansion of thermodynamic potential (TP) looks as follows:

$$\Phi = \Phi_0 + a\varphi^2 + b\varphi^4. \qquad (1)$$

Here lower and upper critical dimensions equal 2 and 4 correspondingly. In more complicated models critical dimensions depend on the model parameters.

There are a few ways to generalize model (1). The first one is to take into account terms with higher powers of the order parameters in the expansion of the TP. It leads one to the notion of the multicritical point. Multicritical points of various types have been observed in many physical systems. The most known example of a system with the tricritical point (TCP) is the mixture of helium isotopes $He^3$-$He^4$. There are TCPs on phase diagrams of some ferroelectrics for example in $KH_2PO_4$ (Fig. 2), antiferromagnetics. Upper CD of systems with TCP is equal to 3, so corresponding models are renormalazed in usual physical space.

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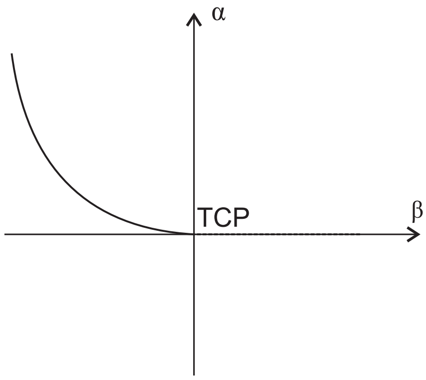


***Fig. 2.*** *Schematic phase diagram for systam with TCP. $\beta > 0$ – second order PT, $\beta < 0$ – first order PT*

Another way is to take into account an inhomogeneity of the order parameters. It leads one to the notion of the Lifshitz point (LP). The theoretical investigations of PT in systems with anisotropic modulations of the order parameters near the Lifshitz points began in the 1970th and have continued up to date. Initially the main fields of application of these theories were systems with anisotropic magnetic ordering. But recently systems with Lifshitz points have increasingly arisen interest in such fields of physics as particle physics, quantum gravity and cosmology [7,8]. Lifshitz points arise in systems with modulated structures of OP. There are 3 phases coexist in LP: initial ordering phase, homogeneous disordered phase and modulated phase (Fig. 3). Modulated structures of different types have been observed in a lot of magnetic and ferroelectric crystals (MnP, $NaNO_2$, $K_2SeO_4$ etc.).

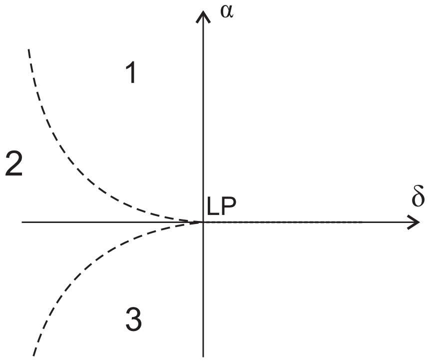


***Fig. 3.*** *Schematic phase diagram for system with Lifshitz point. 1 – ordered state, 2 – state with modulated structures, disordered state*

This paper is about CDs of systems with mutual multicritical and Lifshitz behavior. The simplest example of such point is tricritical Lifshitz point (Fig.4). Point of that kind has been studied in ferroelectrics of type $Sn_2P_2(Se_xS_{1-x})_6$. Corresponding TP:

$$\Phi = \Phi_0 + \frac{\alpha}{2}\varphi^2 + \frac{\beta}{4}\varphi^4 + \frac{\gamma}{6}\varphi^6 + \frac{\delta}{2}\varphi'^2 + \frac{g}{4}\varphi''^2. \quad (2)$$

We introduce a model that allows one to study a system with a new type of critical point. The gradients of OP and the highest power of OP in such model may take arbitrary values.

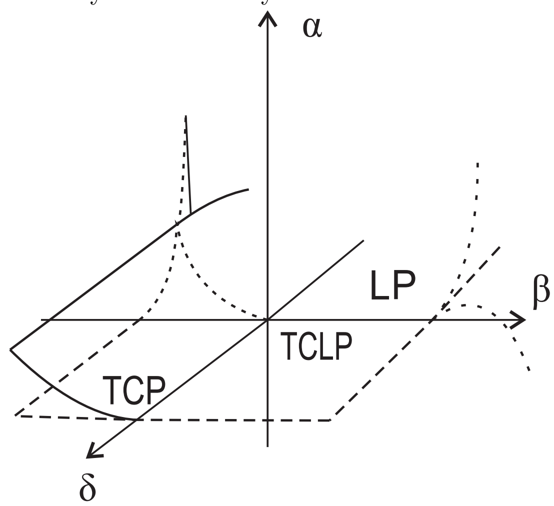


***Fig. 4.*** *Phase diagram for* $Sn_2P_2(Se_xS_{1-x})_6$ *in the vicinity of TCLP*

In the vicinity of the critical point under consideration the effective Hamiltonian can be written as follows:

$$H = \int d^m x_i d^{d-m} x_c \left\{ \frac{r}{2}\eta^2 + \frac{\gamma}{2}\left(\Delta_i^{\frac{1}{2}}\eta\right)^2 + \right.$$
$$\left. + \frac{\delta}{2}\left(\Delta_c^{\frac{1}{2}}\eta\right)^2 + \frac{\beta}{2}\left(\Delta_i^{\frac{p}{2}}\eta\right)^2 + u\eta^{N+1} \right\}, \quad (3)$$

where $\eta$ is a one-component order parameter, $d$ is the space dimension and $r, \gamma, \delta, \beta, u$ are the model parameters. We assume that the space can be divided into two subspaces of dimensions $m$ and $d-m$ denoted by $i$ and $c$ respectively. There are wave modulation vectors in the first subspace and none in the second one. Let us assume $d$ and $m$ to be continuous variables and $d > m$. $\Delta_c$ and $\Delta_i$ are the Laplacian operators available in the corresponding subspaces. In this case the operators $\Delta^l$ are defined as $\Delta^l = \Delta\left(\Delta^{l-1}\right)$. For non-integer values of l and m the corresponding operators are determined using the inverse Fourier transformation. In the critical point $r = \gamma = 0$, and the other parameters in (3) are positive quantities.

Using the Fourier transformation :

$$\eta(x) = \int d^d q \cdot \eta(q) e^{iqx}, \quad (4)$$

we rewrite the Hamiltonian (3) in momentum space:

$$H = \frac{1}{2}\int d^d q \nu(q) \eta(q) \eta(-q) +$$
$$+ u \int d^d q_1 \cdot \ldots \cdot d^d q_{N+1} \quad (5)$$
$$\times \delta(q_1 + \ldots + q_{N+1}) (\eta(q_1) \ldots \eta(q_{N+1})).$$

Here $\nu(q) = r + \gamma q_i^2 + \beta q_i^{2p} + \delta q_c^2$, $q_i$ and $q_c$ are the absolute values of the wave vector $\overrightarrow{q}$ being in the sectors $i$ and $c$ respectively, and $q_i^2 = \sum\limits_{\alpha=1}^{m} q_\alpha^2$, $q_c^{\,2} = \sum\limits_{\alpha=m+1}^{d} q_\alpha^2$.

The main parameters which determine the critical behavior of the system with the Hamiltonian (3) are the order of higher gradients $p$, the dimension of the subspace of modulation, and the order of nonlinearity of the model.

## 3. THE CRITICAL DIMENSIONS OF THE ANISOTROPIC SYSTEMS

The lower critical dimension $d_l$ is defined by the following condition: there are no ordering states in space with $d < d_l$ under condition of nonzero temperature. From the thermodynamic point of view it means that fluctuation contribution to entropy is a divergent function of temperature. The fluctuation contribution to entropy looks as follows:

$$S_{fl} = s\tau^{\sigma(d)}, \tag{6}$$

here $\tau = (T - T_c)/T_c$ is the reduced temperature, $\sigma(d)$ is a function of space dimensionality and does not depend on T. We are interested in the critical behavior of $S_{fl}$, so:

$$\lim_{\tau\to 0} S_{fl} = \begin{cases} 0 & \sigma(d) < 0, \\ \infty & \sigma(d) > 0. \end{cases} \tag{7}$$

One can see that under condition $T_c \neq 0$ the fluctuation contribution to entropy is a divergent function if $\sigma(d) > 0$ otherwise it goes to zero. Thus one can find the lower critical dimension from the following condition:

$$\sigma(d_l) = 0. \tag{8}$$

In order to calculate $\sigma(d_l)$ let us find how the fluctuation contribution to entropy depends on the temperature. By definition:

$$S_{fl}(\tau) = \frac{\partial G_{fl}}{\partial \tau}, \tag{9}$$

here $G_{fl}$ is a fluctuation contribution to the thermodynamic potential, in model (3) it looks as:

$$G_{fl} = A \int d^m q_i \cdot d^{d-m} q_c \ln \frac{\beta\, q_i^{2p} + \delta\, q_c^2 + \alpha\,\tau}{\pi T}, \tag{10}$$

where A is a parameter that does not depend on temperature. From (9) and (10):

$$S_{fl} = \alpha A \int d^m q_i \cdot d^{d-m} q_c \left(\beta\, q_i^{2p} + \delta\, q_c^2 + \alpha\,\tau\right)^{-1}, \tag{11}$$

after some manipulations:

$$S_{fl} = \tau^{-1}\alpha A \int d^m q_i \cdot d^{d-m} q_c^2 \left\{\beta \cdot \left(q_i \cdot \tau^{-\frac{1}{2p}}\right)^{2p} + \right.$$
$$\left. +\gamma \cdot \left(q_c \cdot \tau^{-\frac{1}{2}}\right)^2 + \alpha\right\}^{-2}. \tag{12}$$

After the change of variables $\kappa_i = q_i \cdot \tau^{-\frac{1}{2p}}, \kappa_c = q_c \cdot \tau^{-\frac{1}{2}}$ the final expression for $S_{fl}$ takes the form:

$$S_{fl} = \tau^{\sigma(d)} \cdot I(\kappa_i, \kappa_c), \tag{13}$$

here $\sigma(d) = 2 - (m/2p + (d - m)/2)$ and the function $I(\kappa_i, \kappa_c)$ does not depend on $t$. And finally for $\sigma(d)$:

$$\sigma(d) = \frac{m}{2p} + \frac{d - m}{2} - 1, \tag{14}$$

From (8) and (14):

$$d_l = m\left(1 - \frac{1}{p}\right) + 2. \tag{15}$$

There are several ways to calculate the upper critical dimension. First is similar to way we calculated $d_l$ - comparing the fluctuation contribution to a heat capcity:

$$C_V = T\frac{\partial^2 \Phi}{\partial T^2} = \Phi = B\frac{2\,(N+1)}{(N-1)^2} \cdot t^{-\alpha_l}, \tag{16}$$

where $\alpha_l = 2 - \frac{N+1}{N-1}$.
With fluctuation correction to the heat capacity:

$$C_{1,fl} = t^{-\alpha_{fl}} \cdot I(\kappa_i, \kappa_c), \tag{17}$$

where $\alpha_{fl} = 2 - \left(\frac{m}{2p} + \frac{d-m}{2}\right)$ and the function $I(\kappa_i, \kappa_c)$ does not depend on $t$.

Second is to find $d_u$ from the stability condition of the fixed point of corresponding renomgroup transformation:

$$\nu'(q) = z^2 a^{-m} b^{-(d-m)}$$
$$\times \left(r + \delta\frac{q_c^2}{b^2} + \gamma\frac{q_i^2}{a^2} + \beta\frac{q_i^{2p}}{a^{2p}} + \ldots\right), \tag{18}$$

$$u'(q) = z^{N+1} a^{-N} b^{-N(d-m)} \cdot (u + \ldots). \tag{19}$$

Here we take into account that the scale parameters changing the scale for the momenta are independent in the sectors $i$ and $c$. We denote them $a$ and $b$ respectively. The scale parameter for $u$ is denoted by $z$. Both of those ways lead us to the following expression for $d_u$ [9]:

$$d_u = m\left(1 - \frac{1}{p}\right) + 2\frac{N+1}{N-1}. \tag{20}$$

Also one can find $d_u$ from the condition of scale variation invariance of the model (3) under transformation with generator:

$$X = \frac{N-1}{2}\frac{\partial}{\partial x_c} + \frac{N-1}{2p}\frac{\partial}{\partial x_i} - \frac{\partial}{\partial \varphi}. \tag{21}$$

## 4. DISCUSSION

The spatial distribution of an order parameter is defined by the solution of the variational equation for the functional of the free energy. This equation is a nonlinear differential equation (NDE). In our model the order of corresponding equation is $2p$. At present there are no universal methods for solving the NDE, therefore of fundamental importance are the methods enabling to simplify the NDE analysis. One of the most effective ways for the NDE analysis is the analysis of its symmetries. Investigation in the group NDE structure allows one to determine the behavior of solutions without defining their explicit form, to find different NDE invariants, in particular, to construct the conservation laws. Knowing the symmetry groups of the assumed NDE enables also to construct important classes of particular solutions. Every additional NDE-admitted Lie group allows reducing the equation order by one. The variational symmetries play here a very important role. These symmetries belong to the variational equation, as well as, to the functional, for which the given equation is variational. The presence of the variational symmetry in the NDE makes it possible to reduce its order by 2. The importance of the variational symmetries is due to the conservation laws related to them. In paper [9] was shown that model (3) is invariant under scale variational transformations with generators:

$$\hat{X} = \frac{N-1}{2}\frac{\partial}{\partial x_c} + \frac{N-1}{2p}\frac{\partial}{\partial x_i} - \frac{\partial}{\partial \varphi}. \quad (22)$$

Let us find the range of the fluctuation region:

$$\Delta d \equiv d_u - d_l = \frac{4}{N-1}, \text{and} \lim_{N\to\infty}(d_u - d_l) = 0. \quad (23)$$

So, the fluctuation region decreases as a function of power of nonlinearity (see Fig. 5). This fact is physically reasonable. Strong coupling supresses the fluctuations. As it is expected, the lower critical dimension of any systems is not less then 2.

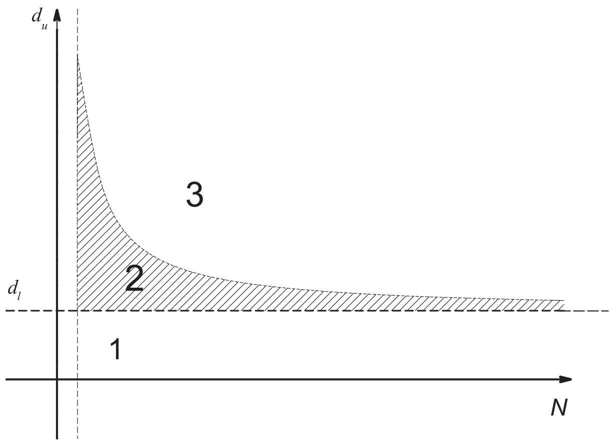


***Fig. 5.*** *Dependence of $\Delta d$ on the order of nonlinearity of the model*

Let us compare the obtained results with those known previously. As mentioned above, in the case of the simple critical point ($N = 3, p = 1, m = 0$) the lower and upper CDs are equal to 2 and 4 respectively. The CDs of the system with $m$-axial Lifshitz point ($N = 3, p = 2$) depend on $m$ linearly (some special cases in 3 and 4 dimensional spaces will be discussed later) . An interesting case is the model with an isotropic Lifshitz-like point with derivatives of order up to $p$ (Lifshitz point of order $p$). The variational equations in this model coincide with Multidimensional Polywave Equation that is invariant under conformal transformations.

Let us consider what types of anisotropic system in 3 and 4 dimensional spaces are possible according to the condition $d > d_l$. There are 2 types of anisotropic systems in three-dimensional space:

1) usual critical point (without anisotropy) $d_l = 2$, and

2) 1-axial Lifshitz point $d_l = 2,5$. In other cases $d_l \geq 3$.

In the case of 4-dimensional space:

1) usual critical point (without anisotropy) $d_l = 2$, and

2) 1-, 2-, 3-axial Lifshitz point of order 2 $d_l = 2.5,\ 3,\ 3.5$.

3) 1-, 2-axial Lifshits point of order $p > 2$, $d_l = 3 - 1/p,\ 4 - 1/p$. In other cases $d_l \geq 4$.

In general cases: for some initial order $p$ of the Lifshitz point, the ordering in the space with dimensionality $d > 2p$ with any kind of anisotropy is possible. In case of $d \leq 2p$, there are various situations and this case demands additional investigation.

The obtained results are correct for classical PTs. As we know, in the theory of quantum PTs the effective dimension of a system in the vicinity of the quantum critical point is higher than a dimension of space. So it is apparent that there are more possible types of PTs in a quantum case. In particular, our results do not contradict a possibility of quantum PTs in 2-dimensional systems.

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## КРИТИЧЕСКИЕ ЯВЛЕНИЯ И КРИТИЧЕСКИЕ РАЗМЕРНОСТИ АНИЗОТРОПНЫХ НЕЛИНЕЙНЫХ СИСТЕМ

***А.В. Бабич, С.В. Березовский, Л.Н. Киценко, В.Ф. Клепиков***

Предложена модель, позволяющая обобщить понятия точек Лифшица и мультикритических точек. Предложенная модель учитывает в термодинамическом потенциале как высшие градиенты параметров порядка, так и высшие нелинейности. Рассчитаны верхняя и нижняя критические размерности для такой модели. Полученные результаты позволяют определить флуктуационную область, в которой приближение среднего поля не работает.

## КРИТИЧНІ ЯВИЩА І КРИТИЧНІ РОЗМІРНОСТІ АНІЗОТРОПНИХ НЕЛІНІЙНИХ СИСТЕМ

***А.В. Бабіч, С.В. Березовський, Л.М. Кіценко, В.Ф. Клепіков***

Запропоновано модель, яка дозволяє узагальнити поняття точок Ліфшица и мультикритичних точок. Модель, що запропоновано, враховує в термодинамичному потенціалі як вищи градієнти параметрів порядку, так і вищи нелінійності. Розраховано верхню і нижчу критичні размірності для такої моделі. Здобуті результати дозволяють визначити флуктуаційну область, в якій наближення середнього поля не дійсно.